Some results for graphs without long cycles

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joint work with Zoltán Füredi, Alexandr Kostochka, and Jacques Verstraëte

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Graphs with circumference $< k$

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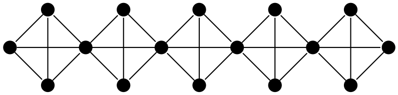

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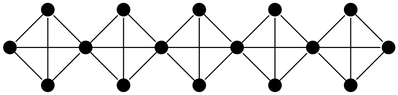
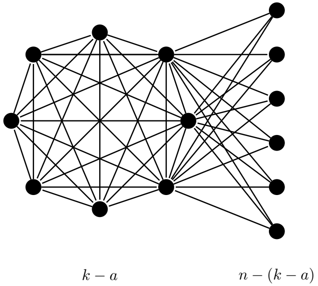


Note that this example is not 2-connected.

Classical theorems

Fix $n \geq k, k/2 > a \geq 1$. Define $H_{n,k,a} = A \cup B$ where A is a complete graph on $k - a$ vertices, B is an independent set on $n - (k - a)$ vertices, and each vertex in B has the same a neighbors in A.



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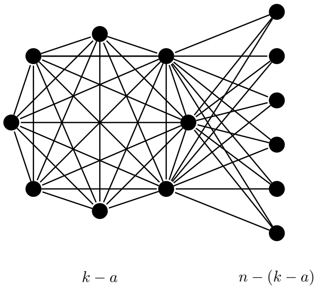

Note that $H_{n,k,a}$ has no cycle of length at least k.

Classical theorems

Define $h(n, k, a) := e(H_{n,k,a}) = \binom{k-a}{2} + a(n-k+a)$.
If n and k are fixed, then $h(n, k, a)$ is convex in a. Therefore $h(n, k, a)$ is maximized at one of the endpoints of its domain, either $a \in \{2, \lfloor \frac{k-1}{2} \rfloor\}$.

Theorem (Kopylov (1977))

Let $n \geq k \geq 5$. If G is an n-vertex 2-connected graph with circumference less than k, then

$$e(G) \leq \max\{h(n, k, 2), h(n, k, \lfloor \frac{k-1}{2} \rfloor)\}$$

with equality only if $G = H_{n,k,2}$ or $G = H_{n,k,\lfloor \frac{k-1}{2} \rfloor}$.

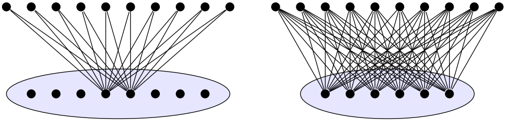

Figure: $H_{n,k,2}$ and $H_{n,k,\lfloor \frac{k-1}{2} \rfloor}$($k$ odd).

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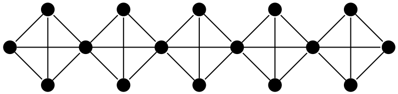

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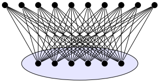
Let $t \geq 4$, $n \geq 3t$, and $k = 2t + 1$. Let G be a 2-connected n-vertex graph with circumference less than k. If $e(G) > h(n, k, t - 1)$, then $G \subseteq H_{n,k,t}$.



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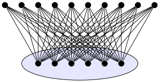

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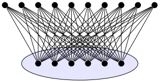
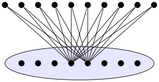


Note that $h(n,k,t) - h(n,k,t-1) = n - t - 3$.

Is it true for all n?
No! For n small ($n < 5t/2$), $e(H_{n,k,2}) > h(n,k,t-1)$, $H_{n,k,2}$ is 2-connected and has no cycle of length at least k.



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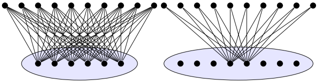

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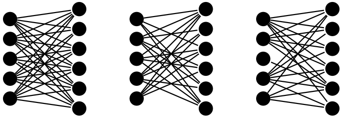
Let $\mathcal{F}$ denote the class of graphs obtained by removing at most $t-3$ edges from $K_{t,t+1}$.



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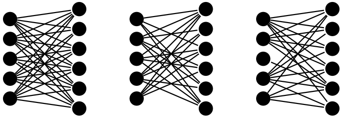

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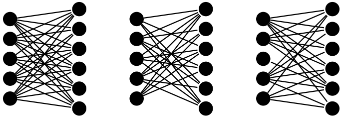

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Lemma 2 (folklore): Let $n \geq 4$ and let G be an n-vertex 2-connected graph. For every $v \in V(G)$, there exists a $w \in N(v)$ such that G/vw is 2-connected.

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Lemma 3 (Main lemma on contraction) (F-K-V): Let $k \geq 9$ and suppose G and G' are 2-connected graphs such that $G = G'/xy$ and G' has circumference less than k. If G contains a subgraph $F \in \mathcal{F}$ then G' also contains a subgraph of $F' \in \mathcal{F}$.

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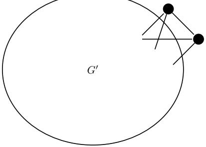

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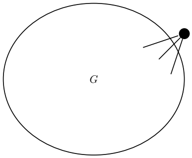

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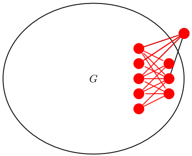

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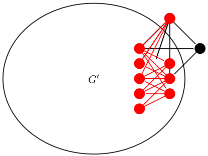

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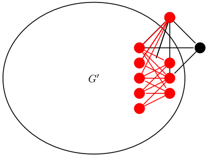

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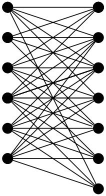

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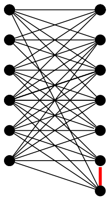

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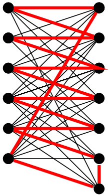

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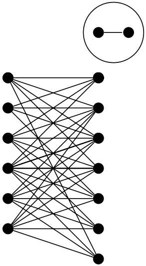

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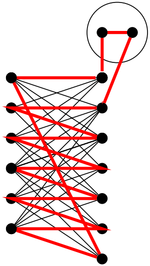

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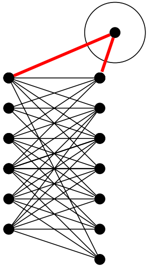

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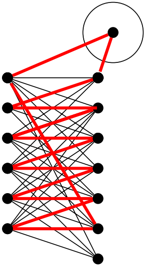

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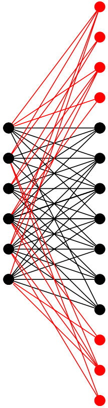

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Note that we cannot create a cycle of length at least k after contracting.

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We show that either $G_k \subseteq H_{k,k,t}$ or G_k has small minimum degree $(\delta(G_k) < (t+4)/3)$.

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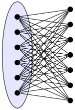

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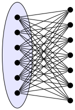

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In particular, $G_n = G$ contains a subgraph $F' \in \mathcal{F}$. Then by Lemma 4, $G \subseteq H_{n,k,t}$.

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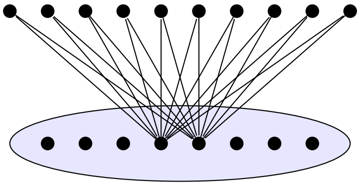
Goal: prove that $G^* = H_{n,k,2}$, so therefore $G \subseteq H_{n,k,2}$.



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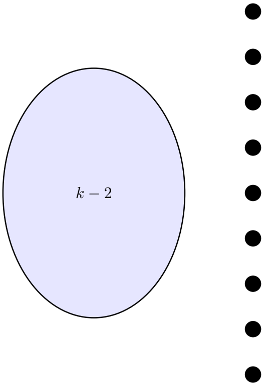

We show that G^* must contain a large complete graph K_{k-2}.

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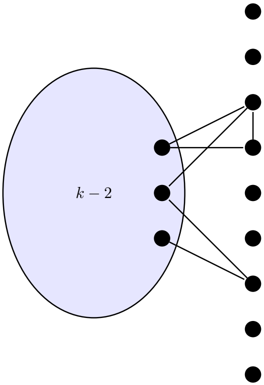
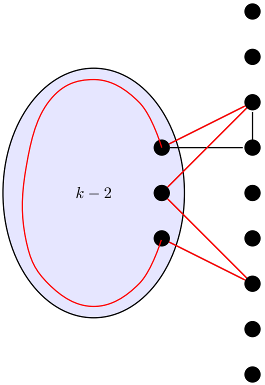


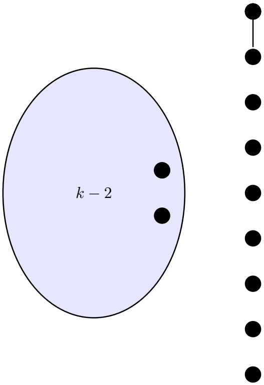
Suppose there exists at least 3 vertices in the K_{k-2} with outside neighbors.

Proof idea



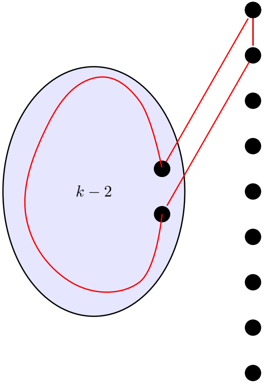
Then because G is 2-connected, we can find a path with at least 2 vertices outside K_{k-2} and easily extend it to a cycle with $\geq k$ vertices.

Proof idea



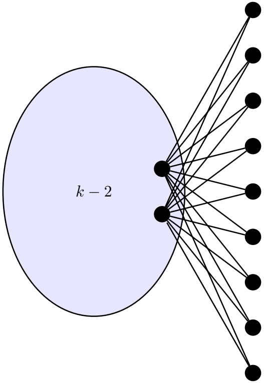
So only two vertices in the K_{k-2} have outside neighbors. Suppose there is an edge between vertices outside of the K_{k-2}.

Proof idea



Then again we may find a long cycle

Proof idea



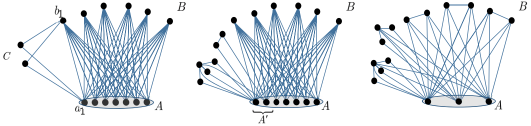
Therefore $G^* = H_{n,k,2}$.

Full version of theorem

Theorem (Füredi, Kostochka, L., Verstraëte (2017+))

Let $t \geq 4$ *and* $k \in \{2t+1, 2t+2\}$*. If* G *is a 2-connected graph on* $n \geq k$ *vertices and circumference less than* k*, then either* $e(G) \leq \max\{h(n,k,t-1), h(n,k,3)\}$ *or*

(a) $k = 2t+1$, *and* $G \subseteq H_{n,k,t}$ *or*
(b) $k = 2t+2$, *and* $G - A$ *is a star forest for some* $A \subseteq V(G)$ *of size at most* t.
(c) $G \subseteq H_{n,k,2}$.



Full version of theorem

Theorem (Füredi, Kostochka, L., Verstraëte (2017+))

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(a) $k = 2t+1$, *and* $G \subseteq H_{n,k,t}$ *or*
(b) $k = 2t+2$, *and* $G - A$ *is a star forest for some* $A \subseteq V(G)$ *of size at most* t.
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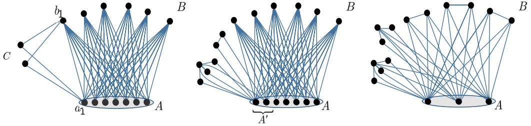

Thanks for listening!