# The Erdős–Gallai theorem for cycles in hypergraphs

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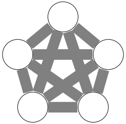
**Theorem (Turán 1941)**

$ex(n, K_r) = \binom{r-1}{2}\left(\frac{n}{r-1}\right)^2 = \left(\frac{r-2}{r-1}\right)\binom{n}{2}$.



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*Let $F$ be any graph with chromatic number $\chi(F) \geq 3$. Then* $ex(n, F) = \left(\frac{\chi(F)-2}{\chi(F)-1}\right)\binom{n}{2} + o(n^2)$.

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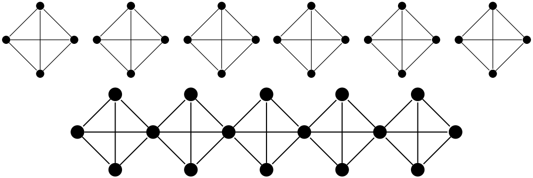
- (Erdős 1964): $ex(n, C_{2k}) \leq \gamma_k n^{1+1/k}$.
- Still open: is $ex(n, C_{2k}) = \Theta(n^{1+1/k})$??

# Paths and long cycles

**Theorem (Erdős and Gallai 1959)**

*1.* $ex(n, P_k) \leq \frac{n}{k-1}\binom{k-1}{2} = \frac{1}{2}n(k-2)$ *where* $P_k$ *is the path on* $k$ *vertices.*

*2. Let* $G$ *be an* $n$*-vertex graph with no cycle of length at least* $k$*. Then* $e(G) \leq \frac{n-1}{k-2}\binom{k-1}{2} = \frac{1}{2}(n-1)(k-1)$*.*



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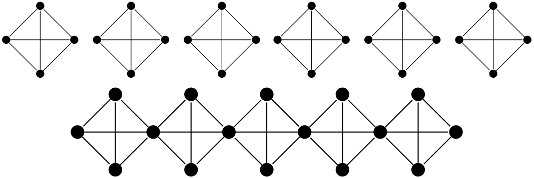

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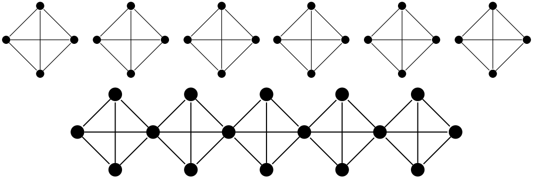

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Our goal: a hypergraph analogue of Erdős–Gallai.

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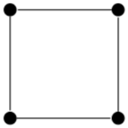
A *berge cycle* of length $\ell$ in a hypergraph is a set of $\ell$ distinct hyperedges $e_1, \ldots, e_\ell$ and $\ell$ distinct vertices $v_1, \ldots, v_\ell$ such that $v_i \in e_i \cap e_{i+1}$ (with indices modulo $\ell$).



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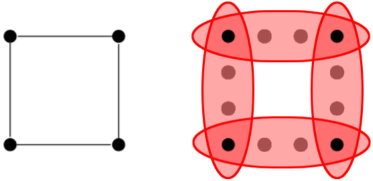

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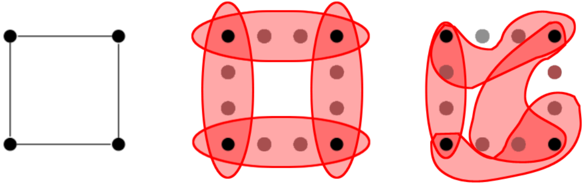

Figure: a cycle and two different berge cycles

For general graphs: let $F$ be a graph. A hypergraph $\mathcal{H}$ contains a *berge F* if there exists an injective function $f : E(F) \to E(\mathcal{H})$ such that $e \subset f(e)$ for all $e \in E(F)$.

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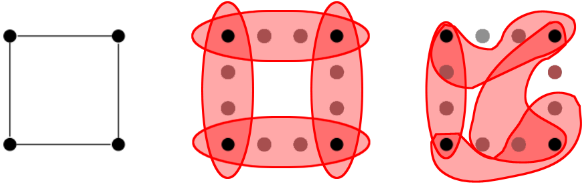

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Analogue: forbid berge copies of a graph in a hypergraph, count the maximum number of hyperedges.

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**Theorem (Gerbner-Methuku-Vizer 2017)**

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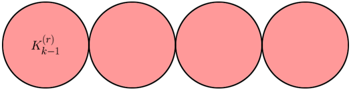
# Hypergraph analogue of Erdős–Gallai, part I

**Theorem (Füredi-Kostochka-L. and Kostochka-L. 2018+)**

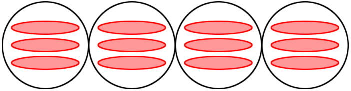
*Let $\mathcal{H}$ be an $n$-vertex $r$-uniform hypergraph with Berge cycle of length $k$ or longer. Then*

1. $r \le k-3$ *and* $e(\mathcal{H}) \le \frac{n-1}{k-2}\binom{k-1}{r}$, *or*
2. $r \ge k+1$ *and* $e(\mathcal{H}) \le \frac{n-1}{r}(k-1)$.

Extremal examples:




$r \le k-3$: each "block" is a $K_{k-1}^{(r)}$.



$r \ge k+1$: each "block" contains $r+1$ vertices and $k-1$ hyperedges.

# Proof idea

For the small $r$ case ($r \leq k-3$): we use a "careful" version of

$$ex_r(n, C_{\geq k}) \leq ex(n, K_r, C_{\geq k}) + ex(n, C_{\geq k}),$$

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$$ex_r(n, C_{\geq k}) \leq ex(n, K_r, C_{\geq k}) + ex(n, C_{\geq k}),$$

in the end we get $ex_r(n, C_{\geq k}) \leq \frac{n-1}{k-2}\binom{k-1}{r} (= ex(n, K_r, C_{\geq k}))$.

# Proof idea

For the small $r$ case ($r \leq k-3$): we use a "careful" version of

$$ex_r(n, C_{\geq k}) \leq ex(n, K_r, C_{\geq k}) + ex(n, C_{\geq k}),$$

in the end we get $ex_r(n, C_{\geq k}) \leq \frac{n-1}{k-2}\binom{k-1}{r} (= ex(n, K_r, C_{\geq k}))$.

For large $r$ ($r \geq k+1$), this method does not work:

$$ex_r(n, C_{\geq k}) \leq ex(n, K_r, C_{\geq k}) + ex(n, C_{\geq k})$$

$$= 0 + \frac{n-1}{2}(k-1).$$

But we want $ex_r(n, C_{\geq k}) \leq \frac{n-1}{r}(k-1)$.

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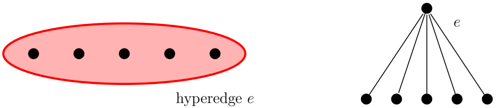

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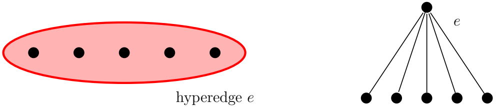


Suppose $C = v_1 e_1 v_2 e_2 \ldots e_\ell v_1$ is a berge cycle in $\mathcal{H}$ of length $\ell$. In the bigraph $G_{\mathcal{H}}$, $C$ is a *cycle of length* $2\ell$.

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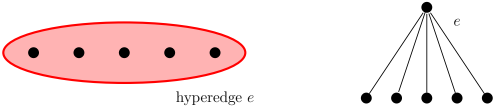


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$\mathcal{H}$ has no cycle of length $\geq k \Leftrightarrow G_{\mathcal{H}}$ has no cycle of length $\geq 2k$.

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Properties of $G_{\mathcal{H}}$:

- $G_{\mathcal{H}}$ has circumference $< 2k$,
- for every $x \in X$, $d_{G_{\mathcal{H}}}(x) = r$ ($r$-uniformity),
- for every $x, x' \in X$, $N_{G_{\mathcal{H}}}(x) \neq N_{G_{\mathcal{H}}}(x')$ (no multi-hyperedges).

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**Theorem (Kostochka-L. 2018+)**

*If $r \geq k+1$, and $G = (X, Y; E)$ is a bipartite graph with $|Y| = n$ satisfying the three conditions above, then $|X| \leq \frac{(k-1)(n-1)}{r}$.*

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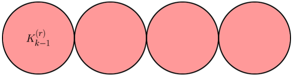
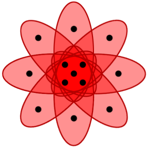



"small" $r$: $\frac{n-1}{k-2}\binom{k-1}{r}$ edges



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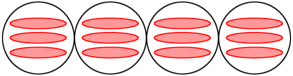

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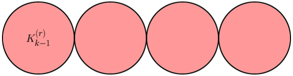


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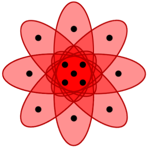

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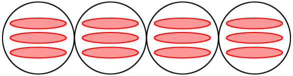

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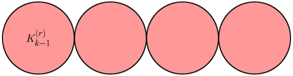


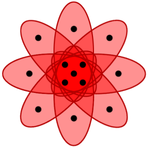

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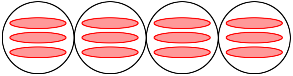

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Thanks for listening!