Linear Algebra review concepts

Linear independence, dependence, and bases

Let \{v_1, v_2, ..., v_n\} be a set of vectors of a vector space V.

- \{v_1, v_2, ..., v_n\} are linearly independent if \(c_1v_1 + c_2v_2 + ... + c_nv_n = 0\) has only the trivial solution \(c_1 = c_2 = ... = c_n = 0\).

- \{v_1, v_2, ..., v_n\} are linearly dependent if they are not linearly independent. I.e., if \(c_1v_1 + c_2v_2 + ... + c_nv_n = 0\) has a nontrivial solution. Note that \(c_1 = c_2 = ... = c_n = 0\) is still a solution; therefore we have at least 2 solutions. It follows that there must be infinitely many solutions to the system.

- If \{v_1, v_2, ..., v_n\} are linearly dependent, there must exist a \(v_i\) that is a linear combination of the other vectors. I.e., \(v_i \in \text{span}\{v_1, ..., v_{i-1}, v_{i+1}, ..., v_n\}\).

- If \(v_i \in \text{span}\{v_1, ..., v_{i-1}, v_{i+1}, ..., v_n\}\), then \(\text{span}\{v_1, ..., v_{i-1}, v_i, v_{i+1}, ..., v_n\} = \text{span}\{v_1, ..., v_{i-1}, v_{i+1}, ..., v_n\}\).

- \{v_1, v_2, ..., v_n\} form a basis of V if the following hold:
  - \(\text{span}\{v_1, v_2, ..., v_n\} = V\),
  - \{v_1, v_2, ..., v_n\} are linearly independent.

- Given a vector space, there are infinitely many choices for a basis. However every basis of the vector space will have the same number of vectors.

- The dimension of V, denoted \(\dim(V)\), is the number of vectors in a basis of V.

- Suppose V has dimension k.
  - Any set of k linearly independent vectors of V form a basis of V (the k vectors therefore span V).
  - Any set of k vectors that span V form a basis of V (the k vectors are therefore linearly independent).

- Given a basis \{v_1, ..., v_n\} of V, any vector \(u \in V\) can be written as a UNIQUE linear combination of the basis vectors.

- If \(B = \{v_1, ..., v_n\}\) is a basis of V and \(u = a_1v_1 + ... + a_nv_n\), then the coordinate vector of u with respect to basis B is \([u]_B = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\), i.e., the vector that records the weights of the linear combination. Sometimes we denote the coordinate vector as just \(u_B\).

Four fundamental subspaces of a matrix

Let A be an \(m \times n\) matrix. \(A = \begin{bmatrix} a_1 & a_2 & ... & a_n \end{bmatrix}\) where \(a_i\) is the \(i\)th column of A.

- The null space of A is \(\text{null}(A) = \{x \in \mathbb{R}^n : Ax = 0\}\).

- The column space of A is \(\text{col}(A) = \text{span}\{a_1, ..., a_n\}\).

- The row space of A is \(\text{col}(A^T)\).

- The left null space of A is \(\text{null}(A^T)\).

- Let A be an \(m \times n\) matrix, and let U be any echelon form of A.

\(^1\)there may be typos.
A linear transformation can be represented as a matrix as follows: Let \[ A \] be a \( m \times n \) matrix and \( x \) a \( n \)-dimensional vector, then \[ T(x) = Ax. \]

### Orthogonality

- Let \( x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \) and \( y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \) be two vectors in \( \mathbb{R}^n \). \( x \) and \( y \) are orthogonal if
  \[ x \cdot y = x_1y_1 + x_2y_2 + \ldots + x_ny_n = 0. \]

- The length or norm of a vector \( x \) is \( ||x|| = \sqrt{x \cdot x} = \sqrt{x_1^2 + \ldots + x_n^2}. \)

- A set of vectors \( \{v_1, \ldots, v_n\} \) is orthogonal if each pair of vectors is orthogonal.

- A set of vectors \( \{v_1, \ldots, v_n\} \) is orthonormal if it is orthogonal and every vector is a unit vector (\( ||v_i|| = 1 \) for all \( i \)).

- Let \( \{v_1, \ldots, v_n\} \) be an orthonormal basis of \( \mathbb{R}^n \). Then for any \( w \in \mathbb{R}^n \), \( w = c_1v_1 + \ldots + c_nv_n \), where \( c_1 = w \cdot v_1 \), \( c_n = w \cdot v_n \).

- Let \( V \) be a subspace of \( \mathbb{R}^n \). The orthogonal complement of \( V \) is\[ V^\perp = \{ w \in \mathbb{R}^n : v \cdot w = 0 \text{ for all } v \in V \}. \]

- \( \dim(V) + \dim(V^\perp) = \dim(\mathbb{R}^n) = n. \)

- **The Fundamental Theorem of Linear Algebra:** for any matrix \( A \),
  - \( \text{null}(A) = \text{col}(A^T). \)
  - \( \text{null}(A^T) = \text{col}(A). \)
An important application of the FTLA: Let $V$ be a subspace of $\mathbb{R}^n$ with a basis $\{v_1, ..., v_k\}$. To find $V^\perp$, we can construct a matrix

$$A = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_k^T \end{bmatrix},$$

where $\text{col}(A^T) = \text{span}\{v_1, ..., v_n\} = V$. Then $V^\perp = \text{col}(A^T)^\perp = \text{nul}(A)$. I.e., we can compute the orthogonal complement of $V$ by finding the null space of $A$.

Graphs

A graph is a set of nodes and edges. Each edge connects two nodes. In our case, the edges are directed.

- If $G$ is a graph with $m$ edges and $n$ nodes, we can represent it by an $m \times n$ matrix $A$ called the incidence matrix. $A$ is defined as follows:

$$A_{i,j} = \begin{cases} 1 & \text{if edge } i \text{ enters node } j \\ -1 & \text{if edge } i \text{ leaves node } j \\ 0 & \text{else} \end{cases}$$

- The rows of $A$ correspond to the edges and the columns correspond to the nodes. Because every edge leaves one node and enters another node, each row of $A$ has one $-1$, one $1$, and all other entries are $0$.

- A vector $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ assigns potential $x_i$ to node $i$. $Ax = 0$ if and only if the difference of potentials across each edge is $0$. This means that in a connected component of $G$, each node receives the same potential.

- A graph $G$ is connected if and only if $\text{nul}(A) = \text{span}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$.

- A basis of $\text{nul}(A)$ corresponds to the connected components of $G$.

- $\text{dim(\text{nul}(A))}$ is the number of connected components of $G$.

- A vector $y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$ assigns current $y_i$ to edge $i$. $A^T y = 0$ if and only if at every node, the current in is equal to the current out.

- One way to ensure "current in = current out" for every vertex is to find the independent loops in the graph and assign each edge in a given loop the same current (where we must consider the direction of the edge; see example below).

- A basis of $\text{nul}(A^T)$ corresponds to independent loops in $G$.

- $\text{dim(\text{nul}(A^T))}$ is the number of independent loops in $G$.

- Example: consider the following graph and its incidence matrix $A$. 

![Graph example](image-url)
- There are 3 connected components given by vertices \{1, 2, 3, 4\}, \{5, 6\}, and \{7\}. Therefore a basis of \(nul(A)\) is

\[
\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1
\end{bmatrix}
\]

Note that a vector in \(nul(A)\) has the same number of coordinates as the number of nodes of the graph.

- There is 1 independent loop in the graph given by edge1, -edge3, -edge2. Therefore a basis of \(nul(A^T)\) is

\[
\begin{bmatrix}
1 \\
-1 \\
0
\end{bmatrix}
\]

Note that a vector in \(nul(A^T)\) has the same number of coordinates as the number of edges of the graph.

More facts about invertible matrices
Let \(A\) be an \(n \times n\) matrix. The following are equivalent:

- \(A\) is invertible.
- There exists a matrix \(A^{-1}\) such that \(AA^{-1} = A^{-1}A = I\).
- \(A\) has \(n\) pivots in echelon form (every row and every column has a pivot).
- The reduced row echelon form of \(A\) is \(I\).
- \(A\) is a product of elementary matrices.
- There is a unique solution to \(Ax = 0\) (\(x = 0\)).
- For any \(b \in \mathbb{R}^n\), \(Ax = b\) has a unique solution.
- \(Nul(A) = \{0\}\).
- \(Col(A) = \mathbb{R}^n\).
- The columns of \(A\) are linearly independent.
- The rows of \(A\) are linearly independent.
- The columns of \(A\) span \(\mathbb{R}^n\).
- The rows of \(A\) span \(\mathbb{R}^n\).
- The columns of \(A\) form a basis of \(\mathbb{R}^n\).
- The rows of \(A\) form a basis of \(\mathbb{R}^n\).
- \(rank(A) = n\).
- \(dim(Nul(A)) = 0\).