A topological proof of the boundedness theorem for $\mathbf{\Sigma}^1_1$ well-founded relations

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Let $\cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$ be an inverse sequence of Polish spaces and continuous maps. We can regard this as a rooted forest, where the $x \in X_0$ are the roots, $y \in f_1^{-1}(x)$ are their children, etc. Then the **inverse limit** $\varprojlim_n X_n = \{(x_0, x_1, \dots) \in \prod_n X_n \mid \forall n\, (x_n = f_{n+1}(x_{n+1}))\}$ is the set of branches through this forest.

Proposition. Suppose each f_i has dense image. Then so does each projection $p_i : \varprojlim_n X_n \to X_i$.

Note that when each f_i is an embedding, this reduces to the Baire category theorem.

Proof. We may assume $i = 0$. Let $U \subseteq X_0$ be nonempty open; we must show that $\mathrm{im}(p_0) \cap U \neq \varnothing$. By replacing X_0 with U, X_1 with $f_1^{-1}(U)$, X_2 with $f_2^{-1}(f_1^{-1}(U))$, etc., it is enough to assume $X_0 = U \neq \varnothing$, and prove that $\varprojlim_n X_n \neq \varnothing$.

Fix a compatible complete metric on each X_n. Let $x_0^0 \in X_0$, then use density of $\mathrm{im}(f_1) \subseteq X_0$ to find $x_1^1 \in X_1$ such that $x_0^1 := f_1(x_1^1)$ is within distance $1/2$ of x_0^0, then use density of $\mathrm{im}(f_2) \subseteq X_1$ to find $x_2^2 \in X_2$ such that $x_1^2 := f_2(x_2^2)$ and $x_0^2 := f_1(x_1^2)$ are within distance $1/4$ of x_1^1, x_0^1 respectively, etc. Then letting $x_n := \lim_{k\to\infty} x_n^k$, we have $(x_n)_n \in \varprojlim_n X_n$. $\square$

Now define inductively

$$\begin{aligned} X_n^0 &:= X_n, \\ X_n^{\alpha+1} &:= \overline{f_{n+1}(X_{n+1}^\alpha)}^{X_n^\alpha}, \\ X_n^\lambda &:= \textstyle\bigcap_{\alpha<\lambda} X_n^\alpha \qquad \text{for } \lambda \text{ limit.} \end{aligned}$$

In other words, at each successor stage we remove the interior of the set of leaves. By induction, $\mathrm{im}(p_n) \subseteq X_n^\alpha$ for all n, α. Each $(X_n^\alpha)_\alpha$ is a descending sequence of closed sets in X_n, hence there is some least countable stage θ at which all the X_n^θ stabilize. Then each $f_{n+1} : X_{n+1}^\theta \to X_n^\theta$ has dense image, whence each $p_i : \varprojlim_n X_n \to X_i^\theta$ has dense image. It follows that each $X_i^\theta = \overline{\mathrm{im}(p_i)}^{X_i}$.

For each n, put

$$\begin{aligned} \rho_n : X_n &\longrightarrow \theta \sqcup \{\infty\} \\ x &\longmapsto \begin{cases} \text{unique } \alpha \text{ s.t. } x \in X_n^\alpha \setminus X_n^{\alpha+1} & \text{if } x \notin X_n^\theta, \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

So $\rho_n^{-1}(<\alpha) = X_n \setminus X_n^\alpha$. If $\rho_{n+1}(x) = \alpha < \infty$, then $x \in X_{n+1}^\alpha$, whence $f_{n+1}(x) \in f_{n+1}(X_{n+1}^\alpha) \subseteq X_n^{\alpha+1}$, i.e., $\rho_n(f_{n+1}(x)) \geq \alpha + 1$. Thus $\rho := \bigsqcup_n \rho_n : \bigsqcup_n X_n \to \theta \sqcup \{\infty\}$ is a monotone map from the forest consisting of the $(X_n, f_n)_n$ as described above to $\theta \sqcup \{\infty\}$. In particular, if $\varprojlim_n X_n = \varnothing$, i.e., the forest is well-founded, then every node in it has rank $< \theta < \omega_1$.

Corollary (boundedness theorem for $\mathbf{\Sigma}^1_1$ well-founded relations)**.** Let X be a Polish space and $R \subseteq X \times X$ be a $\mathbf{\Sigma}^1_1$ well-founded relation. Then R has countable rank.

Proof. Let $g : Y \to X \times X$ be a continuous map from Polish Y with image R. Let

$$X_n := \{(x_0, y_1, x_1, y_2, x_2, \ldots, x_n) \in X \times (Y \times X)^n \mid g(y_1) = (x_1, x_0) \ \& \ \cdots \ \& \ g(y_n) = (x_n, x_{n-1})\}.$$

Then (with the obvious projection maps $p_{n+1} : X_{n+1} \to X_n$) $\varprojlim_n X_n = \varnothing$, since R is well-founded. By an easy induction, for each $(x_0, y_1, x_1, \ldots, x_n) \in X_n$ we have $\rho_n(x_0, \ldots, x_n) \geq$ the R-rank of x_n. So R has rank $\leq \theta < \omega_1$ where θ is as defined above. $\square$