A topological proof of the boundedness theorem for $\Sigma^1_1$ well-founded relations

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Let $\cdots \xrightarrow{f_{n+1}} X_{n+1} \xrightarrow{f_n} X_n$ be an inverse sequence of Polish spaces and continuous maps. We can regard this as a rooted forest, where the $x \in X_0$ are the roots, $y \in f_1^{-1}(x)$ are their children, etc. Then the inverse limit $\lim_{\leftarrow} X_n = \{ (x_0, x_1, \ldots) \in \prod_n X_n \mid \forall n \, (x_n = f_{n+1}(x_{n+1})) \}$ is the set of branches through this forest.

Proposition. Suppose each $f_i$ has dense image. Then so does each projection $p_i : \lim_{\leftarrow} X_n \to X_i$.

Note that when each $f_i$ is an embedding, this reduces to the Baire category theorem.

Proof. We may assume $i = 0$. Let $U \subseteq X_0$ be nonempty open; we must show that $\text{im}(p_0) \cap U \neq \emptyset$.

By replacing $X_0$ with $U$, $X_1$ with $f_1^{-1}(U)$, $X_2$ with $f_2^{-1}(f_1^{-1}(U))$, etc., it is enough to assume $X_0 = U \neq \emptyset$, and prove that $\lim_{\leftarrow} X_n \neq \emptyset$.

Fix a compatible complete metric on each $X_n$. Let $x_0^0 \in X_0$, then use density of $\text{im}(f_1) \subseteq X_0$ to find $x_1^1 \in X_1$ such that $x_1^1 := f_1(x_0^1)$ is within distance $1/2$ of $x_0^0$, then use density of $\text{im}(f_2) \subseteq X_1$ to find $x_2^2 \in X_2$ such that $x_2^2 := f_2(x_1^2)$ and $x_3^2 := f_1(x_2^2)$ are within distance $1/4$ of $x_1^1, x_0^0$ respectively, etc. Then letting $x_n := \lim_{k \to \infty} x_n^k$, we have $(x_n)_n \in \lim_{\leftarrow} X_n$. \hfill $\Box$

Now define inductively

\[
\begin{align*}
X_0^n := X_n, \\
X_{\alpha+1}^n := f_{n+1}(X_{\alpha+1}^{n+1}), \\
X_\lambda^n := \bigcap_{\alpha < \lambda} X_\alpha^n \quad \text{for } \lambda \text{ limit.}
\end{align*}
\]

In other words, at each successor stage we remove the interior of the set of leaves. By induction, $\text{im}(p_n) \subseteq X_\alpha^n$ for all $n, \alpha$. Each $(X_\alpha^n)_\alpha$ is a descending sequence of closed sets in $X_n$, hence there is some least countable stage $\theta$ at which all the $X^\theta_n$ stabilize. Then each $f_{n+1} : X^\theta_{n+1} \to X^\theta_n$ has dense image, whence each $p_i : \lim_{\leftarrow} X_n \to X^\theta_i$ has dense image. It follows that each $X^\theta_i = \text{im}(p_i)^X_i$.

For each $n$, put

\[
\rho_n : X_n \to \theta \sqcup \{ \infty \}
\]

\[
x \mapsto \begin{cases} 
\text{unique } \alpha \text{ s.t. } x \in X^n_\alpha \setminus X^{\alpha+1}_n & \text{if } x \notin X^\theta_n, \\
\infty & \text{otherwise.}
\end{cases}
\]

So $\rho_n^{-1}(\alpha) = X_n \setminus X^n_\alpha$. If $\rho_{n+1}(x) = \alpha < \infty$, then $x \in X^n_{\alpha+1}$, whence $f_{n+1}(x) \in f_{n+1}(X^n_{\alpha+1}) \subseteq X^n_{\alpha+1}$, i.e., $\rho_n(f_{n+1}(x)) \geq \alpha + 1$. Thus $\rho := \bigsqcup_n \rho_n : \bigsqcup_n X_n \to \theta \sqcup \{ \infty \}$ is a monotone map from the forest consisting of the $(X_n, f_n)_n$ as described above to $\theta \sqcup \{ \infty \}$. In particular, if $\lim_{\leftarrow} X_n = \emptyset$, i.e., the forest is well-founded, then every node in it has rank $\theta < \omega_1$.\hfill $\Box$
Corollary (boundedness theorem for $\Sigma_1^1$ well-founded relations). Let $X$ be a Polish space and $R \subseteq X \times X$ be a $\Sigma_1^1$ well-founded relation. Then $R$ has countable rank.

Proof. Let $g : Y \to X \times X$ be a continuous map from Polish $Y$ with image $R$. Let

$$X_n := \{ (x_0, y_1, x_1, y_2, x_2, \ldots, x_n) \in X \times (Y \times X)^n \mid g(y_1) = (x_1, x_0) \wedge \cdots \wedge g(y_n) = (x_n, x_{n-1}) \}.$$ 

Then (with the obvious projection maps $p_{n+1} : X_{n+1} \to X_n$) $\lim_{\leftarrow n} X_n = \emptyset$, since $R$ is well-founded. By an easy induction, for each $(x_0, y_1, x_1, \ldots, x_n) \in X_n$ we have $\rho_n(x_0, \ldots, x_n) \geq$ the $R$-rank of $x_n$. So $R$ has rank $\leq \theta < \omega_1$ where $\theta$ is as defined above. \qed