A simple proof of the Lusin–Suslin theorem

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Lemma. Let $f : X \to Y$ be a continuous map between Polish spaces and Z be Y with a finer Polish topology. Then the finer topology on X obtained by adjoining $f^{-1}(V)$ for open $V \subseteq Z$ is Polish.

Proof. Let $g : Z \to Y$ be the identity map, which is continuous. The new topology on X is induced by the bijection

$$\begin{aligned} X &\cong \{(x, z) \in X \times Z \mid f(x) = g(z)\} \\ x &\mapsto (x, f(x)). \end{aligned}$$

□

Theorem (Lusin–Suslin)**.** Let $f : X \to Y$ be a continuous injection between Polish spaces. Then $f(X) \subseteq Y$ is Borel.

Proof. Let $\mathcal{U}$ be a countable basis of open sets in X. For each $U \in \mathcal{U}$, since f is injective, $f(U), f(X \setminus U) \subseteq Y$ are disjoint, so by the Lusin separation theorem, there is a Borel set $B_U \subseteq Y$ such that $f(U) \subseteq B_U \subseteq X \setminus f(X \setminus U)$, i.e., $U = f^{-1}(B_U)$. Let Y' be Y with a finer Polish topology making each B_U open (and the same Borel sets as Y), and let X' be X with $f^{-1}(V)$ adjoined to its topology for each open $V \subseteq Y'$. Then X' is Polish by the above, and $f : X' \to Y'$ is a continuous embedding, so $f(X) = f(X')$ is $\mathbf{\Pi}^0_2$ in Y', hence Borel in Y. □