On Lusin–Novikov uniformization

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In this note, we prove the Lusin–Novikov uniformization theorem by applying the boundedness theorem for $\mathbf{\Sigma}^1_1$ to the fiberwise Cantor–Bendixson derivative. See [Kec, §34–35] for a detailed treatment of these methods, especially [Kec, 35.43] which is a far-reaching generalization of Lusin–Novikov, Arsenin–Kunugui, and many similar results. The goal of this note is to take a straight-line route to the Lusin–Novikov theorem, developing as little background as is needed.

We will in fact give two (closely related) versions of the proof. The first is shorter and more combinatorial, while the second is more abstract and topological in flavor.

1 Borel derivatives

Let P be a standard Borel poset with

- a least element $\bot$;
- a Borel countable decreasing meet operation $\bigwedge$;
- no uncountable strictly decreasing transfinite sequences.

A **Borel derivative** on P is a Borel map $\partial : P \to P$ which is monotone and obeys $\partial(p) \le p$ for all $p \in P$. Given ∂, we define the transfinite iterates $\partial^\alpha : P \to P$ for all $\alpha < \omega_1$ in the usual way, using that $\bigwedge$ is Borel:

$$\partial^0 := 1_P, \qquad \partial^{\alpha+1} := \partial \circ \partial^\alpha, \qquad \partial^\alpha := \bigwedge_{\beta<\alpha} \partial^\beta \quad \text{for } \alpha \text{ limit.}$$

Since P has no uncountable strictly decreasing transfinite sequences, for each $p \in P$, there is a least $\alpha < \omega_1$ such that $\partial^\alpha(p) = \partial^{\alpha+1}(p)$; we write $\partial^\infty(p) := \partial^\alpha(p)$, and call this α the **rank** $\rho(p)$ of p if $\partial^\infty(p) = \bot$. If $\partial^\infty(p) \ne \bot$, then we define $\rho(p) := \infty$.

Lemma 1 ([Kec, 34.10])**.** For any Borel derivative ∂ as above, there is an analytic binary relation on P which restricts to the relation $\{(p, q) \in P \times P \mid \rho(p) < \rho(q) < \infty\}$ on $\{p \in P \mid \rho(p) < \infty\}$.

Proof. We claim that for $\rho(p), \rho(q) < \infty$, we have $\rho(p) < \rho(q)$ iff there is a countable linear order $(I, <)$ (where $I \subseteq \mathbb{N}$, say) with a least element 0 and a greatest element $\top$ as well as countable families $(p_i)_i, (q_i)_i \in P^I$ such that

(i) $p_0 = p \;\&\; q_0 = q$,
(ii) $p_\top = \bot \ne q_\top$,
(iii) $\forall i \le j\, (p_i \ge p_j \;\&\; q_i \ge q_j)$,
(iv) $\forall i > 0\, (p_i = \bigwedge_{j<i} \partial(p_j) \;\&\; q_i = \bigwedge_{j<i} \partial(q_j))$.

Indeed, if $\rho(p) < \rho(q)$, we may take I to be (an isomorphic copy of) $\rho(p) + 1$, $p_i := \partial^i(p)$, and $q_i := \partial^i(q)$. Conversely, suppose such $I, (p_i)_i, (q_i)_i$ exist. Then I must be a well-order: if there were a strictly descending sequence $i_0 > i_1 > \cdots$, we would have $q_{i_k} \le q_0 = q$ by (iii) and $q_{i_k} \le \partial(q_{i_{k+1}})$ by (iv) for all k, whence $q_\top \le q_{i_k} \le \partial^\alpha(q)$ for all $\alpha < \omega_1$ and all k by induction on α, contradicting that $\rho(q) < \infty$ so $\partial^{\rho(q)}(q) = \bot$ but $q_\top \ne \bot$. Thus since I has a greatest element, $I \cong \alpha + 1$ for some $\alpha < \omega_1$, such that $\alpha \ge \rho(p)$ since $\partial^\alpha(p) = p_\top = \bot$, but $\alpha < \rho(q)$ since $\partial^\alpha(q) = q_\top \ne \bot$. ☐

Corollary 2. Let ∂ be a Borel derivative on P as above, and X be a standard Borel space equipped with a Borel map $f : X \to P$ such that $\rho(f(x)) < \infty$ for all $x \in X$. Then there is some $\alpha < \omega_1$ such that $\rho(f(x)) < \alpha$ for all $x \in X$.

Proof. The relation $\{(x, y) \mid \rho(f(x)) < \rho(f(y))\}$ is analytic and well-founded, hence has rank $< \omega_1$, which means that $\{\rho(f(x)) \mid x \in X\}$ is countable. ☐

2 Lusin–Novikov via trees

Let $\mathrm{Tr} \subseteq \mathcal{P}(\mathbb{N}^{<\omega})$ be the standard Borel space of trees on $\mathbb{N}$, ordered by inclusion. The **Cantor–Bendixson derivative** of $T \in \mathrm{Tr}$ is

$$\partial_{\mathrm{CB}}(T) := \{t \in T \mid t \text{ has two incomparable extensions}\}.$$

Clearly, ∂_{CB} is a Borel derivative on Tr satisfying the assumptions of the preceding section, and $\partial_{\mathrm{CB}}(T) = T$ iff $[T]$ is perfect. The associated **Cantor–Bendixson rank** is denoted ρ_{CB}. Thus $\rho_{\mathrm{CB}}(T) < \infty$ iff $[T]$ is countable.

Theorem 3 (Lusin–Novikov uniformization)**.** Let $f : X \to Y$ be a countable-to-1 Borel map between standard Borel spaces. Then there are countably many partial Borel maps $g_n : Y \rightharpoonup X$ such that $f^{-1}(y) = \{g_n(y) \mid n \in \mathbb{N}\}$ for each $y \in Y$.

Proof. By the usual change of topology, we may assume $X, Y \subseteq \mathbb{N}^{\mathbb{N}}$ are closed subspaces and f is continuous. Then there is a Borel map $Y \ni y \mapsto T_y \in \mathrm{Tr}$ such that $[T_y] = f^{-1}(y) \subseteq \mathbb{N}^{\mathbb{N}}$ (namely $T_y := \{t \in \mathbb{N}^{<\omega} \mid N_t \cap f^{-1}(N_{y||t|}) \ne \varnothing\}$ where $N_t := \{x \in \mathbb{N}^{\mathbb{N}} \mid t \subseteq x\}$). Since f is countable-to-1, each $[T_y]$ is countable, i.e., each $\rho_{\mathrm{CB}}(T_y) < \infty$. Thus by Corollary 2, $\rho_{\mathrm{CB}}(T_y)$ is bounded by some $\alpha < \omega_1$. For each $\beta < \alpha$ and $t \in \mathbb{N}^{<\omega}$, let

$$Y_{\beta,t} := \{y \in Y \mid \forall n \ge |t| \, \exists! t \subseteq s \in \partial^\beta_{\mathrm{CB}}(T_y) \, (|s| = n)\}.$$

Then $Y_{\beta,t} \subseteq Y$ is Borel, as is the map $g_{\beta,t} : Y_{\beta,t} \to \mathbb{N}^{\mathbb{N}}$ taking y to the unique $t \subseteq g_{\beta,t}(y) \in [\partial^\beta_{\mathrm{CB}}(T_y)]$. Each $g_{\beta,t}(y) \in [\partial^\beta_{\mathrm{CB}}(T_y)] \subseteq [T_y] = f^{-1}(y)$, and each $x \in f^{-1}(y)$ is some such $g_{\beta,t}(y)$, namely for any β and finite initial segment $t \subseteq x$ such that $t \in \partial^\beta_{\mathrm{CB}}(T_y) \setminus \partial^{\beta+1}_{\mathrm{CB}}(T_y)$. Let g_n enumerate the $g_{\beta,t}$. ☐

3 Lusin–Novikov via compactifications

Let X be a Polish space, $\widehat{X}$ be a compact Polish space containing X as a subspace, $\mathcal{K}(\widehat{X})$ be the hyperspace of closed sets in $\widehat{X}$.

Lemma 4 ([Kec, 34.11])**.** $\bigcap : \mathcal{K}(\widehat{X})^{\mathbb{N}} \to \mathcal{K}(\widehat{X})$ is Borel.

Proof. $\bigcap_n F_n \ne \varnothing$ iff for every k, there is a basic open set U_k of radius $< 1/k$ which intersects every F_n (given this, take a limit point of a sequence of points $x_k \in U_k$). Now for open U, we have $U \cap \bigcap_n F_n \ne \varnothing$ iff some $G_k \cap \bigcap_n F_n \ne \varnothing$, where $U = \bigcup_k G_k$ with G_k closed. ☐

Since $X \subseteq \widehat{X}$ is G_δ, let $X = \bigcap \mathcal{U}$ for a countable family $\mathcal{U}$ of open sets in $\widehat{X}$. For $F \in \mathcal{K}(\widehat{X})$, let

$$\partial_{\mathrm{CB}}(F) := \{y \in F \mid y \text{ is not isolated in } F\},$$
$$\partial_{\mathrm{CB},\mathcal{U}}(F) := \bigcap_{U \in \mathcal{U}} \overline{U \cap \partial_{\mathrm{CB}}(F)}.$$

Lemma 5. $\partial_{\mathrm{CB},\mathcal{U}} : \mathcal{K}(\widehat{X}) \to \mathcal{K}(\widehat{X})$ is Borel.

Proof. ∂_{CB} is Borel, since $U \cap \partial_{\mathrm{CB}}(F) \neq \varnothing$ iff $\exists$ basic open V with $\overline{V} \subseteq U$ such that $V \cap F$ is infinite, and $V \cap F$ is infinite iff $\forall n\, \exists$ pairwise disjoint $V_1, \dots, V_n \subseteq V$ all intersecting F.

$F \mapsto \overline{U \cap F}$ is Borel, since $V \cap \overline{U \cap F} \neq \varnothing \iff V \cap U \cap F \neq \varnothing$.

$\bigcap$ is Borel by the preceding lemma. □

Lemma 6. $\partial_{\mathrm{CB},\mathcal{U}}(F) = F$ iff $X \cap F$ is perfect and $F = \overline{X \cap F}$.

Proof. Suppose $\partial_{\mathrm{CB},\mathcal{U}}(F) = F$. Then $F \subseteq \overline{U \cap \partial_{\mathrm{CB}}(F)} \subseteq \overline{U \cap F}$ for every $U \in \mathcal{U}$, whence by Baire category, $F \subseteq \overline{X \cap F}$. Since $\partial_{\mathrm{CB},\mathcal{U}}(F) \subseteq \partial_{\mathrm{CB}}(F)$, F is perfect, whence so is the dense subset $X \cap F$. □

It follows that $\partial_{\mathrm{CB},\mathcal{U}} : \mathcal{K}(\widehat{X}) \to \mathcal{K}(\widehat{X})$ is a Borel derivative, whose fixed points are $\overline{F}$ for perfect closed $F \subseteq X$. The associated rank is denoted $\rho_{\mathrm{CB},\mathcal{U}}$. Thus $\rho_{\mathrm{CB},\mathcal{U}}(F) < \infty$ iff $X \cap F$ is countable.

Theorem 7 (Lusin–Novikov uniformization)**.** Let $f : X \to Y$ be a countable-to-1 Borel map between standard Borel spaces. Then there are countably many partial Borel maps $g_n : Y \rightharpoonup X$ such that $f^{-1}(y) = \{g_n(y) \mid n \in \mathbb{N}\}$ for each $y \in Y$.

Proof. By the usual change of topology, we may assume X, Y are Polish and f is continuous. Let $\widehat{X}, \widehat{Y}$ be compact Polish spaces containing X, Y respectively such that f extends to a continuous map $\widehat{f} : \widehat{X} \to \widehat{Y}$ (e.g., embed X, Y in $[0,1]^{\mathbb{N}} =: \widehat{Y}$, and let $\widehat{X}$ be the closure in $([0,1]^{\mathbb{N}})^2$ of the graph of f), and let $X = \bigcap \mathcal{U}$ for a countable family $\mathcal{U}$ of open sets in $\widehat{X}$. Then $Y \ni y \mapsto F_y := \widehat{f}^{-1}(y) \in \mathcal{K}(\widehat{X})$ is a Borel map (indeed, $U \cap F_y \neq \varnothing \iff y \in \widehat{f}(U) = \bigcup_k \widehat{f}(G_k)$ where $U = \bigcup_k G_k$ with G_k closed), such that $X \cap F_y = f^{-1}(y)$. Since each $f^{-1}(y)$ is countable, each $\rho_{\mathrm{CB},\mathcal{U}}(F_y) < \omega_1$. Thus by Corollary 2, $\rho_{\mathrm{CB},\mathcal{U}}(F_y)$ is bounded by some $\alpha < \omega_1$. Let $\mathcal{V}$ be a countable basis of open sets in $\widehat{X}$. For each $\beta < \alpha$ and $V \in \mathcal{V}$, let

$$Y_{\beta,V} := \{y \in Y \mid V \cap \partial^{\beta+1}_{\mathrm{CB},\mathcal{U}}(F_y) = \varnothing \;\&\; |V \cap \partial^{\beta}_{\mathrm{CB},\mathcal{U}}(F_y)| = |X \cap V \cap \partial^{\beta}_{\mathrm{CB},\mathcal{U}}(F_y)| = 1\}.$$

Then $Y_{\beta,V}$ is Borel, as is the map $g_{\beta,V} : Y_{\beta,V} \to X$ taking y to the unique point in $X \cap V \cap \partial^{\beta}_{\mathrm{CB},\mathcal{U}}(F_y)$ (indeed, for any $F \in \mathcal{K}(\widehat{X})$ and open $V \subseteq \widehat{X}$, we have $|V \cap F| = 1$ iff $V \cap F \neq \varnothing$ and there do not exist two disjoint basic open subsets of V which both intersect F). Each $g_{\beta,V}(y) \in X \cap \partial^{\beta}_{\mathrm{CB},\mathcal{U}}(F_y) \subseteq X \cap F_y = f^{-1}(y)$, and each $x \in f^{-1}(y)$ is some such $g_{\beta,V}(y)$, namely for the unique β such that $x \in \partial^{\beta}_{\mathrm{CB},\mathcal{U}}(F_y) \setminus \partial^{\beta+1}_{\mathrm{CB},\mathcal{U}}(F_y)$ (since $\rho_{\mathrm{CB},\mathcal{U}}(F_y) < \alpha$) and for any V witnessing that $x \in \partial^{\beta}_{\mathrm{CB},\mathcal{U}}(F_y)$ is isolated (since $x \in X \setminus \partial^{\beta+1}_{\mathrm{CB},\mathcal{U}}(F_y)$). Let g_n enumerate the $g_{\beta,V}$. □

References

[Kec] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, 1995.