On uniform ergodic decomposition

Ruiyuan Chen

In this note, we prove the Farrell–Varadarajan uniform ergodic decomposition for countable Borel equivalence relations. Essentially the same proof is contained in the notes of Slutsky [Sl], who derives it as a corollary of the Becker–Kechris proof [BK, §4.5] of Nadkarni’s theorem; we have made some simplifications to obtain a shorter direct proof of uniform ergodic decomposition.

Fix a countable Borel equivalence relation \((X, E)\). Let \(\text{INV}_E\) (resp., \(\text{EINV}_E\)) denote the space of Borel \(E\)-invariant (ergodic) probability measures on \(X\). For Borel \(A, B \subseteq X\), \(A \preceq E B\) means that there is a Borel injection \(f : A \rightarrow B\) with graph contained in \(E\).

**Lemma 1** (Becker–Kechris comparability [BK, 4.5.1]). For any Borel \(A, B \subseteq X\), there is a Borel \(E\)-invariant partition \(X = Y \sqcup Z\) such that

\[
Y \cap A \preceq E Y \cap B, \quad Z \cap A \succeq E Z \cap B.
\]

*Proof.* Using Feldman–Moore, it is easy to find a maximal partial Borel injection \(f : A \hookrightarrow B\) with graph contained in \(E\). Then for each \(E\)-class \(C \in X/E\), either \(C \cap A \subseteq \text{dom}(f)\) or \(C \cap B \subseteq \text{im}(f)\).

Letting \(Z := [A \setminus \text{dom}(f)]_E\) and \(Y := X \setminus Z\), we have \(f|(Y \cap A)\) witnessing \(Y \cap A \preceq E Y \cap B\) and \(f^{-1}|(Z \cap B)\) witnessing \(Z \cap A \succeq E Z \cap B\). \(\square\)

**Lemma 2.** For any Borel \(A, B \subseteq X\) and rational \(r > 0\), there is a Borel \(E\)-invariant partition \(X = Y \sqcup Z\) such that for any Borel \(E\)-invariant measure \(\mu\),

\[
\mu(Y \cap A) \leq r\mu(Y \cap B), \quad \mu(Z \cap A) \geq r\mu(Z \cap B).
\]

(These then also hold for \(E\)-invariant \(Y' \subseteq Y\) and \(Z' \subseteq Z\), by replacing \(\mu\) with \(\mu|Y', \mu|Z'\).)

*Proof.* By induction on the Euclidean algorithm applied to the numerator and denominator of \(r\).

The base case is \(r = 1\): apply Lemma 1. If \(r < 1\), then reduce to the case \(r^{-1}\) with \(A, B\) swapped and \(Y, Z\) swapped. Suppose \(r > 1\); we will reduce to the case \(r - 1\). Apply Lemma 1 to get a Borel \(E\)-invariant partition \(X = Y' \sqcup Z'\) with \(Y' \cap A \preceq E Y' \cap B\) and \(f : Z' \cap B \preceq E Z' \cap A\), whence for any \(E\)-invariant \(\mu\),

\[
\mu(Y' \cap A) \leq \mu(Y' \cap B) \leq r\mu(Y' \cap B).
\]

Now apply the induction hypothesis (the case \(r - 1\)) to \((Z' \cap A \setminus \text{im}(f))\), \(Z' \cap B\) to get a Borel \(E\)-invariant partition \(Z'' = Y'' \sqcup Z''\) such that for any \(E\)-invariant \(\mu\),

\[
\mu(Y'' \cap A \setminus \text{im}(f)) \leq (r - 1)\mu(Y'' \cap B), \quad \mu(Z'' \cap A \setminus \text{im}(f)) \geq (r - 1)\mu(Z'' \cap B).
\]

Then \(Y := Y' \sqcup Y''\) and \(Z := Z''\) works. \(\square\)
Lemma 3. Let $X$ be a measurable space, $\mu$, $\nu$ be $\sigma$-finite measures on $X$, and $f : X \to [0, \infty)$ be a measurable function. Then $d\nu = f \, d\mu$ iff for all $r > 0$,

$$\nu(f^{-1}([0, r])) \leq r\mu(f^{-1}([0, r])), \quad \nu(f^{-1}((r, \infty])) \geq r\mu(f^{-1}((r, \infty))).$$

Proof. $\implies$ is obvious. For $\impliedby$, by replacing $\mu, \nu$ with their restrictions to measurable $A \subseteq X$, it is enough to prove $\nu(X) = \int f \, d\mu$. By the first inequality and $\sigma$-finiteness, we have $\nu(f^{-1}(0)) = 0 = \int_{f^{-1}(0)} f \, d\mu$; thus we may restrict attention to $f^{-1}((0, \infty))$, i.e., assume $f : X \to (0, \infty)$.

Suppose $\nu(X) > \int f \, d\mu$. Let $c > 1$ such that $\nu(X) > c \int f \, d\mu$. By tiling $(0, \infty)$, we may find $0 < a < b < \infty$ such that $b/a \leq c$ and

$$\nu(f^{-1}([a, b])) > c \int_{f^{-1}([a, b])} f \, d\mu \geq ca\mu(f^{-1}([a, b])) \geq ca\nu(f^{-1}([a, b])/b \geq \nu(f^{-1}([a, b]))),$$

a contradiction. Similarly, $\nu(X) < \int f \, d\mu$ leads to a contradiction. \hfill \Box

Lemma 4 (uniform conditional probability). For any Borel $A \subseteq X$, there is a Borel $E$-invariant function $f : X \to [0, 1]$ such that for any Borel $E$-invariant finite measure $\mu$,

$$\mu(A) = \int_X f \, d\mu.$$

Proof. For each rational $r \in (0, 1)$, let $X = Y_r \sqcup Z_r$ be given by Lemma 2 with $B = \emptyset$, so that for any Borel $E$-invariant $\mu$ and Borel $E$-invariant $Y' \subseteq Y_r$ and $Z' \subseteq Z_r$,

$$(*) \quad \mu(Y' \cap A) \leq r\mu(Y'), \quad \mu(Z' \cap A) \geq r\mu(Z').$$

Put

$$f(x) := \sup\{r \in \mathbb{Q} \cap (0, 1) \mid x \in Z_r\}.$$ 

Then for all $r \in (0, 1)$ and Borel $E$-invariant $Y' \subseteq f^{-1}([0, r])$ and $Z' \subseteq f^{-1}((r, 1])$, $(*)$ holds. Now apply Lemma 3 to the measures $\mu, \mu|A$ restricted to the $E$-invariant Borel $\sigma$-algebra on $X$. \hfill \Box

Theorem 5 (Farrell–Varadarajan uniform ergodic decomposition). There is a Borel $E$-invariant $Y \subseteq X$ and a Borel $E$-invariant function $p : Y \to \text{EINV}_E$ such that for any Borel $E$-invariant probability measure $\mu \in \text{INV}_E$, we have

$$\mu = \int_Y p \, d\mu.$$ 

(In particular, $\text{INV}_E \neq \emptyset \implies \text{EINV}_E \neq \emptyset$; and for $\mu \in \text{EINV}_E$, $\mu$ concentrates on $p^{-1}(\mu)$.)

Proof. We may assume that $X$ is compact Polish zero-dimensional and $E$ is induced by a continuous action of a countable group $\Gamma \curvearrowright X$, e.g., $\mathbb{F}_\omega \curvearrowright (2^\omega)^{\mathbb{F}_\omega}$. Let $\mathcal{A} := \{\text{clopens in } X\}$. Then by
measure regularity, Carathéodory, and compactness, Borel finite measures on $X$ are in bijection (via restriction) with finitely additive finite measures $\mathcal{A} \to [0, \infty)$, so that we can regard

$$\text{EINV}_E \subseteq \text{INV}_E \subseteq [0, 1]^\mathcal{A}.$$ 

For each $A \in \mathcal{A}$, let $p_A : X \to [0, 1]$ be $f$ given by Lemma 4, and put

$$p \equiv (p_A)_A : X \to [0, 1]^\mathcal{A}.$$ 

Then $p$ is $E$-invariant. Let $Y := p^{-1}(\text{EINV}_E)$. We claim that this works.

Let $\mu \in \text{INV}_E$. By Lemma 4, we have

$$(*) \quad \mu = \int_X p \, d\mu.$$ 

Thus, it suffices to check that $\mu$ concentrates on $Y$, i.e., that for $\mu$-a.e. $x$, we have $p(x) \in \text{EINV}_E$. For instance, to check that $p(x) : A \to [0, 1]$ is additive for $\mu$-a.e. $x$: let $A, B \in \mathcal{A}$ be disjoint, and let

$$Z := \{x \in X \mid p(x)(A) + p(x)(B) < p(x)(A \cup B)\};$$ 

then $Z$ is $E$-invariant, whence by $(*)$ applied to $\mu|Z$,

$$\int_Z (p(x)(A) + p(x)(B)) \, d\mu(x) = \mu(Z \cap A) + \mu(Z \cap B) = \mu(Z \cap (A \cup B)) = \int_Z p(x)(A \cup B) \, d\mu(x),$$ 

whence $Z$ is $\mu$-null since the left integrand is everywhere strictly less than the right integrand. The rest of the verification that $p(x)$ is $\mu$-a.e. a $\Gamma$-invariant finitely additive probability measure on $\mathcal{A}$ is similar. Thus $p(x) \in \text{INV}_E$ for $\mu$-a.e. $x$.

Now for any Borel $E$-invariant $Z \subseteq X$, by $(*)$ applied to $\mu, \mu|Z$, we have

$$\int_X p(x)(Z) \, d\mu(x) = \mu(Z) = \int_Z p(x)(Z) \, d\mu(x),$$ 

whence $p(x)(Z) = 0$ for $\mu$-a.e. $x \in X \setminus Z$; replacing $Z$ with $X \setminus Z$, we get $p(x)(Z) = 1$ for $\mu$-a.e. $x \in Z$. Taking $Z := p^{-1}(D)$ for a countable generating family of Borel $D \subseteq \text{INV}_E$, we get that for $\mu$-a.e. $x$, for all Borel $D \subseteq \text{INV}_E$,

$$p(x)(p^{-1}(D)) = \chi_D(p(x));$$ 

in particular, taking $D := \{p(x)\}$, we get that $p(x)$ concentrates on $p^{-1}(p(x))$ for $\mu$-a.e. $x \in p^{-1}(\text{INV}_E)$. For such $x$, for Borel $E$-invariant $Z \subseteq X$, we have

$$p(x)|Z = \int_Z p \, dp(x) = p(x)(Z)p(x),$$ 

whence $p(x)(Z) = p(x)(Z)^2$. Thus $p(x)$ is ergodic for $\mu$-a.e. $x \in p^{-1}(\text{INV}_E)$.

\begin{flushright}
\text{\hfill \Box}
\end{flushright}

\textbf{References}
