On uniform ergodic decomposition

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In this note, we prove the Farrell–Varadarajan uniform ergodic decomposition for countable Borel equivalence relations. Essentially the same proof is contained in the notes of Slutsky [Sl], who derives it as a corollary of the Becker–Kechris proof [BK, §4.5] of Nadkarni's theorem; we have made some simplifications to obtain a shorter direct proof of uniform ergodic decomposition.

Fix a countable Borel equivalence relation (X, E). Let INV_E (resp., EINV_E) denote the space of Borel E-invariant (ergodic) probability measures on X. For Borel $A, B \subseteq X$, $A \preceq_E B$ means that there is a Borel injection $f : A \to B$ with graph contained in E.

Lemma 1 (Becker–Kechris comparability [BK, 4.5.1])**.** For any Borel $A, B \subseteq X$, there is a Borel E-invariant partition $X = Y \sqcup Z$ such that

$$Y \cap A \preceq_E Y \cap B, \qquad Z \cap A \succeq_E Z \cap B.$$

Proof. Using Feldman–Moore, it is easy to find a maximal partial Borel injection $f : A \rightharpoonup B$ with graph contained in E. Then for each E-class $C \in X/E$, either $C \cap A \subseteq \mathrm{dom}(f)$ or $C \cap B \subseteq \mathrm{im}(f)$. Letting $Z := [A \setminus \mathrm{dom}(f)]_E$ and $Y := X \setminus Z$, we have $f|(Y \cap A)$ witnessing $Y \cap A \preceq_E Y \cap B$ and $f^{-1}|(Z \cap B)$ witnessing $Z \cap A \succeq_E Z \cap B$. □

Lemma 2. For any Borel $A, B \subseteq X$ and rational $r > 0$, there is a Borel E-invariant partition $X = Y \sqcup Z$ such that for any Borel E-invariant measure μ,

$$\mu(Y \cap A) \le r\mu(Y \cap B), \qquad \mu(Z \cap A) \ge r\mu(Z \cap B).$$

(These then also hold for E-invariant $Y' \subseteq Y$ and $Z' \subseteq Z$, by replacing μ with $\mu|Y', \mu|Z'$.)

Proof. By induction on the Euclidean algorithm applied to the numerator and denominator of r. The base case is $r = 1$: apply Lemma 1. If $r < 1$, then reduce to the case r^{-1} with A, B swapped and Y, Z swapped. Suppose $r > 1$; we will reduce to the case $r - 1$. Apply Lemma 1 to get a Borel E-invariant partition $X = Y' \sqcup Z'$ with $Y' \cap A \preceq_E Y' \cap B$ and $f : Z' \cap B \preceq_E Z' \cap A$, whence for any E-invariant μ,

$$\mu(Y' \cap A) \le \mu(Y' \cap B) \le r\mu(Y' \cap B).$$

Now apply the induction hypothesis (the case $r - 1$) to $(Z' \cap A) \setminus \mathrm{im}(f), Z' \cap B$ to get a Borel E-invariant partition $Z' = Y'' \sqcup Z''$ such that for any E-invariant μ,

$$\mu(Y'' \cap A \setminus \mathrm{im}(f)) \le (r-1)\mu(Y'' \cap B), \qquad \mu(Z'' \cap A \setminus \mathrm{im}(f)) \ge (r-1)\mu(Z'' \cap B).$$

Then $Y := Y' \sqcup Y''$ and $Z := Z''$ works. □

Lemma 3. Let X be a measurable space, μ, ν be σ-finite measures on X, and $f : X \to [0, \infty)$ be a measurable function. Then $d\nu = f\, d\mu$ iff for all $r > 0$,

$$\nu|f^{-1}([0,r)) \le r\mu|f^{-1}([0,r)), \qquad \nu|f^{-1}((r,\infty)) \ge r\mu|f^{-1}((r,\infty)).$$

Proof. $\Longrightarrow$ is obvious. For $\Longleftarrow$, by replacing μ, ν with their restrictions to measurable $A \subseteq X$, it is enough to prove $\nu(X) = \int f\, d\mu$. By the first inequality and σ-finiteness, we have $\nu(f^{-1}(0)) = 0 = \int_{f^{-1}(0)} f\, d\mu$; thus we may restrict attention to $f^{-1}((0,\infty))$, i.e., assume $f : X \to (0, \infty)$.

Suppose $\nu(X) > \int f\, d\mu$. Let $c > 1$ such that $\nu(X) > c \int f\, d\mu$. By tiling $(0, \infty)$, we may find $0 < a < b < \infty$ such that $b/a \le c$ and

$$\begin{aligned}\nu(f^{-1}([a,b))) &> c \int_{f^{-1}([a,b))} f\, d\mu \\ &\ge ca\mu(f^{-1}([a,b))) \\ &\ge ca\nu(f^{-1}([a,b)))/b \\ &\ge \nu(f^{-1}([a,b))),\end{aligned}$$

a contradiction. Similarly, $\nu(X) < \int f\, d\mu$ leads to a contradiction. ☐

Lemma 4 (uniform conditional probability)**.** For any Borel $A \subseteq X$, there is a Borel E-invariant function $f : X \to [0,1]$ such that for any Borel E-invariant finite measure μ,

$$\mu(A) = \int_X f\, d\mu.$$

Proof. For each rational $r \in (0,1)$, let $X = Y_r \sqcup Z_r$ be given by Lemma 2 with $B = X$, so that for any Borel E-invariant μ and Borel E-invariant $Y' \subseteq Y_r$ and $Z' \subseteq Z_r$,

$$(*) \qquad \mu(Y' \cap A) \le r\mu(Y'), \qquad \mu(Z' \cap A) \ge r\mu(Z').$$

Put

$$f(x) := \sup\{r \in \mathbb{Q} \cap (0,1) \mid x \in Z_r\}.$$

Then for all $r \in (0,1)$ and Borel E-invariant $Y' \subseteq f^{-1}([0,r))$ and $Z' \subseteq f^{-1}((r,1])$, $(*)$ holds. Now apply Lemma 3 to the measures $\mu, \mu|A$ restricted to the E-invariant Borel σ-algebra on X. ☐

Theorem 5 (Farrell–Varadarajan uniform ergodic decomposition)**.** There is a Borel E-invariant $Y \subseteq X$ and a Borel E-invariant function $p : Y \to \mathrm{EINV}_E$ such that for any Borel E-invariant probability measure $\mu \in \mathrm{INV}_E$, we have

$$\mu = \int_Y p\, d\mu.$$

(In particular, $\mathrm{INV}_E \ne \varnothing \implies \mathrm{EINV}_E \ne \varnothing$; and for $\mu \in \mathrm{EINV}_E$, μ concentrates on $p^{-1}(\mu)$.)

Proof. We may assume that X is compact Polish zero-dimensional and E is induced by a continuous action of a countable group $\Gamma \curvearrowright X$, e.g., $\mathbb{F}_\omega \curvearrowright (2^{\mathbb{N}})^{\mathbb{F}_\omega}$. Let $\mathcal{A} := \{\text{clopens in } X\}$. Then by measure regularity, Carathéodory, and compactness, Borel finite measures on X are in bijection (via restriction) with finitely additive finite measures $\mathcal{A} \to [0, \infty)$, so that we can regard

$$\mathrm{EINV}_E \subseteq \mathrm{INV}_E \subseteq [0,1]^{\mathcal{A}}.$$

For each $A \in \mathcal{A}$, let $p_A : X \to [0,1]$ be f given by Lemma 4, and put

$$p := (p_A)_A : X \longrightarrow [0,1]^{\mathcal{A}}.$$

Then p is E-invariant. Let $Y := p^{-1}(\mathrm{EINV}_E)$. We claim that this works.

Let $\mu \in \mathrm{INV}_E$. By Lemma 4, we have

$$(*) \qquad \mu = \int_X p \, d\mu.$$

Thus, it suffices to check that μ concentrates on Y, i.e., that for μ-a.e. x, we have $p(x) \in \mathrm{EINV}_E$. For instance, to check that $p(x) : \mathcal{A} \to [0,1]$ is additive for μ-a.e. x: let $A, B \in \mathcal{A}$ be disjoint, and let

$$Z := \{x \in X \mid p(x)(A) + p(x)(B) < p(x)(A \cup B)\};$$

then Z is E-invariant, whence by $(*)$ applied to $\mu|Z$,

$$\int_Z (p(x)(A) + p(x)(B)) \, d\mu(x) = \mu(Z \cap A) + \mu(Z \cap B) = \mu(Z \cap (A \cup B)) = \int_Z p(x)(A \cup B) \, d\mu(x),$$

whence Z is μ-null since the left integrand is everywhere strictly less than the right integrand. The rest of the verification that $p(x)$ is μ-a.e. a Γ-invariant finitely additive probability measure on $\mathcal{A}$ is similar. Thus $p(x) \in \mathrm{INV}_E$ for μ-a.e. x.

Now for any Borel E-invariant $Z \subseteq X$, by $(*)$ applied to $\mu, \mu|Z$, we have

$$\int_X p(x)(Z) \, d\mu(x) = \mu(Z) = \int_Z p(x)(Z) \, d\mu(x),$$

whence $p(x)(Z) = 0$ for μ-a.e. $x \in X \setminus Z$; replacing Z with $X \setminus Z$, we get $p(x)(Z) = 1$ for μ-a.e. $x \in Z$. Taking $Z := p^{-1}(D)$ for a countable generating family of Borel $D \subseteq \mathrm{INV}_E$, we get that for μ-a.e. x, for all Borel $D \subseteq \mathrm{INV}_E$,

$$p(x)(p^{-1}(D)) = \chi_D(p(x));$$

in particular, taking $D := \{p(x)\}$, we get that $p(x)$ concentrates on $p^{-1}(p(x))$ for μ-a.e. $x \in p^{-1}(\mathrm{INV}_E)$. For such x, for Borel E-invariant $Z \subseteq X$, we have

$$p(x)|Z = \int_Z p \, dp(x) = p(x)(Z)p(x),$$

whence $p(x)(Z) = p(x)(Z)^2$. Thus $p(x)$ is ergodic for μ-a.e. $x \in p^{-1}(\mathrm{INV}_E)$. □

References

[BK] H. Becker and A. S. Kechris, *The Descriptive Set Theory of Polish Group Actions*, London Math. Soc. Lecture Note Series **232**, Cambridge University Press, 1996.

[Sl] K. Slutsky, *Countable Borel equivalence relations*, lecture notes, http://www.kslutsky.com/lecture-notes/cber.pdf