On uniform ergodic decomposition

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In this note, we prove the Farrell–Varadarajan uniform ergodic decomposition for countable Borel equivalence relations. Essentially the same proof is contained in the notes of Slutsky [Sl], who derives it as a corollary of the Becker–Kechris proof [BK, §4.5] of Nadkarni’s theorem; we have made some simplifications to obtain a shorter direct proof of uniform ergodic decomposition.

Fix a countable Borel equivalence relation \((X, E)\). Let \(\text{INV}_E\) (resp., \(\text{EINV}_E\)) denote the space of Borel \(E\)-invariant (ergodic) probability measures on \(X\). For Borel \(A, B \subseteq X\), \(A \preceq E B\) means that there is a Borel injection \(f : A \hookrightarrow B\) with graph contained in \(E\).

**Lemma 1** (Becker–Kechris comparability [BK, 4.5.1]). For any Borel \(A, B \subseteq X\), there is a Borel \(E\)-invariant partition \(X = Y \sqcup Z\) such that

\[
Y \cap A \preceq E Y \cap B, \quad \quad \quad Z \cap A \succeq E Z \cap B.
\]

**Proof.** Using Feldman–Moore, it is easy to find a maximal partial Borel injection \(f : A \hookrightarrow B\) with graph contained in \(E\). Then for each \(E\)-class \(C \in X/E\), either \(C \cap A \subseteq \text{dom}(f)\) or \(C \cap B \subseteq \text{im}(f)\). Letting \(Z := A \setminus \text{dom}(f)\) and \(Y := X \setminus Z\), we have \(f|(Y \cap A)\) witnessing \(Y \cap A \preceq E Y \cap B\) and \(f^{-1}|(Z \cap B)\) witnessing \(Z \cap A \succeq E Z \cap B\). \(\square\)

**Lemma 2.** For any Borel \(A, B \subseteq X\) and rational \(r > 0\), there is a Borel \(E\)-invariant partition \(X = Y \sqcup Z\) such that for any Borel \(E\)-invariant measure \(\mu\),

\[
\mu(Y \cap A) \leq r \mu(Y \cap B), \quad \quad \quad \mu(Z \cap A) \geq r \mu(Z \cap B).
\]

(These then also hold for \(E\)-invariant \(Y' \subseteq Y\) and \(Z' \subseteq Z\), by replacing \(\mu, \nu\) with \(\mu|Y', \nu|Z'\).)

**Proof.** By induction on the Euclidean algorithm applied to the numerator and denominator of \(r\). The base case is \(r = 1\): apply Lemma 1. If \(r < 1\), then reduce to the case \(r^{-1}\) with \(A, B\) swapped and \(Y, Z\) swapped. Suppose \(r > 1\); we will reduce to the case \(r - 1\). Apply Lemma 1 to get a Borel \(E\)-invariant partition \(X = Y' \sqcup Z'\) with \(Y' \cap A \preceq E Y' \cap B\) and \(f : Z' \cap B \preceq E Z' \cap A\), whence for any \(E\)-invariant \(\mu\),

\[
\mu(Y' \cap A) \leq \mu(Y' \cap B) \leq r \mu(Y' \cap B).
\]

Now apply the induction hypothesis (the case \(r - 1\)) to \((Z' \cap A) \setminus \text{im}(f), Z' \cap B\) to get a Borel \(E\)-invariant partition \(Z' = Y'' \sqcup Z''\) such that for any \(E\)-invariant \(\mu\),

\[
\mu(Y'' \cap A \setminus \text{im}(f)) \leq (r - 1)\mu(Y'' \cap B), \quad \quad \quad \mu(Z'' \cap A \setminus \text{im}(f)) \geq (r - 1)\mu(Z'' \cap B).
\]

Then \(Y := Y' \cup Y''\) and \(Z := Z''\) works. \(\square\)
Lemma 3. Let $X$ be a measurable space, $\mu, \nu$ be $\sigma$-finite measures on $X$, and $f : X \to [0, \infty)$ be a measurable function. Then $d\nu = f \, d\mu$ iff for all $r > 0$,

$$
\nu| f^{-1}([0, r)) \leq r\mu| f^{-1}([0, r)), \quad \nu| f^{-1}((r, \infty)) \geq r\nu| f^{-1}((r, \infty)).
$$

Proof. $\Longrightarrow$ is obvious. For $\Longleftrightarrow$, by replacing $\mu, \nu$ with their restrictions to measurable $A \subseteq X$, it is enough to prove $\nu(X) = \int f \, d\mu$. By the first inequality and $\sigma$-finiteness, we have $\nu(f^{-1}(0)) = 0 = \int_{f^{-1}(0)} f \, d\mu$; thus we may restrict attention to $f^{-1}((0, \infty))$, i.e., assume $f : X \to (0, \infty)$.

Suppose $\nu(X) > \int f \, d\mu$. Let $c > 1$ such that $\nu(X) > c \int f \, d\mu$. By tiling $(0, \infty)$, we may find $0 < a < b < \infty$ such that $b/a \leq c$ and

$$
\nu(f^{-1}((a, b))) > c \int_{f^{-1}((a, b))} f \, d\mu
\geq ca\mu(f^{-1}((a, b)))
\geq ca\nu(f^{-1}((a, b)))/b
\geq \nu(f^{-1}((a, b))),
$$

a contradiction. Similarly, $\nu(X) < \int f \, d\mu$ leads to a contradiction. \hfill \square

Lemma 4 (uniform conditional probability). For any Borel $A \subseteq X$, there is a Borel $E$-invariant function $f : X \to [0, 1]$ such that for any Borel $E$-invariant finite measure $\mu$,

$$
\mu(A) = \int_X f \, d\mu.
$$

Proof. For each rational $r \in (0, 1)$, let $X = Y_r \cup Z_r$ be given by Lemma 2 with $B = \emptyset$, so that for any Borel $E$-invariant $\mu$ and Borel $E$-invariant $Y' \subseteq Y_r$ and $Z' \subseteq Z_r$,

$$(*) \quad \mu(Y' \cap A) \leq r\mu(Y'), \quad \mu(Z' \cap A) \geq r\mu(Z').
$$

Put

$$
\quad f(x) : = \sup\{r \in \mathbb{Q} \cap (0, 1) \mid x \in Z_r\}.
$$

Then for all $r \in (0, 1)$ and Borel $E$-invariant $Y' \subseteq f^{-1}([0, r))$ and $Z' \subseteq f^{-1}((r, 1))$, $(*)$ holds. Now apply Lemma 3 to the measures $\mu, \mu|A$ restricted to the $E$-invariant Borel $\sigma$-algebra on $X$. \hfill \square

Theorem 5 (Farrell–Varadarajan uniform ergodic decomposition). There is a Borel $E$-invariant $Y \subseteq X$ and a Borel $E$-invariant function $p : Y \to \text{EINV}_E$ such that for any Borel $E$-invariant probability measure $\mu \in \text{INV}_E$, we have

$$
\mu = \int_Y p \, d\mu.
$$

(In particular, $\text{INV}_E \neq \emptyset \implies \text{EINV}_E \neq \emptyset$; and for $\mu \in \text{EINV}_E$, $\mu$ concentrates on $p^{-1}(\mu)$.)

Proof. We may assume that $X$ is compact Polish zero-dimensional and $E$ is induced by a continuous action of a countable group $\Gamma \act X$, e.g., $\mathbb{F}_\omega \act (2^\mathbb{N})^{\mathbb{F}_\omega}$. Let $\mathcal{A} := \{\text{clopens in } X\}$. Then by
measure regularity, Carathéodory, and compactness, Borel finite measures on $X$ are in bijection (via restriction) with finitely additive finite measures $A \to [0, \infty)$, so that we can regard

$$EINV_E \subseteq INV_E \subseteq [0,1]^A.$$ 

For each $A \in \mathcal{A}$, let $p_A : X \to [0,1]$ be $f$ given by Lemma 4, and put

$$p := (p_A)_A : X \longrightarrow [0,1]^A.$$ 

Then $p$ is $E$-invariant. Let $Y := p^{-1}(EINV_E)$. We claim that this works.

Let $\mu \in INV_E$. By Lemma 4, we have

$$(*) \quad \mu = \int_X p \, d\mu.$$

Thus, it suffices to check that $\mu$ concentrates on $Y$, i.e., that for $\mu$-a.e. $x$, we have $p(x) \in EINV_E$. For instance, to check that $p(x) : A \to [0,1]$ is additive for $\mu$-a.e. $x$: let $A, B \in \mathcal{A}$ be disjoint, and let

$$Z := \{ x \in X \mid p(x)(A) + p(x)(B) < p(x)(A \cup B) \};$$

then $Z$ is $E$-invariant, whence by $(*)$ applied to $\mu|Z$,

$$\int_Z (p(x)(A) + p(x)(B)) \, d\mu(x) = \mu(Z \cap A) + \mu(Z \cap B) = \mu(Z \cap (A \cup B)) = \int_Z p(x)(A \cup B) \, d\mu(x),$$

whence $Z$ is $\mu$-null since the left integrand is everywhere strictly less than the right integrand. The rest of the verification that $p(x)$ is $\mu$-a.e. a $\Gamma$-invariant finitely additive probability measure on $\mathcal{A}$ is similar. Thus $p(x) \in INV_E$ for $\mu$-a.e. $x$.

Now for any Borel $E$-invariant $Z \subseteq X$, by $(*)$ applied to $\mu, \mu|Z$, we have

$$\int_X p(x)(Z) \, d\mu(x) = \mu(Z) = \int_Z p(x)(Z) \, d\mu(x),$$

whence $p(x)(Z) = 0$ for $\mu$-a.e. $x \in X \setminus Z$; replacing $Z$ with $X \setminus Z$, we get $p(x)(Z) = 1$ for $\mu$-a.e. $x \in Z$. Taking $Z := p^{-1}(D)$ for a countable generating family of Borel $D \subseteq INV_E$, we get that for $\mu$-a.e. $x$, for all Borel $D \subseteq INV_E$,

$$p(x)(p^{-1}(D)) = \chi_D(p(x));$$

in particular, taking $D := \{p(x)\}$, we get that $p(x)$ concentrates on $p^{-1}(p(x))$ for $\mu$-a.e. $x \in p^{-1}(INV_E)$. For such $x$, for Borel $E$-invariant $Z \subseteq X$, we have

$$p(x)|Z = \int_Z p \, dp(x) = p(x)(Z)p(x),$$

whence $p(x)(Z) = p(x)(Z)^2$. Thus $p(x)$ is ergodic for $\mu$-a.e. $x \in p^{-1}(INV_E)$.

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**References**
