Decompositions and measures on countable Borel equivalence relations

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Abstract

We show that the uniform measure-theoretic ergodic decomposition of a countable Borel equivalence relation (X, E) may be realized as the topological ergodic decomposition of a continuous action of a countable group $\Gamma \curvearrowright X$ generating E. We then apply this to the study of the cardinal algebra $\mathcal{K}(E)$ of equidecomposition types of Borel sets with respect to a compressible countable Borel equivalence relation (X, E). We also make some general observations regarding quotient topologies on topological ergodic decompositions, with an application to weak equivalence of measure-preserving actions.

1 Introduction

In this paper, we study several related constructions on a countable Borel equivalence relation.

In Section 3, we study the relation between two different notions of ergodic decomposition. An action of a group G via homeomorphisms on a Polish space X is **minimal** if $X \neq \varnothing$ and each orbit is dense; the **topological ergodic decomposition** of an arbitrary action $G \curvearrowright X$ is the standard Borel decomposition of X into minimal invariant G_δ subsets. A countable Borel equivalence relation E on a standard Borel space X is **uniquely ergodic** if it admits a unique ergodic invariant probability measure; the **measure-theoretic ergodic decomposition** of an arbitrary (X, E) is the standard Borel decomposition of X into E-invariant, uniquely ergodic pieces.

We show that for a countable Borel equivalence relation (X, E), generated by a countable Borel group action $\Gamma \curvearrowright X$, the measure-theoretic ergodic decomposition of E may be realized *via* the topological ergodic decomposition, with respect to some Polish topology on X making the Γ-action continuous. Moreover, we may pick the topology so as to include in the decomposition not only the invariant ergodic probability measures, but also all invariant ergodic σ-finite measures which are regular with respect to the topology, where "regular" means in the weak sense that there is some open set with finite positive measure. Here is a rough statement of the result; see Theorem 3.2.

Theorem 1.1. *Let (X, E) be a countable Borel equivalence relation, induced by a countable Borel group action $\Gamma \curvearrowright X$. For cofinally many Polish topologies on X inducing the Borel structure and making the Γ-action continuous, we have the following:*

- *Each component of the topological ergodic decomposition of the action $\Gamma \curvearrowright X$ admits, up to scaling, at most one E-invariant σ-finite measure which is regular with respect to the topology; and such a measure is E-ergodic (if it exists).*

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- *The set R of components admitting such a measure, as well as the set $P \subseteq R$ of components admitting an E-invariant probability measure, are Borel.*
- *We have a Borel bijection between R and the space of E-invariant ergodic regular σ-finite measures modulo scaling, taking a component in R to the unique such measure on that component. This restricts to a Borel bijection between P and the space EINV_E of E-invariant ergodic probability measures, yielding the usual measure-theoretic ergodic decomposition.*

In Section 4, we study the following canonical algebraic structure associated to a *compressible* countable Borel equivalence relation (X, E). A **cardinal algebra** [Tar] is a set equipped with a countable addition operation, satisfying certain axioms motivated by cardinal arithmetic. Cardinal algebras appear naturally in the study of group actions and paradoxical decompositions (see e.g., [Chu]), as well as the classification of Borel equivalence relations [KMd]. An example belonging to both of these contexts is the **algebra $\mathcal{K}(E)$ of equidecomposition types** of Borel sets $A \subseteq X$ with respect to a compressible equivalence relation E on X. We show that several well-known results about countable Borel equivalence relations translate to nice algebraic properties of $\mathcal{K}(E)$ and a related "completion" algebra $\mathcal{L}(E)$; see Theorems 4.16, 4.27 and 4.29.

Theorem 1.2. *Let (X, E) be a compressible countable Borel equivalence relation.*

- *$\mathcal{K}(E)$ is a cardinal algebra with finite meets, countable joins, and real multiples of elements represented by E-aperiodic Borel sets $A \subseteq X$, and obeys all Horn axioms involving these operations which hold in the algebra $[0, \infty]$.*
- *The completion $\mathcal{L}(E) \supseteq \mathcal{K}(E)$ by adjoining real multiples for* all *elements of $\mathcal{K}(E)$ can be naturally viewed as a cardinal algebra of E-equidecomposition types of Borel real-valued functions on X. $\mathcal{L}(E)$ has finite meets, countable joins, and real multiples of all elements, and obeys all Horn axioms involving these operations which hold in the algebra $[0, \infty]$.*
- *Homomorphisms $\mathcal{K}(E) \to [0, \infty]$ preserving the above operations are in canonical bijection with E-invariant E-ergodic measures; the same holds for $\mathcal{L}(E)$ in place of $\mathcal{K}(E)$.*
- *The space EINV_E^σ of all σ-finite such measures forms a "dual" of $\mathcal{L}(E)$, from which $\mathcal{L}(E)$ may be recovered as the "double dual".*

We begin in Section 2 with some general observations regarding topological ergodic decompositions. Let G be a group acting via homeomorphisms on a Polish space X. By passing to the realm of **quasi-Polish spaces** [deB], a possibly non-Hausdorff generalization of Polish spaces, we may realize the topological ergodic decomposition of $G \curvearrowright X$ in a canonical way: as the quasi-Polish T_0-quotient of the quotient space X/G. We give a simple application of this fact: in the case of the space $\underset{\sim}{A}(\Gamma, X, \mu)$ of weak equivalence classes of measure-preserving actions of a countable group Γ (recently studied by several authors; see [BuK] for a survey), the quasi-Polish topology encodes both the usual compact Hausdorff topology and the weak containment partial ordering.

The appendix contains some technical facts about quasi-Polish spaces which are needed in the rest of the paper.

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2 Topological ergodic decompositions

Let X be a Polish space and G be a group acting via homeomorphisms on X. Define the preordering $\preccurlyeq_G$ on X by

$$x \preccurlyeq_G y \iff x \in \overline{G \cdot y} \iff \overline{G \cdot x} \subseteq \overline{G \cdot y}.$$

The symmetric part $\approx_G$ of $\preccurlyeq_G$, given by

$$x \approx_G y \iff x \preccurlyeq_G y \;\&\; y \preccurlyeq_G x \iff \overline{G \cdot x} = \overline{G \cdot y},$$

is a G_δ equivalence relation, hence smooth (see e.g., [Mil]). The quotient space $X/{\approx_G}$ is called the **topological ergodic decomposition** of the action $G \curvearrowright X$, and is a standard Borel space, partially ordered by (the quotient of) $\preccurlyeq_G$, and equipped with the projection map $X \twoheadrightarrow X/{\approx_G}$ which is invariant Borel and whose fibers are the minimal G-invariant G_δ subsets of X. See e.g., [K10, 10.3].

Recall that on an arbitrary topological space X, the **specialization preordering** is defined by

$$x \lesssim y \iff x \in \overline{\{y\}} \iff \forall \text{open } U \subseteq X \, (x \in U \implies y \in U).$$

The specialization preordering is a partial order iff X is T_0, and is discrete iff X is T_1. Note that the specialization preordering on the quotient space X/G (with the quotient topology) is given by

$$[x]_G \lesssim [y]_G \iff x \in \overline{G \cdot y} \iff x \preccurlyeq_G y.$$

Hence, the T_0-quotient of X/G, which we denote by

$$X /\!\!/ G,$$

is in canonical bijection with the topological ergodic decomposition $X/{\approx_G}$; and its specialization ordering agrees with $\preccurlyeq_G$. We henceforth identify $X /\!\!/ G$ with $X/{\approx_G}$ (i.e., we regard the elements of $X /\!\!/ G$ as equivalence classes of elements of X, not of X/G).

Note that open sets in $X /\!\!/ G$ lift to G-invariant open sets $U \subseteq X$; for such U, we write the corresponding open set in $X /\!\!/ G$ as

$$U /\!\!/ G \subseteq X /\!\!/ G,$$

and similarly for closed sets.

Next, we observe that the quotient topology on $X /\!\!/ G$, though not necessarily Hausdorff, is nonetheless well-behaved. A **quasi-Polish space** [deB] is a $\mathbf{\Pi}^0_2$ subset of $\mathbb{S}^{\mathbb{N}}$, where $\mathbb{S} = \{0 < 1\}$ with the Sierpiński topology ($\{0\}$ closed but not open), and where $\mathbf{\Pi}^0_2$ means a countable intersection of sets of the form $U \cup F$ with U open and F closed. Quasi-Polish spaces are closed under countable products, countable disjoint unions, $\mathbf{\Pi}^0_2$-subsets, and continuous open T_0 images; are Polish iff they are regular; can be made Polish by adjoining countably many closed sets to the topology; and induce a standard Borel structure (see [deB] or [Ch] for proofs of these basic facts). Since the projection $X \twoheadrightarrow X /\!\!/ G$ is clearly open, $X /\!\!/ G$ is quasi-Polish, hence standard Borel. It is easily seen that the Borel structure agrees with the quotient Borel structure induced from X, i.e., the usual Borel structure on $X/{\approx_G}$.

Finally, we note that we may consider the following slightly more general context. Let X be a quasi-Polish space and E be an equivalence relation on X such that the E-saturation of every open set $U \subseteq X$ is open. Then X/E has specialization preorder

$$[x]_E \lesssim [y]_E \iff x \in \overline{[y]_E};$$

we denote this by $x \preccurlyeq_E y$ and its symmetric part by $x \approx_E y$. So the T_0-quotient, denoted

$$X /\!\!/ E \cong X/{\approx_E},$$

is the topological ergodic decomposition of X into **E-minimal** (meaning each E-class is dense) components. The condition on saturations of open sets ensures that the projection $X \twoheadrightarrow X /\!\!/ E$ is open, whence $X /\!\!/ E$ is quasi-Polish (in particular standard Borel). As before, we identify $X /\!\!/ E$ with $X/{\approx_E}$, and we write $U /\!\!/ E \subseteq X /\!\!/ E$ for the open set corresponding to E-invariant open $U \subseteq X$. For $x \in X$, we put

$$[\![x]\!]_E := [x]_{\approx_E}.$$

We recover the earlier case of a G-action by taking E to be an orbit equivalence relation E_G.

We summarize these observations as follows:

Proposition 2.1. *Let X be a (quasi-)Polish space and E be an equivalence relation on X such that the E-saturation of every open $U \subseteq X$ is open. Then the T_0-quotient $X /\!\!/ E$ of the quotient X/E is a quasi-Polish space, and the projection $p : X \twoheadrightarrow X /\!\!/ E$ is open with kernel*

$$x \approx_E y \iff \overline{[x]_E} = \overline{[y]_E},$$

hence $X /\!\!/ E$ is the topological ergodic decomposition of (X, E). Moreover, the specialization order on $X /\!\!/ E$, given by

$$[\![x]\!]_E \leq [\![y]\!]_E \iff x \preccurlyeq_E y \iff x \in \overline{[y]_E}$$

(where $[\![x]\!]_E := [x]_{\approx_E}$), is the canonical partial order on the topological ergodic decomposition. Finally, $\approx_E$ has a Borel selector, i.e., the projection p has a Borel section $s : X /\!\!/ E \hookrightarrow X$.

Proof. The last statement about the Borel section is a general fact about continuous open maps between quasi-Polish spaces; see e.g., [Ch, 7.9]. Everything else follows from the above discussion. □

Remark 2.2. De Brecht has pointed out that conversely, every quasi-Polish space can be expressed as the topological ergodic decomposition (indeed, the quotient) of some Polish space by a Polish group action. This may be seen as follows: we have a continuous open surjection $q : [0, \infty) \twoheadrightarrow \mathbb{S}$ sending 0 to 0 and $(0, \infty) \subseteq [0, \infty)$ to 1, which is the quotient of $[0, \infty)$ by the multiplicative action of $(0, \infty)$; then $q^{\mathbb{N}} : [0, \infty)^{\mathbb{N}} \twoheadrightarrow \mathbb{S}^{\mathbb{N}}$ is the quotient of the product action of $(0, \infty)^{\mathbb{N}}$, and so a $\mathbf{\Pi}^0_2$ subset $X \subseteq \mathbb{S}^{\mathbb{N}}$ is the quotient $(q^{\mathbb{N}})^{-1}(X)/(0, \infty)^{\mathbb{N}}$.

2.1 Change of topology

We now record some technical facts, needed in Section 3, concerning the behavior of $X /\!\!/ E$ upon changing the topology of X.

Let X be a quasi-Polish space and E be an equivalence relation on X such that the E-saturation of every open set is open, as in Proposition 2.1. Let τ be the topology of X. Since we will be considering other topologies, we write $[\![x]\!]_E^\tau$ for $[\![x]\!]_E \in (X, \tau) /\!\!/ E$ when necessary to avoid confusion; similarly, we write $\approx_E^\tau$ for $\approx_E$, etc.

Lemma 2.3. *Let $F_0, F_1, \dots \subseteq X$ be countably many E-invariant τ-closed sets, and let $\tau' \supseteq \tau$ be the finer topology obtained by adjoining the F_i to τ. Then the E-saturation of every τ'-open set is τ'-open (as in Proposition 2.1), and $(X, \tau') /\!\!/ E$ is $(X, \tau) /\!\!/ E$ with the closed sets $F_i /\!\!/ E$ adjoined to its topology. (In particular, $(X, \tau) /\!\!/ E = (X, \tau') /\!\!/ E$ as sets, i.e., the topological ergodic decompositions with respect to τ, τ' have the same components.)*

Proof. A basic τ'-open set is of the form $V = U \cap F_{i_0} \cap \cdots \cap F_{i_{n-1}}$ for a τ-open set U; since the F_i are E-invariant, the saturation of V is $[U]_E \cap F_{i_0} \cap \cdots \cap F_{i_{n-1}}$ which is τ'-open. This shows that $(X, \tau'), E$ also obey the hypotheses of Proposition 2.1, as well as that every E-invariant τ'-open set belongs to the topology generated by the E-invariant τ-open sets along with the F_i; the latter easily implies that $(X, \tau) /\!\!/ E$ and $(X, \tau') /\!\!/ E$ are related in the claimed manner. $\square$

Corollary 2.4. *Under the hypotheses of Proposition 2.1, we may adjoin countably many E-invariant closed sets to the topology of X, such that $X /\!\!/ E$ retains the same elements but becomes Polish.*

Proof. Find countably many closed sets $F_i /\!\!/ E \subseteq X /\!\!/ E$, the quotients of E-invariant closed sets $F_i \subseteq X$, such that adjoining the $F_i /\!\!/ E$ to the topology of $X /\!\!/ E$ makes $X /\!\!/ E$ Polish (e.g., by embedding $X /\!\!/ E$ as a $\mathbf{\Pi}_2^0$ subspace of $\mathbb{S}^\mathbb{N}$); then adjoin the F_i to the topology of X. $\square$

If $\tau \subseteq \tau'$ are two topologies on X, both satisfying the hypotheses of Proposition 2.1, then the two topological ergodic decompositions are related by a quotient map

$$
\begin{aligned}
(X, \tau') /\!\!/ E &\twoheadrightarrow (X, \tau) /\!\!/ E \\
[\![x]\!]_E^{\tau'} &\mapsto [\![x]\!]_E^\tau.
\end{aligned}
$$

Suppose now that we have a sequence of quasi-Polish topologies $\tau_0 \subseteq \tau_1 \subseteq \cdots$ on X, each with the property that the E-saturation of an open set is open. We then have an inverse sequence

$$\cdots \twoheadrightarrow (X, \tau_2) /\!\!/ E \twoheadrightarrow (X, \tau_1) /\!\!/ E \twoheadrightarrow (X, \tau_0) /\!\!/ E,$$

of which we may take the **inverse limit**

$$
\begin{aligned}
\varprojlim_i (X, \tau_i) /\!\!/ E &= \{([\![x_i]\!]_E^{\tau_i})_i \in \textstyle\prod_i (X, \tau_i) /\!\!/ E \mid \forall i\, ([\![x_{i+1}]\!]_E^{\tau_i} = [\![x_i]\!]_E^{\tau_i})\} \\
&= \{([\![x_i]\!]_E^{\tau_i})_i \in \textstyle\prod_i (X, \tau_i) /\!\!/ E \mid \forall i\, (x_{i+1} \approx_E^{\tau_i} x_i)\},
\end{aligned}
$$

equipped with the subspace topology (which is quasi-Polish, since equality is $\mathbf{\Pi}_2^0$). The union of the τ_i generates a quasi-Polish topology τ [deB, Lemma 72] on X, the **join** of the τ_i. We have the quotient maps $(X, \tau) /\!\!/ E \twoheadrightarrow (X, \tau_i) /\!\!/ E$ for each i; these induce a comparison map

$$
\begin{aligned}
h : (X, \tau) /\!\!/ E &\longrightarrow \varprojlim_i (X, \tau_i) /\!\!/ E \\
[\![x]\!]_E^\tau &\longmapsto ([\![x]\!]_E^{\tau_i})_i.
\end{aligned}
$$

Lemma 2.5. *Under the above hypotheses, h is a homeomorphism.*

Proof. First, we check that h is an embedding. Let $U /\!\!/ E \subseteq (X, \tau) /\!\!/ E$ be an open set. Since τ is generated by $\bigcup_i \tau_i$, we have $U = \bigcup_i U_i$, where each U_i is τ_i-open. Then $[U_i]_E$ is E-invariant τ_i-open, hence descends to an open $[U_i]_E /\!\!/ E \subseteq (X, \tau_i) /\!\!/ E$. Let

$$\begin{aligned} V_i &:= \{(\llbracket x_j \rrbracket_E^{\tau_j})_j \in \varprojlim_j (X, \tau_j) /\!\!/ E \mid \llbracket x_i \rrbracket_E^{\tau_i} \in [U_i]_E /\!\!/ E\} \\ &= \{(\llbracket x_j \rrbracket_E^{\tau_j})_j \in \varprojlim_j (X, \tau_j) /\!\!/ E \mid x_i \in [U_i]_E\} \end{aligned}$$

be the preimage of $[U_i]_E /\!\!/ E$ under the ith projection $\varprojlim_j (X, \tau_j) /\!\!/ E \to (X, \tau_i) /\!\!/ E$. Since U is E-invariant, $U = \bigcup_i U_i = \bigcup_i [U_i]_E$, whence it is easily seen that $U /\!\!/ E = h^{-1}(\bigcup_i V_i)$. We have shown that every open set in $(X, \tau) /\!\!/ E$ is the h-preimage of an open set in $\varprojlim_i (X, \tau_i) /\!\!/ E$; since the former space is T_0, this means that h is an embedding, as desired.

To check that h is surjective (which is not needed in what follows), we use Proposition A.3, with $X_i := (X, \tau_i)$ and $Y_i := (X, \tau_i) /\!\!/ E$. The Beck–Chevalley condition in the hypotheses of that result amounts to the trivial fact that for τ_i-open $U \subseteq X$, its E-saturation is the same whether we regard U as τ_i-open or τ_{i+1}-open. ☐

2.2 Weak equivalence of measure-preserving actions

We give here a simple example of the extra information that may be contained in the quasi-Polish topology on $X /\!\!/ E$.

Let (X, μ) be a nonatomic standard probability space and Γ be a countable group. The set of measure-preserving actions $a : \Gamma \curvearrowright (X, \mu)$, where two actions are identified if they agree modulo μ-null sets, is denoted

$$A(\Gamma, X, \mu).$$

For an action $a : \Gamma \curvearrowright (X, \mu)$, we write $\gamma^a \cdot x := a(\gamma, x)$. There is a canonical Polish topology on $A(\Gamma, X, \mu)$ (see [K10, II §10(A)]), generated by the maps $a \mapsto \gamma^a \cdot B$ to the measure algebra MALG_μ of μ, for $\gamma \in \Gamma$ and Borel $B \subseteq X$. The Polish group $\mathrm{Aut}(X, \mu)$ of measure-preserving automorphisms of (X, μ) acts continuously on $A(\Gamma, X, \mu)$ via conjugation. The resulting topological ergodic decomposition

$$\utilde{A}(\Gamma, X, \mu) := A(\Gamma, X, \mu) /\!\!/ \mathrm{Aut}(X, \mu)$$

is the **space of weak equivalence classes** of measure-preserving actions $\Gamma \curvearrowright (X, \mu)$; and the associated preordering $\preccurlyeq$ and equivalence relation $\approx$ on $A(\Gamma, X, \mu)$ are called **weak containment** and **weak equivalence**, respectively. See [K10, II §10(C)] or [BuK, §2.1].

There is a natural compact Polish topology on $\utilde{A}(\Gamma, X, \mu)$, due to Abért–Elek [AE]; various equivalent descriptions of this topology are known (see [BuK, §10.1]). Denote this topology by τ. The weak containment partial ordering $\preccurlyeq$ is closed as a subset of $\utilde{A}(\Gamma, X, \mu)^2$ with the τ-product topology (see [BuK, §10.3]). We also have the quasi-Polish quotient topology on $\utilde{A}(\Gamma, X, \mu)$ induced by $A(\Gamma, X, \mu)$; denote this topology by σ. (Note that σ is not T_1, since the specialization order $\preccurlyeq$ is not discrete; see [BuK, §10.3].)

In the theory of topological posets, there is a well-known bijective correspondence between compact Hausdorff spaces equipped with a closed partial order, and the following class of T_0-spaces. A topological space X is **stably compact** if

- it is **locally compact**, i.e., every point has a basis of compact neighborhoods;
- it is **strongly sober**, i.e., every ultrafilter has a unique greatest limit (in the specialization preorder).

(See [GHK+, VI-6.15], or [GHK+, VI-6.7] for an equivalent definition.) The **patch topology** on a stably compact space X has basic closed sets consisting of closed sets in X together with compact sets which are upward-closed in the specialization order. Given an arbitrary topological space Y with a partial order $\leq$, the **upper topology** on Y consists of all $\leq$-upward-closed open sets.

Theorem 2.6 ([GHK+, VI-6.18])**.** *For any set X, there is a bijection*

$$\begin{aligned}\{\textit{stably compact topologies } \sigma \textit{ on } X\} \cong \left\{ (\tau, \leq) \;\middle|\; \begin{array}{l} \tau\textit{: compact Hausdorff topology on } X, \\ \leq\textit{: } \tau\textit{-closed partial order on } X \end{array} \right\} \\ \sigma \mapsto (\textit{patch topology, specialization order}) \\ \textit{upper topology} \leftarrow\!\shortmid (\tau, \leq).\end{aligned}$$

Using this, we have yet another description of the compact Polish topology τ on $\underset{\sim}{A}(\Gamma, X, \mu)$:

Proposition 2.7. *The quasi-Polish quotient topology σ on $\underset{\sim}{A}(\Gamma, X, \mu)$ induced by $A(\Gamma, X, \mu)$ corresponds, via Theorem 2.6, to the compact Polish topology τ and the weak containment order $\preccurlyeq$.*

Proof. It suffices to check that the upper topology of $(\tau, \preccurlyeq)$ is σ. By [BuK, 10.4], every σ-open set is τ-open, as well as $\preccurlyeq$-upward closed (by definition of the specialization preorder). For the converse, we use the following description of τ (see [BuK, 10.3]). For each $k \in \mathbb{N}$, finite subset $\Delta \subseteq \Gamma$, and $\vec{r} = (r_{\gamma,i,j})_{\gamma \in \Delta; i,j<k} \in [0,1]^{\Delta \times k \times k}$, define the upper semicontinuous map

$$\begin{aligned} f_{\Delta,k,\vec{r}} : A(\Gamma, X, \mu) &\longrightarrow [0,1] \\ a &\longmapsto \inf_{\vec{B} \in \mathrm{MALG}_\mu^k} \max_{\gamma \in \Delta; i,j<k} |\mu(\gamma(B_i) \cap B_j) - r_{\gamma,i,j}|. \end{aligned}$$

Let Φ denote the set of all such tuples $(\Delta, k, \vec{r})$. Then $(f_{\Delta,k,\vec{r}})_{\Delta,k,\vec{r}} : A(\Gamma, X, \mu) \to [0,1]^\Phi$ descends to an order-*reversing* embedding $\underset{\sim}{A}(\Gamma, X, \mu) \to [0,1]^\Phi$ onto a closed subposet of $[0,1]^\Phi$ (see [BuK, 2.9]); and τ is obtained by pulling back the usual compact Hausdorff topology on $[0,1]^\Phi$. It follows that the upper topology of $(\tau, \preccurlyeq)$ is obtained by pulling back the *lower* topology on $[0,1]^\Phi$ (induced by the open sets $[0, r)$ in $[0,1]$): indeed, for each closed, $\preccurlyeq$-downward closed $F \subseteq \underset{\sim}{A}(\Gamma, X, \mu)$, the upward closure of its image in $[0,1]^\Phi$ is a closed (by compactness), upward-closed set whose pullback to $\underset{\sim}{A}(\Gamma, X, \mu)$ is F (because $(f_{\Delta,k,\vec{r}})_{\Delta,k,\vec{r}}$ is order-reversing). Since each $f_{\Delta,k,\vec{r}} : A(\Gamma, X, \mu) \to [0,1]$ is upper semicontinuous, i.e., continuous with respect to the lower topology on $[0,1]$, it follows that the upper topology of $(\tau, \preccurlyeq)$ is contained in the quotient topology σ, as desired. □

In other words, the quasi-Polish quotient topology on $\underset{\sim}{A}(\Gamma, X, \mu)$ contains exactly the same information as the usual compact Polish topology together with the weak containment order.

3 Topological versus measure-theoretic ergodic decompositions

Let E be a countable Borel equivalence relation on a standard Borel space X. Recall (see e.g., [KM, I §2]) that a Borel measure μ on X is ***E*-invariant** if the following equivalent conditions hold:

- μ is invariant with respect to some Borel action of a countable group $\Gamma \curvearrowright X$ inducing E;
- μ is invariant with respect to any Borel action of a countable group $\Gamma \curvearrowright X$ inducing E;
- for any two Borel sets $A, B \subseteq X$ such that there is a Borel bijection $f : A \to B$ with graph contained in E (denoted $A \sim_E B$; see [DJK, §2] or Section 4), we have $\mu(A) = \mu(B)$.

A Borel measure μ on X is **E-ergodic** if for any E-invariant Borel set $A \subseteq X$, we have $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. We say that E is **uniquely ergodic** if it admits a unique ergodic invariant probability Borel measure. (Henceforth, by "measure" we mean Borel measure.) Let $\mathcal{P}(X)$ denote the standard Borel **space of probability measures on** X (see [K95, §17.E]), $\mathrm{INV}_E \subseteq \mathcal{P}(X)$ denote the subset of E-invariant measures, and $\mathrm{EINV}_E \subseteq \mathrm{INV}_E$ denote the subset of E-ergodic E-invariant measures. It is well-known that $\mathrm{INV}_E, \mathrm{EINV}_E$ are Borel (see [K95, 17.33], [KM, I 3.3]).

Recall (see e.g., [DJK, §2]) that E is **compressible** if the following equivalent conditions hold:

- there is a Borel injection $f : X \to X$ with graph contained in E such that $X \setminus f(X)$ is an **E-complete section** (i.e., $[X \setminus f(X)]_E = X$);
- $E \cong E \times I_{\mathbb{N}}$, where $I_{\mathbb{N}}$ is the indiscrete equivalence relation $\mathbb{N}^2$ on $\mathbb{N}$;
- there are no E-invariant probability measures (**Nadkarni's theorem**; see [Nad], [BK, §4.3]).

The **uniform (measure-theoretic) ergodic decomposition theorem** of Farrell and Varadarajan states that there is a standard Borel decomposition of non-compressible E into invariant, uniquely ergodic pieces (see e.g., [KM, I 3.3], [DJK, 9.5]):

Theorem 3.1 (Farrell, Varadarajan)**.** *Let E be a countable Borel equivalence relation on X. Suppose E is not compressible. Then there is a Borel E-invariant surjection $p : X \twoheadrightarrow \mathrm{EINV}_E$, such that*

(i) for each $\mu \in \mathrm{EINV}_E$, $\mu|p^{-1}(\mu)$ is the unique $E|p^{-1}(\mu)$-ergodic invariant probability measure;

(ii) for each $\mu \in \mathrm{INV}_E$, we have $\mu = \int p \, d\mu$.

Moreover, such p (satisfying only (i)) is unique modulo compressible sets.

In this section, we show that this measure-theoretic ergodic decomposition may be realized in a particularly nice way: namely, as an instance of the *topological* ergodic decomposition of Proposition 2.1, for a suitably chosen Polish topology on X. Furthermore, we may include in the decomposition not only the E-invariant *probability* measures, but also all E-invariant σ-finite measures which are "regular" with respect to the topology, in the following weak sense. We say that a σ-finite measure μ on a Polish space X is **totally singular** if for every open $U \subseteq X$, either $\mu(U) = 0$ or $\mu(U) = \infty$. By a "regular" measure, we mean one that is not totally singular.

Let (X, E) be a countable Borel equivalence relation, and fix a countable group Γ with a Borel action on X inducing E. We say that a Polish topology on X is **good** if it generates the Borel structure on X and makes the Γ-action continuous (hence makes E satisfy the hypotheses of Proposition 2.1). In the following, by "cofinally many", we mean that any good Polish topology may be refined to one with the specified properties.

Theorem 3.2. *For cofinally many good Polish topologies on X, the topological ergodic decomposition $p : X \twoheadrightarrow X /\!\!/ E$ has the following properties:*

(i) $X /\!\!/ E$ *is Polish;*

(ii) *each component* $C \in X /\!\!/ E$ *admits, modulo scaling, at most one non-totally-singular* $E|C$*-invariant* σ*-finite measure, and this measure is ergodic (if it exists);*

(iii) *the sets*

$$R := \{C \in X /\!\!/ E \mid E|C \text{ admits a non-totally-singular invariant } \sigma\text{-finite measure}\},$$
$$P := \{C \in X /\!\!/ E \mid E|C \text{ admits an invariant probability measure}\}$$

are clopen in $X /\!\!/ E$*;*

(iv) *there is an open set* $S \subseteq X$*, such that* $p(S) = R$*,* S *is a complete* $E|p^{-1}(R)$*-section,* $p^{-1}(P) \subseteq S$*, and there is a Borel isomorphism*

$$\begin{aligned} R &\xrightarrow{\cong} \mathrm{EINV}_{E|S} \\ C &\longmapsto \mu_C \end{aligned}$$

taking each component $C \in R$ *admitting a non-totally-singular invariant (ergodic)* σ*-finite measure to the restriction of such a measure to* $S \cap C$*, with the resulting measure* μ_C *an ergodic probability measure such that* $\mu_C(S \cap C) = 1$*.*

Remark 3.3. Condition (iv) says, informally, that we may identify the set $R \subseteq X /\!\!/ E$ with the space of non-totally-singular ergodic invariant σ-finite measures modulo scaling, so that the projection map p takes a point in a component C supporting such a measure to the unique such measure, as in Theorem 3.1. However, the space of σ-finite measures does not have a natural standard Borel structure, so we have to represent such measures via their finite restrictions to some open set S.

By restricting the set S and the map $C \mapsto \mu_C$ to $P \subseteq R$, we recover the usual ergodic decomposition (Theorem 3.1): the composite

$$p^{-1}(P) \xrightarrow{p} P \xrightarrow{C \mapsto \mu_C} \mathrm{EINV}_{E|p^{-1}(P)} \cong \mathrm{EINV}_E$$

has property (i) in Theorem 3.1, hence (by uniqueness) may be identified with the map p in Theorem 3.1. (The last isomorphism above follows from observing that $E|(X \setminus p^{-1}(P))$ is compressible. To extend the above composite to a map defined on all of X as in Theorem 3.1, simply absorb this compressible set into any component $C \in p^{-1}(P)$.)

We devote the rest of this section to the proof of Theorem 3.2, which essentially consists of repeatedly taking the usual ergodic decomposition (Theorem 3.1) and refining the topology to make all of the desired properties hold. As a way of organizing this iteration, we introduce the following notion: we say that a class $\mathcal{C}$ of good Polish topologies on X is a **club** if it is cofinal in the above sense (i.e., any (good) Polish topology may be refined to one in $\mathcal{C}$), as well as closed under countable increasing joins (i.e., if $\tau_0 \subseteq \tau_1 \subseteq \tau_2 \subseteq \cdots \in \mathcal{C}$, then the (good Polish) topology generated by $\bigcup_i \tau_i$ is in $\mathcal{C}$). Similar terminology is used in [BK, 5.1.4].

Lemma 3.4. *For countably many clubs* $\mathcal{C}_i$ *of good Polish topologies on* X*,* $\bigcap_i \mathcal{C}_i$ *is still a club.*

Proof. Clearly $\bigcap_i \mathcal{C}_i$ is closed under countable increasing joins. To show that it is cofinal, let $f : \mathbb{N} \to \mathbb{N}$ be a surjection taking each value infinitely often, let τ_0 be a good Polish topology on X, and recursively let $\tau_{i+1} \in \mathcal{C}_{f(i)}$ refine τ_i; then the join of the τ_i refines τ_0 and is in each $\mathcal{C}_i$. $\square$

Lemma 3.5. *The class of good Polish topologies τ on X such that $(X, \tau) /\!\!/ E$ is Polish is a club.*

Proof. By Corollary 2.4 and Lemma 2.5. $\square$

We will need the following standard fact on extending invariant measures; see [DJK, 3.2].

Proposition 3.6. *Let $A \subseteq X$ be a Borel set and μ be an $E|A$-invariant σ-finite measure. Then there is a unique E-invariant σ-finite measure $[\mu]_E$ such that $[\mu]_E|A = \mu$ and $[\mu]_E(X \setminus [A]_E) = 0$.*

Explicitly, $[\mu]_E$ is given as follows: enumerate $\Gamma = \{\gamma_0, \gamma_1, \dots\}$, and let $B_i := (\gamma_i \cdot A) \setminus \bigcup_{j<i}(\gamma_j \cdot A)$. Then

$$[\mu]_E(B) := \textstyle\sum_i \mu(\gamma_i^{-1} \cdot (B \cap B_i)). \qquad \square$$

For a Borel set $A \subseteq X$, we say that a good Polish topology τ on X **splits** A if for every component $C \in (X, \tau) /\!\!/ E$, $E|(A \cap C)$ admits at most one invariant probability measure.

Lemma 3.7. *For every Borel $A \subseteq X$, all sufficiently fine good Polish topologies on X split A.*

Proof. If $E|A$ is compressible, then clearly any topology splits A. Otherwise, let $q : A \twoheadrightarrow \mathrm{EINV}_{E|A}$ be an ergodic decomposition as in Theorem 3.1, and let $B_i \subseteq \mathrm{EINV}_{E|A}$ be a countable separating family of Borel sets. Then any good Polish topology τ making each $[q^{-1}(B_i)]_E \subseteq X$ clopen splits A. Indeed, given such τ, for any $C \in (X, \tau) /\!\!/ E$, if there is some $x \in A \cap C$, then putting

$$F := \textstyle\bigcap_{B_i \ni q(x)} [q^{-1}(B_i)]_E \cap \bigcap_{B_i \not\ni q(x)} (X \setminus [q^{-1}(B_i)]_E),$$

F is τ-closed and E-invariant and contains x, hence contains its component $[\![x]\!]_E^\tau = C$; but clearly $q^{-1}(q(x)) = A \cap F$, whence $A \cap C \subseteq A \cap F = q^{-1}(q(x))$, whence $E|(A \cap C)$ has at most one invariant probability measure (namely, $q(x)$) by definition of q. $\square$

We say that a good Polish topology τ on X is **very good** if τ splits every τ-open set.

Lemma 3.8. *If a good Polish topology τ on X splits every set in a basis for τ, then τ is very good.*

Proof. For any component $C \in (X, \tau) /\!\!/ E$, since $E|C$ is minimal, every τ-open $U \subseteq X$ which intersects C intersects every equivalence class in C. So if for some $C \in (X, \tau) /\!\!/ E$ and τ-open $U \subseteq X$, $E|(U \cap C)$ had two distinct invariant probability measures μ, ν, then letting $V \subseteq U$ be basic open with $V \cap C \neq \varnothing$, we have that $\mu|(V \cap C), \nu|(V \cap C)$ are nonzero finite $E|(V \cap C)$-invariant measures, which must be distinct modulo scaling by Proposition 3.6 (since they extend to μ, ν respectively), whence $E|(V \cap C)$ also has two distinct invariant probability measures. $\square$

Lemma 3.9. *The class of very good Polish topologies on X is a club.*

Proof. Closure under countable increasing join follows from Lemma 3.8. To check cofinality: given any good τ_0, repeatedly apply Lemma 3.7 to obtain $\tau_1 \supseteq \tau_0$ which splits all sets in a countable basis for τ_0, then similarly obtain $\tau_2 \supseteq \tau_1$, etc.; the join τ of $\tau_0 \subseteq \tau_1 \subseteq \tau_2 \subseteq \cdots$ then splits every τ-open set, by Lemma 3.8. $\square$

Lemma 3.10. *Every very good Polish topology τ on X satisfies condition (ii) in Theorem 3.2.*

Proof. Suppose $C \in (X,\tau)/\!\!/E$ has two non-totally-singular $E|C$-invariant σ-finite measures μ, ν. Then $\mu(U \cap C), \nu(V \cap C) \in (0,\infty)$ for some τ-open $U, V \subseteq X$. Since $E|C$ is minimal, U, V intersect every $E|C$-class. So there is a group element $\gamma \in \Gamma$ such that $(\gamma \cdot U) \cap V \cap C \neq \varnothing$. Then $W := (\gamma \cdot U) \cap V$ still intersects every $E|C$-class, whence $\mu(W \cap C), \nu(W \cap C) \in (0,\infty)$. Since τ splits W, $\mu|(W \cap C) = r\nu|(W \cap C)$ for some $r \in (0,\infty)$; since W intersects every $E|C$-class, by Proposition 3.6 we have $\mu = [\mu|(W \cap C)]_{E|C} = r[\nu|(W \cap C)]_{E|C} = r\nu$.

To check that a non-totally-singular $E|C$-invariant measure μ is necessarily ergodic, let $U \subseteq X$ be τ-open so that $\mu(U \cap C) \in (0,\infty)$; since τ splits U, $\mu|(U \cap C)$ is $E|(U \cap C)$-ergodic, whence since U intersects every $E|C$-class, μ is $E|C$-ergodic. $\square$

For a very good Polish topology τ on X (thus Theorem 3.2(ii) holds by Lemma 3.10), let

$$R(\tau), P(\tau) \subseteq (X,\tau)/\!\!/E$$

denote the sets R, P defined in Theorem 3.2(iii). For a τ-open $U \subseteq X$, let

$$P(U,\tau) := \{C \in (X,\tau)/\!\!/E \mid \mathrm{INV}_{E|(U\cap C)} \neq \varnothing\}.$$

Clearly, $P(X,\tau) = P(\tau)$.

Lemma 3.11. *For any basis $\mathcal{U}$ for τ, $R(\tau) = \bigcup_{U \in \mathcal{U}} P(U,\tau)$.*

Proof. $\subseteq$ is because for $C \in (X,\tau)/\!\!/E$, any $E|C$-invariant non-totally-singular measure μ restricts to an $E|(U \cap C)$-invariant finite measure for some $U \in \mathcal{U}$; $\supseteq$ is because any $E|(U \cap C)$-invariant probability measure extends to an $E|C$-invariant non-totally-singular measure (Proposition 3.6). $\square$

Lemma 3.12. *The sets $R(\tau), P(\tau), P(U,\tau) \subseteq (X,\tau)/\!\!/E$ above are Borel.*

Proof. By the above, it is enough to check that $P(U,\tau) \subseteq (X,\tau)/\!\!/E$ is Borel. Indeed, it is the preimage, under the embedding

$$(X,\tau)/\!\!/E \hookrightarrow \mathcal{P}((X,\tau)/\!\!/E)$$

taking $C \in (X,\tau)/\!\!/E$ to the Dirac delta δ_C, of the image of the measure pushforward map

$$p_* : \mathrm{EINV}_{E|U} \longrightarrow \mathcal{P}((X,\tau)/\!\!/E)$$

(where $p : U \subseteq X \twoheadrightarrow (X,\tau)/\!\!/E$ is the projection), which is injective because τ splits U. $\square$

Lemma 3.13. *Let $\tau_0 \subseteq \tau_1 \subseteq \cdots$ be a sequence of very good Polish topologies on X, and let τ be their join (which is very good by Lemma 3.9). Let*

$$p_i : (X,\tau_i) \twoheadrightarrow (X,\tau_i)/\!\!/E, \qquad\qquad p : (X,\tau) \twoheadrightarrow (X,\tau)/\!\!/E$$

denote the quotient projections. Then

$$p^{-1}(R(\tau)) = \textstyle\bigcap_i p_i^{-1}(R(\tau_i)), \qquad\qquad p^{-1}(P(\tau)) = \textstyle\bigcap_i p_i^{-1}(P(\tau_i)),$$

and for τ_0-open $U \subseteq X$,

$$p^{-1}(P(U,\tau)) = \textstyle\bigcap_i p_i^{-1}(P(U,\tau_i)).$$

Proof. We check the last case; the other two are similar. For $x \in X$, we have $x \in p^{-1}(P(U,\tau))$ iff $E|(U \cap [\![x]\!]^\tau_E)$ admits an invariant probability measure, while $x \in \bigcap_i p_i^{-1}(P(U,\tau_i))$ iff for every i, $E|(U \cap [\![x]\!]^{\tau_i}_E)$ admits an invariant probability measure. The former clearly implies the latter. Conversely, if the latter holds, let μ_i be the measure on $E|(U \cap [\![x]\!]^{\tau_i}_E)$. Since each τ_i splits U, each μ_i is the unique $E|(U \cap [\![x]\!]^{\tau_i}_E)$-invariant probability measure, whence in fact all the μ_i are identical and supported on $U \cap \bigcap_i [\![x]\!]^{\tau_i}_E$. By Lemma 2.5, $\bigcap_i [\![x]\!]^{\tau_i}_E = [\![x]\!]^\tau_E$, whence $E|(U \cap [\![x]\!]^\tau_E)$ admits an invariant probability measure. ☐

Lemma 3.14. *The following class of good Polish topologies τ on X forms a club: τ is very good, the sets $R(\tau), P(\tau) \subseteq (X,\tau)/\!\!/E$ are closed, and there is a countable basis $\mathcal{U}$ for τ such that $P(U,\tau) \subseteq (X,\tau)/\!\!/E$ is closed for every $U \in \mathcal{U}$.*

Proof. Closure under countable increasing join follows from Lemma 3.9 and Lemma 3.13. To check cofinality: let τ_0 be any good Polish topology on X; given τ_{2n}, let $\tau_{2n+1} \supseteq \tau_{2n}$ be a very good Polish topology by Lemma 3.9, and let $\tau_{2n+2} \supseteq \tau_{2n+1}$ be a good Polish topology in which the preimages under the projection

$$p_{2n+1} : X \twoheadrightarrow (X, \tau_{2n+1})/\!\!/E$$

of the Borel (by Lemma 3.12) sets $R(\tau_{2n+1}), P(\tau_{2n+1}), P(U, \tau_{2n+1}) \subseteq (X, \tau_{2n+1})/\!\!/E$ are clopen, for all U in some countable basis $\mathcal{U}_{2n+1}$ for τ_{2n+1}. Let τ be the join of the τ_i and $p : (X,\tau) \twoheadrightarrow (X,\tau)/\!\!/E$ be the projection. Then by Lemma 3.9, τ is very good, while by Lemma 3.13, the sets $p^{-1}(R(\tau)), p^{-1}(P(\tau)), p^{-1}(P(U,\tau)) \subseteq X$ are τ-closed, i.e., $R(\tau), P(\tau), P(U,\tau) \subseteq (X,\tau)/\!\!/E$ are closed, for all U in the countable basis $\mathcal{U} := \bigcup_n \mathcal{U}_{2n+1}$ for τ. ☐

Proof of Theorem 3.2. Let τ_0 be a good Polish topology on X, and let $\tau_1 \supseteq \tau_0$ be a finer topology satisfying Lemma 3.5 and Lemma 3.14, with the latter giving a basis $\mathcal{U}$. So (i–ii) hold for the decomposition $p : (X,\tau_1) \twoheadrightarrow (X,\tau_1)/\!\!/E$. Let $\tau_2 \supseteq \tau_1$ be given by adjoining the E-invariant τ_1-closed sets $p^{-1}(R(\tau_1)), p^{-1}(P(\tau_1)), p^{-1}(P(U,\tau_1)) \subseteq X$, for all $U \in \mathcal{U}$; by Lemma 2.3, doing so does not change the components of the topological ergodic decomposition, i.e., $(X,\tau_2)/\!\!/E = (X,\tau_1)/\!\!/E$ as sets. So (ii) continues to hold for τ_2; clearly so does (i), and also (iii) holds since $R(\tau_2), P(\tau_2) \subseteq (X,\tau_2)/\!\!/E$ (which are the same sets as $R(\tau_1), P(\tau_1)$) are now clopen.

Finally, we check (iv). Let $\mathcal{U} = \{U_0, U_1, \dots\}$, and put

$$V_i := (U_i \cap p^{-1}(P(U_i,\tau_1))) \setminus (p^{-1}(P) \cup \textstyle\bigcup_{j<i} p^{-1}(P(U_j,\tau_1))).$$

Since P (i.e., $P(\tau_1) = P(\tau_2)$) and $P(U_i,\tau_1)$ are τ_2-clopen, so is each V_i. It is easily seen that

$$R = P \sqcup \textstyle\bigsqcup_i p(V_i)$$

(using Lemma 3.11), with $p(V_i) = P(U_i,\tau_1) \setminus (P \cup \bigcup_{j<i} P(U_j,\tau_1))$. Put

$$S := p^{-1}(P) \cup \textstyle\bigcup_i V_i.$$

Clearly $p(S) = R$ and $p^{-1}(P) \subseteq S$. The map $C \mapsto \mu_C \in \mathrm{EINV}_{E|S}$ is defined in the obvious way: μ_C is the unique such measure so that $p_*(\mu_C) = \delta_C \in \mathcal{P}((X,\tau_2)/\!\!/E)$. For $C \in P$, μ_C exists by definition of P; similarly, for $C \in p(V_i)$, μ_C exists by definition of $P(U_i,\tau_1) \supseteq p(V_i)$. ☐

Remark 3.15. If in the above proof we do not refine τ_1 to τ_2, then we obtain the following variant of the statement of Theorem 3.2: there is a club of topologies satisfying the conditions (and not just cofinally many); but the sets R, P in (iii) are merely closed (instead of clopen), and the set S in (iv) will only be F_σ with open sections (instead of open).

4 Cardinal algebras of equidecomposition types

4.1 Cardinal algebras

A **cardinal algebra** is an algebraic structure $(A, 0, +, \sum)$, where $(A, 0, +)$ is an abelian monoid and $\sum : A^{\mathbb{N}} \to A$ is a countably infinitary operation on A with $\sum(a_i)_{i\in\mathbb{N}}$ denoted also by $\sum_{i<\infty} a_i$, satisfying the following axioms:

(A) $\sum_{i<\infty} a_i = a_0 + \sum_{i<\infty} a_{i+1}$.

(B) $\sum_{i<\infty}(a_i + b_i) = \sum_{i<\infty} a_i + \sum_{i<\infty} b_i$.

(C) If $a + b = \sum_{i<\infty} c_i$, then there are $(a_i)_{i<\infty}, (b_i)_{i<\infty}$ such that $a = \sum_{i<\infty} a_i$, $b = \sum_{i<\infty} b_i$, and $a_i + b_i = c_i$. This is depicted in the following picture:

$$\begin{array}{c|cccc} & c_0 & c_1 & c_2 & \cdots \\ \hline a & a_0 & a_1 & a_2 & \cdots \\ b & b_0 & b_1 & b_2 & \cdots \end{array}$$

(D) If $(a_i)_{i<\infty}, (b_i)_{i<\infty}$ are such that $a_i = b_i + a_{i+1}$, then there is c such that $a_i = c + \sum_{j<\infty} b_{i+j}$.

Cardinal algebras were introduced and comprehensively studied by Tarski [Tar]; the above axioms are from [KMd] and are equivalent to Tarski's original axioms. These axioms imply many other desirable algebraic properties, of which the following will be most important for our purposes:

(E) [Tar, 1.17, 1.38, 1.42] Addition is well-behaved: for finitely many elements $a_0, a_1, \dots, a_{n-1} \in A$, we may define their sum via the equivalent formulas

$$\textstyle\sum_{i<n} a_i := a_0 + \cdots + a_{n-1} = \sum(a_0, \dots, a_{n-1}, 0, 0, \dots);$$

and both finitary and infinitary addition satisfy all commutativity and associativity laws.

(F) [Tar, 1.31, 1.22] We have a canonical partial order, defined by

$$a \le b \iff \exists c\,(a + c = b),$$

which interacts well with addition: $0 \le a$ for all a; and if $a_i \le b_i$ for each i, then $\sum_i a_i \le \sum_i b_i$.

(G) [Tar, 2.24, 2.21, 3.19] Countable increasing joins exist: given $a_0 \le a_1 \le \cdots \in A$, there is a join (i.e., least upper bound) $\bigvee_i a_i \in A$. Moreover, for all $a_0, a_1, \dots \in A$ we have

$$\textstyle\sum_{i<\infty} a_i = \bigvee_{n<\infty} \sum_{i<n} a_i.$$

(H) [Tar, 3.4] If two elements $a, b \in A$ have a meet $a \wedge b \in A$, then they also have a join $a \vee b \in A$, satisfying

$$a + b = a \wedge b + a \vee b.$$

(I) [Tar, 1.43, 1.45] For $n \le \infty$, put

$$n \cdot a := \textstyle\sum_{i<n} a.$$

This yields an action of the multiplicative monoid $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ (where $0\infty := 0$) on A, which preserves the partial order and countable addition in both $\overline{\mathbb{N}}$ and A.

(J) [Tar, 2.34] For $0 < n < \infty$, we say that $a \in A$ is **divisible by** n if there is a b such that $n \cdot b = a$; such b is necessarily unique, hence may be denoted by a/n. We say that a is **completely divisible** if it is divisible by arbitrarily large n.

(K) [Tar, end of §2] [Chu, 1.1–1.13] For completely divisible $a \in A$, we may define **real multiples** $r \cdot a$ for every $r \in \overline{\mathbb{R}^+} := [0, \infty]$ by

$$r \cdot a := \sum_i (p_i \cdot a/q_i)$$

for any sequence of rationals p_i/q_i with sum r such that a is divisible by each q_i; the definition does not depend on the choice of such sequence. This yields an action of the multiplicative monoid $\overline{\mathbb{R}^+}$ (where $0\infty := 0$) on A, extending the action of $\overline{\mathbb{N}} \subseteq \overline{\mathbb{R}^+}$, which preserves the partial order and countable addition in both $\overline{\mathbb{R}^+}$ and A.

4.2 The algebra $\mathcal{K}(E)$

Fix a compressible countable Borel equivalence relation (X, E). Let $\mathcal{B}(X)$ denote the Borel σ-algebra of X. Recall (see e.g., [DJK, §2]) that for $A, B \in \mathcal{B}(X)$, an E**-equidecomposition**

$$f : A \sim_E B$$

is a Borel bijection $f : A \to B$ with graph contained in E; A, B are E**-equidecomposable**, written $A \sim_E B$, if there is some $f : A \sim_E B$. We also write

$$A \preceq_E B, \qquad\qquad A \prec_E B$$

to mean respectively that $A \sim_E C$ for some Borel $C \subseteq B$, and that such C may be chosen so that $[B \setminus C]_E = [B]_E$. (Note that it is possible to have $A \preceq_E B$ but also $A \succ_E B$, e.g., $X \prec_E X$, since E is compressible.)

Put

$$\mathcal{K}(E) := \mathcal{B}(X)/{\sim_E}.$$

The rest of this paper is devoted to the study of the algebraic structure of $\mathcal{K}(E)$ (and the related $\mathcal{L}(E)$ to be defined in the next section). We will use the following notation: for $A \in \mathcal{B}(X)$, write

$$\widetilde{A} := [A]_{\sim_E}.$$

We define finite and countably infinite sums in $\mathcal{K}(E)$ as follows. For countably many elements $\widetilde{A}_0, \widetilde{A}_1, \ldots \in \mathcal{K}(E)$, by compressibility of E, we may choose the representatives $A_0, A_1, \ldots \in \mathcal{B}(X)$ to be pairwise disjoint; put

$$\sum_i \widetilde{A}_i := \widetilde{\bigcup_i A_i} \qquad \text{for pairwise disjoint } A_0, A_1, \ldots.$$

It is straightforward that this is well-defined (given $f_i : A_i \sim_E B_i$ where the B_i are also pairwise disjoint, we have $\bigcup_i f_i : \bigcup_i A_i \sim_E \bigcup_i B_i$). Put also

$$0 := \widetilde{\varnothing}, \qquad\qquad \infty := \widetilde{X}.$$

Proposition 4.1. *$\mathcal{K}(E)$ is a cardinal algebra, with addition as above and canonical partial order given by*

$$\widetilde{A} \le \widetilde{B} \iff A \preceq_E B.$$

Proof. Recall that for any countable group Γ with a Borel action $\Gamma \curvearrowright X$ inducing E, we have $A \sim_E B$ iff there are Borel partitions $A = \bigsqcup_{\gamma\in\Gamma} A_\gamma$ and $B = \bigsqcup_{\gamma\in\Gamma} B_\gamma$ such that $\gamma \cdot A_\gamma = B_\gamma$; see e.g., [BK, §4.2–3]. Thus $\mathcal{K}(E)$ is an instance of the cardinal algebra of equidecomposition types constructed in [Tar, 16.7] (see also [Chu, 2.4]).

(It is also easy to verify axioms (A–D) from Section 4.1 directly, by picking the Borel sets involved in each axiom to be pairwise disjoint, using compressibility of E.)

That $\widetilde{A} \le \widetilde{B} \iff A \preceq_E B$ is immediate from the definitions. □

A key tool in analyzing the structure of $\mathcal{K}(E)$ is the following lemma, first used by Becker–Kechris [BK, 4.5.1] in their proof of the general case of Nadkarni's theorem:

Lemma 4.2 (Becker–Kechris)**.** *For any $A, B \in \mathcal{B}(X)$, there is an E-invariant Borel partition $X = Y \sqcup Z$ such that $A \cap Y \preceq_E B \cap Y$ and $A \cap Z \succ_E B \cap Z$.*

Proposition 4.3. *$\mathcal{K}(E)$ has finite meets, hence also countable joins.*

Proof. Clearly the greatest element is $\infty \in \mathcal{K}(E)$. To compute the meet of $\widetilde{A}, \widetilde{B} \in \mathcal{K}(E)$, let Y, Z be given by Lemma 4.2; then it is easily seen that

$$\widetilde{A} \wedge \widetilde{B} = [(A \cap Y) \cup (B \cap Z)]_{\sim_E}.$$

By §4.1(H), it follows that $\mathcal{K}(E)$ has binary joins, hence (since every cardinal algebra has least element 0 and countable increasing joins by §4.1(G)) arbitrary countable joins.

Alternatively, we may compute joins directly, as follows. Similarly to meets, for Y, Z as above,

$$\widetilde{A} \vee \widetilde{B} = [(A \cap Z) \cup (B \cap Y)]_{\sim_E}.$$

To compute the increasing join of $\widetilde{A}_0 \le \widetilde{A}_1 \le \cdots \in \mathcal{K}(E)$, using compressibility of E, we may choose the representatives $A_0, A_1, \ldots \in \mathcal{B}(X)$ so that $A_0 \subseteq A_1 \subseteq \cdots$; then

$$\textstyle\bigvee_i \widetilde{A}_i = [\bigcup_i A_i]_{\sim_E} \qquad \text{for } A_0 \subseteq A_1 \subseteq \cdots.$$

To check that this works, use §4.1(G): we have $\bigcup_i A_i = A_0 \sqcup \bigsqcup_i (A_{i+1} \setminus A_i)$, whence $[\bigcup_i A_i]_{\sim_E} = \widetilde{A}_0 + \sum_i [A_{i+1} \setminus A_i]_{\sim_E} = \bigvee_n (\widetilde{A}_0 + [A_1 \setminus A_0]_{\sim_E} + \cdots + [A_n \setminus A_{n-1}]_{\sim_E}) = \bigvee_n \widetilde{A}_n$. Finally, we may compute an arbitrary countable join of $\widetilde{A}_0, \widetilde{A}_1, \ldots \in \mathcal{K}(E)$ as the increasing join of finite joins $\bigvee_n (\widetilde{A}_0 \vee \cdots \vee \widetilde{A}_n)$. □

We next consider divisibility in $\mathcal{K}(E)$, for which we use the following lemma [KM, 7.4]:

Lemma 4.4 (Kechris–Miller)**.** *For every aperiodic countable Borel equivalence relation (X, E) and $n > 0$, there is a finite Borel subequivalence relation $F \subseteq E$ all of whose classes have size n.*

Proposition 4.5. *$\widetilde{A} \in \mathcal{K}(E)$ is completely divisible iff $E|A$ is aperiodic.*

Proof. If $E|A$ is aperiodic, then for any $n > 0$, by Lemma 4.4, we may find a Borel subequivalence relation $F \subseteq E|A$ all of whose classes have size n; letting $A_0, \dots, A_{n-1} \subseteq A$ be disjoint Borel transversals of F, we clearly have $\widetilde{A} = \widetilde{A}_0 + \cdots + \widetilde{A}_{n-1}$ and $A_0 \sim_E \cdots \sim_E A_{n-1}$, whence $\widetilde{A}$ is divisible by n. Conversely, if $E|A$ has a finite class, say of cardinality n, then clearly $\widetilde{A}$ is not divisible by any $m > n$. $\square$

We say that $\widetilde{A} \in \mathcal{K}(E)$ is **finite** if $E|A$ is finite (i.e., has finite classes), and **aperiodic** if $E|A$ is aperiodic, or equivalently if $\widetilde{A}$ is completely divisible by Proposition 4.5. We let $\mathcal{K}^{\mathrm{fin}}(E), \mathcal{K}^{\mathrm{ap}}(E) \subseteq \mathcal{K}(E)$ denote the subsets of finite, respectively aperiodic, elements.

Proposition 4.6. *$\mathcal{K}^{\mathrm{ap}}(E) \subseteq \mathcal{K}(E)$ is a cardinal subalgebra.*

Proof. Clearly $\mathcal{K}^{\mathrm{ap}}(E) \subseteq \mathcal{K}(E)$ is closed under countable addition; so it suffices to check that the existential axioms (C) and (D) from §4.1 still hold in $\mathcal{K}^{\mathrm{ap}}(E)$. For (C), let $\widetilde{A}, \widetilde{B}, \widetilde{C}_i \in \mathcal{K}^{\mathrm{ap}}(E)$ with $\widetilde{A} + \widetilde{B} = \sum_i \widetilde{C}_i$, and let $\widetilde{A}_i, \widetilde{B}_i \in \mathcal{K}(E)$ with $\widetilde{A} = \sum_i \widetilde{A}_i$ and $\widetilde{B} = \sum_i \widetilde{B}_i$ be given by (C) in the cardinal algebra $\mathcal{K}(E)$. Let $Y \subseteq X$ be the union of all E-classes whose intersection with some A_i or B_i is finite nonempty. Clearly Y is E-invariant Borel, whence by restricting E, we may assume either $Y = \varnothing$ or $Y = X$. If $Y = \varnothing$, we have $\widetilde{A}_i, \widetilde{B}_i \in \mathcal{K}^{\mathrm{ap}}(E)$, so (C) holds. If $Y = X$, then clearly E is smooth, whence we may easily find $\widetilde{A}_i, \widetilde{B}_i$ making (C) hold. The proof of (D) is similar. $\square$

Let $\overline{\mathbb{R}^+} := [0, \infty]$, which is also a cardinal algebra with finite meets (and countable joins). We say that a map $f : A \to B$ between cardinal algebras A, B is a **$\sum$-homomorphism** if it preserves countable sums (including zero). Clearly, a $\sum$-homomorphism $\mathcal{K}(E) \to \overline{\mathbb{R}^+}$, i.e., a $\sim_E$-invariant σ-additive map $\mathcal{B}(X) \to \overline{\mathbb{R}^+}$, is the same thing as an E-invariant measure:

Proposition 4.7. *We have a canonical bijection*

$$\begin{aligned} \mathrm{INV}^*_E &\cong \{\textstyle\sum\text{-}\textit{homomorphisms } \mathcal{K}(E) \to \overline{\mathbb{R}^+}\} \\ \mu &\longmapsto (\widetilde{A} \mapsto \mu(A)), \end{aligned}$$

where INV^*_E denotes the set of (not necessarily probability or even σ-finite, and possibly zero) E-invariant measures on X. $\square$

We henceforth identify a measure $\mu \in \mathrm{INV}^*_E$ with the corresponding $\sum$-homomorphism.

Lemma 4.8. *Every $\sum$-homomorphism $f : A \to B$ between cardinal algebras preserves countable increasing joins as well as real multiples of completely divisible elements.*

Proof. Let $a_0 \le a_1 \le \cdots \in A$. Then $a_{i+1} = a_i + b_i$ for some $b_i \in A$. Using §4.1(G), we have $\bigvee_i a_i = \bigvee_i (a_0 + b_0 + \cdots + b_{i-1}) = a_0 + \sum_i b_i$, whence $f(\bigvee_i a_i) = f(a_0) + \sum_i f(b_i) = \bigvee_i (f(a_0) + f(b_0) + \cdots + f(b_{i-1})) = \bigvee_i f(a_0 + b_0 + \cdots + b_{i-1}) = \bigvee_i f(a_i)$.

Let $a \in A$ be completely divisible and $r \in \overline{\mathbb{R}^+}$. Then for every positive integer n, we have $n \cdot f(a/n) = f(n \cdot a/n) = f(a)$, whence $f(a/n) = f(a)/n$. So for any sequence of rationals p_i/q_i with sum r, we have $f(r \cdot a) = f(\sum_i p_i \cdot a/q_i) = \sum_i p_i \cdot f(a)/q_i = r \cdot f(a)$. $\square$

Lemma 4.9. *If a $\sum$-homomorphism $\mu : A \to \overline{\mathbb{R}^+}$ preserves binary meets, then it preserves countable joins.*

Proof. For $a, b \in A$, from $a + b = a \wedge b + a \vee b$ (§4.1(H)), we get $\mu(a \wedge b) + \mu(a \vee b) = \mu(a) + \mu(b) = \mu(a) \wedge \mu(b) + \mu(a) \vee \mu(b) = \mu(a \wedge b) + \mu(a) \vee \mu(b)$. If $\mu(a \wedge b) < \infty$, then we may cancel to get $\mu(a \vee b) = \mu(a) \vee \mu(b)$. Otherwise, since $a \wedge b \le a, b, a \vee b$, we have $\mu(a \vee b) = \infty = \mu(a) \vee \mu(b)$. So μ preserves binary joins. Since μ always preserves 0 and countable increasing joins (Lemma 4.8), it preserves arbitrary countable joins. $\square$

Lemma 4.10. *$\mu \in \mathrm{INV}_E^*$ is ergodic iff $\mu : \mathcal{K}(E) \to \overline{\mathbb{R}^+}$ preserves finite meets.*

Proof. By compressibility of E, every nonzero E-invariant measure is infinite; thus μ preserves the greatest element ∞ iff μ is nonzero. If μ is nonzero and preserves binary meets, then for every E-invariant Borel $A \subseteq X$, we have $\widetilde{A} \wedge \widetilde{X \setminus A} = 0$, whence $\mu(A) \wedge \mu(X \setminus A) = \mu(\widetilde{A} \wedge \widetilde{X \setminus A}) = \mu(0) = 0$, whence either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$, i.e., μ is ergodic. Conversely, if μ is ergodic, then for every $\widetilde{A}, \widetilde{B} \in \mathcal{K}(E)$, letting $X = Y \sqcup Z$ be given by Lemma 4.2, we have either $\mu(Y) = 0$ or $\mu(Z) = 0$; in the former case, we have $\mu(A) = \mu(A \cap Z) \ge \mu(B \cap Z) = \mu(B)$ (since $A \cap Z \succ_E B \cap Z$), whence $\mu(\widetilde{A} \wedge \widetilde{B}) = \mu((A \cap Y) \cup (B \cap Z)) = \mu(B \cap Z) = \mu(B) = \mu(A) \wedge \mu(B)$, while in the latter case we similarly have $\mu(\widetilde{A} \wedge \widetilde{B}) = \mu(A) = \mu(A) \wedge \mu(B)$. $\square$

For a cardinal algebra A, we say that a map $\mu : A \to \overline{\mathbb{R}^+}$ is a $(\sum, \wedge, \bigvee, \overline{\mathbb{R}^+})$**-homomorphism** if it preserves countable sums, finite meets, countable joins, and real multiples of completely divisible elements. The preceding lemmas now give

Proposition 4.11. *We have a canonical bijection*

$$\mathrm{EINV}_E^* \cong \{(\textstyle\sum, \wedge, \bigvee, \overline{\mathbb{R}^+})\text{-}\mathit{homomorphisms}\ \mathcal{K}(E) \to \overline{\mathbb{R}^+}\}$$

where $\mathrm{EINV}_E^* \subseteq \mathrm{INV}_E^*$ denotes the (not necessarily σ-finite) E-ergodic invariant measures. $\square$

Remark 4.12. Non-σ-finite measures are not so tractable: for any σ-complete ultrafilter $\mathcal{U}$ of E-invariant Borel subsets of X, we have an E-ergodic invariant measure $\mu \in \mathrm{EINV}_E^*$, given by $\mu(A) = 0$ if $[A]_E \notin \mathcal{U}$, else $\mu(A) = \infty$.

Finally in this section, we show that there are "enough" homomorphisms $\mathcal{K}(E) \to \overline{\mathbb{R}^+}$. Because of the preceding remark, we will in fact only consider homomorphisms corresponding to σ-finite measures. Let $\mathrm{EINV}_E^\sigma \subseteq \mathrm{EINV}_E^*$ denote the subset of σ-finite measures.

Lemma 4.13. *For any $\widetilde{A} \not\le \widetilde{B} \in \mathcal{K}(E)$, there is a $\mu \in \mathrm{EINV}_E^\sigma$ such that $\mu(\widetilde{A}) > \mu(\widetilde{B})$.*

Proof. Since $A \not\preceq_E B$, by restricting E to the set Z given by Lemma 4.2, we may assume that $A \succ_E B$. If $[A]_E \not\subseteq [B]_E$, then we may let μ be an atomic measure; so we may restrict E to $[A]_E$, and assume both A and B are E-complete sections. If $E|B$ were compressible, then we would have $\widetilde{B} = \infty \ge \widetilde{A}$, a contradiction. Thus $E|B$ is not compressible, hence has an ergodic invariant probability measure μ by Nadkarni's theorem. Extending μ to E using Proposition 3.6, we have $\mu(A) > \mu(B)$ since $A \succ_E B$ and $\mu(B) < \infty$, as desired. $\square$

Proposition 4.14. *We have an embedding*

$$\begin{aligned} \eta : \mathcal{K}(E) &\hookrightarrow \overline{\mathbb{R}^+}^{\mathrm{EINV}_E^\sigma} \\ \widetilde{A} &\mapsto (\mu \mapsto \mu(A)) \end{aligned}$$

preserving countable sums, finite meets, countable joins, and real multiples of completely divisible elements (with the pointwise operations in $\overline{\mathbb{R}^+}^{\mathrm{EINV}_E^\sigma}$). $\square$

By universal algebra, we may rephrase this result as follows. A **Horn axiom** in the operations $\sum, \wedge, \bigvee, \overline{\mathbb{R}^+}$ (countable sums, finite meets, countable joins, and real multiples of completely divisible elements) is an axiom of the form

$$\forall \vec{v}\, [\bigwedge_i (s_i(\vec{v}) = t_i(\vec{v})) \to (s(\vec{v}) = t(\vec{v}))]$$

where s_i, t_i, s, t are terms built from the specified operations and the (possibly infinitely many) variables $\vec{v}$ (in the case of the partially defined operations $\overline{\mathbb{R}^+}$, we interpret the right-hand side of the implication to mean "if both terms are defined, then the equality holds").

Corollary 4.15. *$\mathcal{K}(E)$ obeys all Horn axioms in the operations $\sum, \wedge, \bigvee, \overline{\mathbb{R}^+}$ which hold in the algebra $\overline{\mathbb{R}^+}$.* □

We end this section by summarizing all the properties of the algebra $\mathcal{K}(E)$ we have considered:

Theorem 4.16. *$\mathcal{K}(E)$ is a cardinal algebra, with finite meets and (hence) countable joins, and with completely divisible elements coinciding with the aperiodic ones $\mathcal{K}^{\mathrm{ap}}(E) \subseteq \mathcal{K}(E)$ which form a cardinal subalgebra. We have canonical bijections (where $\bigvee^{\uparrow}$ denotes countable increasing joins)*

$$\begin{aligned}
\mathrm{INV}^*_E &\cong \{\textstyle\sum\text{-}\textit{homomorphisms } \mathcal{K}(E) \to \overline{\mathbb{R}^+}\} \\
&= \{(\textstyle\sum, \bigvee^{\uparrow}, \overline{\mathbb{R}^+})\text{-}\textit{homomorphisms } \mathcal{K}(E) \to \overline{\mathbb{R}^+}\}, \\
\mathrm{EINV}^*_E &\cong \{(\textstyle\sum, \wedge, \bigvee, \overline{\mathbb{R}^+})\text{-}\textit{homomorphisms } \mathcal{K}(E) \to \overline{\mathbb{R}^+}\}.
\end{aligned}$$

There are enough $(\sum, \wedge, \bigvee, \overline{\mathbb{R}^+})$-homomorphisms $\mathcal{K}(E) \to \overline{\mathbb{R}^+}$ to separate points: we have an $(\sum, \wedge, \bigvee, \overline{\mathbb{R}^+})$-embedding

$$\begin{aligned}
\eta : \mathcal{K}(E) &\longrightarrow \overline{\mathbb{R}^+}^{\mathrm{EINV}^\sigma_E} \\
\widetilde{A} &\longmapsto (\mu \mapsto \mu(A)).
\end{aligned}$$

In particular, $\mathcal{K}(E)$ obeys all Horn axioms in the operations $\sum, \wedge, \bigvee, \overline{\mathbb{R}^+}$ that hold in $\overline{\mathbb{R}^+}$. □

4.3 The algebra $\mathcal{L}(E)$

We next consider an algebra $\mathcal{L}(E)$ closely related to $\mathcal{K}(E)$. As before, here (X, E) is a compressible countable Borel equivalence relation.

Let $\mathcal{C}(X)$ denote the set of Borel maps $X \to \overline{\mathbb{R}^+} = [0, \infty]$; we think of $\alpha \in \mathcal{C}(X)$ as a "weighted Borel subset" of X. Given $\alpha, \beta \in \mathcal{C}(X)$, an E-**equidecomposition**

$$\phi : \alpha \sim_E \beta$$

is a Borel map $\phi : E \to \overline{\mathbb{R}^+}$ (where $E \subseteq X^2$) such that

$$\alpha(x) = \textstyle\sum_{yEx} \phi(x, y) =: \mathrm{dom}(\phi)(x), \qquad \beta(y) = \textstyle\sum_{xEy} \phi(x, y) =: \mathrm{rng}(\phi)(y);$$

α, β are E-**equidecomposable**, written $\alpha \sim_E \beta$, if there is some $\phi : \alpha \sim_E \beta$. We may think of such ϕ as a two-dimensional Borel "matrix" on each E-class C, whose row and column sums yield α, β respectively.

Our goal in this section is to show that $\mathcal{L}(E) := \mathcal{C}(X)/{\sim_E}$ is a cardinal algebra satisfying analogous properties to those in Theorem 4.16, and in fact is a "completion" of $\mathcal{K}(E)$ by adjoining divisors for indivisible elements; see Theorem 4.27(ii). This will require several preliminary steps: note that it is not even obvious that $\sim_E$ is an equivalence relation.

The following technical lemma says that the doubly infinitary version of axiom §4.1(C) (which holds in any cardinal algebra [Tar, 2.1]) holds "in a Borel way" in the cardinal algebra $\overline{\mathbb{R}^+}$, depicted as follows:

$$
\begin{array}{c|cccc}
 & v(0) & v(1) & v(2) & \cdots \\
\hline
u(0) & d(u,v)(0,0) & d(u,v)(0,1) & d(u,v)(0,2) & \cdots \\
u(1) & d(u,v)(1,0) & d(u,v)(1,1) & d(u,v)(1,2) & \cdots \\
u(2) & d(u,v)(2,0) & d(u,v)(2,1) & d(u,v)(2,2) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Lemma 4.17. *There is a Borel map*

$$d : \{(u,v) \in \overline{\mathbb{R}^+}^{\mathbb{N}} \times \overline{\mathbb{R}^+}^{\mathbb{N}} \mid \textstyle\sum_i u(i) = \sum_j v(j)\} \longrightarrow \overline{\mathbb{R}^+}^{\mathbb{N}^2}$$

such that for all u, v,

$$u(i) = \textstyle\sum_j d(u,v)(i,j), \qquad\qquad v(j) = \sum_i d(u,v)(i,j).$$

Proof. We define $d(u,v)$ by cases:

(I) Suppose $u(i) = v(j) = \infty$ for some i, j. Let i_0, j_0 be the least such. Put

$$d(u,v)(i,j) := \begin{cases} v(j) & \text{if } i = i_0, \\ u(i) & \text{if } j = j_0, \\ 0 & \text{otherwise.} \end{cases}$$

(II) Suppose $u(i) \in \{0, \infty\}$ for all i, while $v(j) < \infty$ for all j. If $u = v = 0$ then put $d(u,v) := 0$. Otherwise, from $\sum_i u(i) = \sum_j v(j)$ we have $\sum_j v(j) = \infty$. Let $f : \mathbb{N} \to \{i \mid u(i) = \infty\}$ take each value in the codomain infinitely often (clearly such f can be found in a Borel way from u). Put $k_0 := 0$, and inductively let $k_{l+1} > k_l$ be least such that $\sum_{j=k_l}^{k_{l+1}-1} v(j) > 1$ (using that $\sum_j v(j) = \infty$). Put

$$d(u,v)(i,j) := \begin{cases} v(j) & \text{if } u(i) = \infty \text{ and } \exists l\,(f(l) = i \ \&\ k_l \le j < k_{l+1}), \\ 0 & \text{otherwise.} \end{cases}$$

The picture (when $u(i) = \infty$ for $i = 0, 2$) is as follows:

$$
\begin{array}{c|cccccccc}
 & v(0) & v(1) & v(2) & v(3) & v(4) & v(5) & v(6) & \cdots \\
\hline
\infty & v(0) & v(1) & & & & v(5) & & \cdots \\
0 & & & & & & & & \\
\infty & & & v(2) & v(3) & v(4) & & v(6) & \cdots \\
\vdots & & & & & & & &
\end{array}
$$

In each row i with $u(i) = \infty$, we put enough consecutive values of v to achieve a sum > 1 before switching to a different i; since each such row is visited infinitely often, its sum will be ∞.

The case where $u(i) < \infty$ for all i and $v(j) \in \{0, \infty\}$ for all j is symmetric.

(III) Suppose $u(i), v(j) < \infty$ for all i, j. Define $d(u,v)$ as follows:

- Set $i_0 := j_0 := 0$.
- Inductively for each k, put

$$r_k := u(i_k) - \sum_{\substack{l<k \\ i_l = i_k}} d(u,v)(i_l, j_l), \qquad s_k := v(j_k) - \sum_{\substack{l<k \\ j_l = j_k}} d(u,v)(i_l, j_l).$$

  If $r_k \le s_k$, put $d(u,v)(i_k, j_k) := r_k$ and $(i_{k+1}, j_{k+1}) := (i_k + 1, j_k)$. Otherwise, put $d(u,v)(i_k, j_k) := s_k$ and $(i_{k+1}, j_{k+1}) := (i_k, j_k + 1)$.
- For all other (i,j) not equal to some (i_k, j_k), put $d(u,v)(i,j) := 0$.

The picture is as follows:

$$\begin{array}{c|cccccccc}
 & v(0) & v(1) & v(2) & v(3) & v(4) & v(5) & v(6) & \cdots \\
\hline
u(0) & r_0 & & & & & & & \\
u(1) & r_1 & & & & & & & \\
u(2) & s_2 & s_3 & s_4 & r_5 & & & & \\
u(3) & & & & s_6 & s_7 & r_8 & & \\
u(4) & & & & & & r_9 & & \\
u(5) & & & & & & s_{10} & \ddots & \\
\vdots & & & & & & & &
\end{array}$$

The marked entries (i,j) are $(i_0, j_0), (i_1, j_1), (i_2, j_2), \dots$. At each step (i_k, j_k), we place the largest possible value to "fill out" the remaining $u(i_k)$ or $v(j_k)$ (after subtracting the previous values in that row/column), and then move down or right in order to continue "filling out" the other value. Using that $\sum_i u(i) = \sum_j v(j)$, it is straightforward to check that this works. (Note that it is possible to have $\lim_{k\to\infty} i_k < \infty$ or $\lim_{k\to\infty} j_k < \infty$, if $\sum_i u(i) = \sum_j v(j) < \infty$.)

(IV) In the remaining case, u (say) takes infinite and nonzero finite values, while v only takes finite values. Let $u'(i) := u(i)$ if $u(i) < \infty$ and $u'(i) := 0$ otherwise, and put $u'' := u - u'$. Define v', v'' such that $v = v' + v''$, $\sum_i u'(i) = \sum_j v'(j)$, and $\sum_i u''(i) = \sum_j v''(j)$, as follows: if $\sum_i u'(i) < \infty$, then let k be least such that $\sum_{j \le k} v(j) > \sum_i u'(i)$, put

$$v'(j) := \begin{cases} v(j) & \text{if } j < k, \\ \sum_i u'(i) - \sum_{j<k} v(j) & \text{if } j = k, \\ 0 & \text{if } j > k, \end{cases}$$

and $v'' := v - v'$; otherwise find v', v'' with $v = v' + v''$ and $\sum_j v'(j) = \sum_j v''(j) = \infty$ using a procedure similar to case (II). We may then put $d(u,v) := d(u', v') + d(u'', v'')$, where the latter are computed using cases (II) and (III) above. $\square$

Proposition 4.18. *$\sim_E$ is an equivalence relation on $\mathcal{C}(X)$.*

Proof. Reflexivity is easy: for $\alpha \in \mathcal{C}(X)$, we have $\phi : \alpha \sim_E \alpha$ where $\phi(x,x) := \alpha(x)$ and $\phi(x,y) := 0$ for $x \ne y$. Symmetry is obvious. For transitivity, let $\alpha, \beta, \gamma \in \mathcal{C}(X)$ with $\phi : \alpha \sim_E \beta$ and $\psi : \beta \sim_E \gamma$. Let $(e_i^x)_{i \in \mathbb{N}}$ for each $x \in X$ be an injective enumeration of $[x]_E$, Borel in x. For $x \mathrel{E} y$, put

$$i_y^x := \text{the unique } i \text{ such that } e_i^x = y;$$

then clearly $(x, y) \mapsto i_y^x$ is a Borel map $E \to \mathbb{N}$. Define $\theta : E \to \overline{\mathbb{R}^+}$ by

$$\theta(x, z) := \sum_{yEx} d((\phi(e_i^y, y))_i, (\psi(y, e_j^y))_j)(i_x^y, i_z^y)$$

where d is given by Lemma 4.17. Then

$$\begin{aligned}
\operatorname{dom}(\theta)(x) &= \sum_{zEx} \sum_{yEx} d((\phi(e_i^y, y))_i, (\psi(y, e_j^y))_j)(i_x^y, i_z^y) \\
&= \sum_{yEx} \sum_{zEy} d((\phi(e_i^y, y))_i, (\psi(y, e_j^y))_j)(i_x^y, i_z^y) \\
&= \sum_{yEx} \sum_{k} d((\phi(e_i^y, y))_i, (\psi(y, e_j^y))_j)(i_x^y, k) \\
&= \sum_{yEx} \phi(e_{i_x^y}^y, y) \\
&= \sum_{yEx} \phi(x, y) \\
&= \alpha(x),
\end{aligned}$$

and similarly $\operatorname{rng}(\theta) = \gamma$. So $\theta : \alpha \sim_E \gamma$. $\square$

We define

$$\mathcal{L}(E) := \mathcal{C}(X)/{\sim_E}.$$

We equip $\mathcal{C}(X)$ with the pointwise countable addition operation, with respect to which $\sim_E$ is a congruence relation (since if $\phi_i : \alpha_i \sim_E \beta_i$ for each i then $\sum_i \phi_i : \sum_i \alpha_i \sim_E \sum_i \beta_i$). Thus, countable addition on $\mathcal{C}(X)$ descends to the quotient algebra $\mathcal{L}(E)$.

Lemma 4.19. *$\mathcal{C}(X)$ is a cardinal algebra.*

Proof. Axioms (A) and (B) are obvious. For (C), given $\alpha, \beta, \gamma_i \in \mathcal{C}(X)$ with $\alpha + \beta = \sum_i \gamma_i$, let

$$\begin{aligned}
\alpha_i(x) &:= d((\alpha(x), \beta(x), 0, 0, \dots), (\gamma_i(x))_i)(0, i), \\
\beta_i(x) &:= d((\alpha(x), \beta(x), 0, 0, \dots), (\gamma_i(x))_i)(1, i),
\end{aligned}$$

where d is given by Lemma 4.17; then $\alpha = \sum_i \alpha_i$, $\beta = \sum_i \beta_i$, and $\alpha_i + \beta_i = \gamma_i$ by the defining properties of d. For (D), given $\alpha_i, \beta_i \in \mathcal{C}(X)$ with $\alpha_i = \beta_i + \alpha_{i+1}$, put

$$\gamma(x) := \textstyle\bigwedge_i \alpha_i(x);$$

it is easily verified that $\alpha_i = \gamma + \sum_j \beta_{i+j}$. $\square$

Proposition 4.20. *$\mathcal{L}(E)$ is a cardinal algebra.*

Proof. By Lemma 4.19 and [Tar, 6.10], it suffices to check that $\sim_E$ is a **finitely refining** equivalence relation: that for $\alpha_1 + \alpha_2 = \alpha \sim_E \beta \in \mathcal{C}(X)$, there are $\beta_1, \beta_2 \in \mathcal{C}(X)$ such that $\beta = \beta_1 + \beta_2$, $\alpha_1 \sim_E \beta_1$, and $\alpha_2 \sim_E \beta_2$. Let $\phi : \alpha \sim_E \beta$. Let $(e_i^x)_{i \in \mathbb{N}}$ for each $x \in X$ be an injective enumeration of $[x]_E$, Borel in x. Define $\phi_1, \phi_2 : E \to \overline{\mathbb{R}^+}$ by

$$\begin{aligned}
\phi_1(x, e_i^x) &:= d((\alpha_1(x), \alpha_2(x), 0, 0, \dots), (\phi(x, e_j^x))_j)(0, i), \\
\phi_2(x, e_i^x) &:= d((\alpha_1(x), \alpha_2(x), 0, 0, \dots), (\phi(x, e_j^x))_j)(1, i),
\end{aligned}$$

where d is given by Lemma 4.17. Then the definition of d ensures that $\operatorname{dom}(\phi_1) = \alpha_1$, $\operatorname{dom}(\phi_2) = \alpha_2$, and $\phi_1 + \phi_2 = \phi$. Put $\beta_1 := \operatorname{rng}(\phi_1)$ and $\beta_2 = \operatorname{rng}(\phi_2)$. $\square$

For $\alpha \in \mathcal{C}(X)$ and a (not necessarily σ-finite) E-invariant measure $\mu \in \mathrm{INV}_E^*$, put

$$\mu(\alpha) := \int \alpha \, d\mu.$$

Lemma 4.21. *For $\alpha \sim_E \beta$, we have $\mu(\alpha) = \mu(\beta)$.*

Proof. By invariance of μ, we may define the measure M on E by

$$M(A) := \int |A_x| \, d\mu(x) = \int |A^y| \, d\mu(y)$$

where $A_x := \{y \mid (x, y) \in A\}$ and $A^y := \{x \mid (x, y) \in A\}$; see e.g., [KM, §16]. Now letting $\phi : \alpha \sim_E \beta$, we have $\mu(\alpha) = \int \phi \, dM$. Indeed, let $(e_i^x)_{i \in \mathbb{N}}$ for each $x \in X$ be an injective enumeration of $[x]_E$, Borel in x, and let $E_i := \{(x, e_i^x) \mid x \in X\}$, so that $E = \bigsqcup_i E_i$; then

$$\begin{aligned} \mu(\alpha) &= \int \sum_i \phi(x, e_i^x) \, d\mu(x) \\ &= \sum_i \int \phi(x, e_i^x) \, d\mu(x) \\ &= \sum_i \int_{(x,y) \in E_i} \phi(x, y) \, dM(x, y) \\ &= \int \phi(x, y) \, dM(x, y). \end{aligned}$$

Similarly, $\mu(\beta) = \int \phi \, dM$, whence $\mu(\alpha) = \mu(\beta)$. $\square$

It follows that each $\mu \in \mathrm{INV}_E^*$ defines a map $\mathcal{L}(E) \to \overline{\mathbb{R}^+}$, which is a $\sum$-homomorphism since integration is countably additive (by the monotone convergence theorem). Thus, analogously to Proposition 4.7, we have a map

$$\begin{aligned} \mathrm{INV}_E^* &\hookrightarrow \{\textstyle\sum\text{-homomorphisms } \mathcal{L}(E) \to \overline{\mathbb{R}^+}\} \\ \mu &\longmapsto (\widetilde{\alpha} \mapsto \textstyle\int \alpha \, d\mu) \end{aligned}$$

which is in fact a bijection (see Theorem 4.27(iii) below). We also have (analogously to Proposition 4.14) a $\sum$-homomorphism

$$\begin{aligned} \iota : \mathcal{L}(E) &\longrightarrow \overline{\mathbb{R}^+}^{\mathrm{EINV}_E^\sigma} \\ \widetilde{\alpha} &\longmapsto (\mu \mapsto \mu(\alpha)); \end{aligned}$$

we will show below that it preserves finite meets (hence countable joins) and is an embedding.

We now begin the comparison between $\mathcal{K}(E)$ and $\mathcal{L}(E)$. Given a Borel set $A \in \mathcal{B}(X)$, its characteristic function χ_A belongs to $\mathcal{C}(X)$; and if $A, B \in \mathcal{B}(X)$ and $f : A \sim_E B$ is an equidecomposition, then the characteristic function of the graph of f is an equidecomposition $\chi_A \sim_E \chi_B$. Thus $A \mapsto \chi_A$ descends to a map between the quotients

$$\begin{aligned} \chi : \mathcal{K}(E) &\longrightarrow \mathcal{L}(E) \\ \widetilde{A} &\longmapsto \widetilde{\chi}_A \end{aligned}$$

which clearly preserves countable sums, i.e., is a $\sum$-homomorphism.

Proposition 4.22. *χ is an order-embedding.*

(Note that this is not obvious: an equidecomposition $\chi_A \sim_E \chi_B$ need not be the characteristic function of the graph of an equidecomposition $A \sim_E B$.)

Proof. We have a commutative diagram

$$\begin{tikzcd} \mathcal{K}(E) \arrow[r, "\chi"] \arrow[dr, "\eta"'] & \mathcal{L}(E) \arrow[d, "\iota"] \\ & \overline{\mathbb{R}^+}^{\mathrm{EINV}^\sigma_E} \end{tikzcd}$$

where η is from Proposition 4.14 and ι is from above. Since η is an order-embedding and ι is order-preserving, it follows that χ is an order-embedding. □

For any $\alpha \in \mathcal{C}(X)$, put

$$\begin{aligned} \textstyle\sum_E \alpha : X/E &\longrightarrow \overline{\mathbb{R}^+} \\ C &\longmapsto \textstyle\sum_{x\in C} \alpha(x). \end{aligned}$$

We say that $\widetilde{\alpha} \in \mathcal{L}(E)$ (or $\alpha \in \mathcal{C}(X)$) is **($E$-)finite** if α has finite sum on every E-class (i.e., $\sum_E \alpha : X/E \to [0,\infty)$), and **($E$-)aperiodic** if α has sum 0 or ∞ on every E-class (i.e., $\sum_E \alpha : X/E \to \{0,\infty\}$). We let $\mathcal{L}^{\mathrm{fin}}(E), \mathcal{L}^{\mathrm{ap}}(E) \subseteq \mathcal{L}(E)$ denote the subsets of finite, respectively aperiodic, elements. Clearly $\chi(\mathcal{K}^{\mathrm{fin}}(E)) \subseteq \mathcal{L}^{\mathrm{fin}}(E)$ and $\chi(\mathcal{K}^{\mathrm{ap}}(E)) \subseteq \mathcal{L}^{\mathrm{ap}}(E)$.

Lemma 4.23. *Suppose $\alpha \in \mathcal{C}(X)$ is E-finite. Then α has E-smooth support, i.e., $E|\alpha^{-1}((0,\infty])$ is smooth. Moreover, for any $\beta \in \mathcal{C}(X)$ with $\sum_E \alpha \le \sum_E \beta$, we have $\widetilde{\alpha} \le \widetilde{\beta}$.*

Proof. If $\alpha(x) > 0$, then since $\sum_{yEx} \alpha(y) < \infty$, the set $\{y \mathrel{E} x \mid \alpha(y) > 1/n\}$ is finite for each n and nonempty for some n; this easily implies that α has smooth support.

For any $\alpha \in \mathcal{C}(X)$ with smooth support (not necessarily E-finite), letting $A \subseteq \alpha^{-1}((0,\infty])$ be a Borel transversal of $E|\alpha^{-1}((0,\infty])$, it is easily seen that $\alpha \sim_E \alpha'$, where $\alpha'(x) := \sum_{yEx} \alpha(y)$ for $x \in A$ and $\alpha'(x) := 0$ for $x \notin A$, so that α' is nonzero on at most one point per E-class. Now if $\sum_E \alpha \le \sum_E \beta$, then $\gamma := \beta|[\alpha^{-1}((0,\infty])]_E$ also has smooth support, and $\sum_E \alpha \le \sum_E \gamma$; letting $\gamma' \sim_E \gamma$ be nonzero on at most one point per E-class, we have $\sum_E \alpha' = \sum_E \alpha \le \sum_E \gamma = \sum_E \gamma'$, which easily implies $\widetilde{\alpha}' \le \widetilde{\gamma}'$, whence $\widetilde{\alpha} = \widetilde{\alpha}' \le \widetilde{\gamma}' = \widetilde{\gamma} \le \widetilde{\beta}$. □

Proposition 4.24. *We have an order-isomorphism*

$$\textstyle\sum_E : \mathcal{L}^{\mathrm{fin}}(E) \cong \{\textit{Borel maps } X/E \to [0,\infty) \textit{ with smooth support}\}$$

(where $f : X/E \to [0,\infty)$ having smooth support means that $E|\bigcup f^{-1}((0,\infty))$ is smooth).

Proof. By Lemma 4.23, $\sum_E$ is an order-embedding. For surjectivity, given Borel $f : X/E \to [0,\infty)$ with smooth support, letting A be a Borel transversal of $E|\bigcup f^{-1}((0,\infty))$, we have $f = \sum_E \alpha$ where $\alpha(x) := f([x]_E)$ for $x \in A$ and $\alpha(x) := 0$ for $x \notin A$. □

Lemma 4.25. *For every $\alpha \in \mathcal{C}(X)$ and E-complete section $Y \subseteq X$, there is a $\beta \in \mathcal{C}(X)$ supported on Y such that $\alpha \sim_E \beta$.*

Proof. Let $f : X \to Y$ be Borel with $f(x) \mathrel{E} x$. Put $\beta(y) := \sum_{x\in f^{-1}(y)} \alpha(x)$ for $y \in Y$ and $\beta(x) = 0$ for $x \notin Y$. Put $\phi(x, f(x)) := \alpha(x)$ and $\phi(x,y) := 0$ for $y \in [x]_E \setminus \{f(x)\}$. Then $\phi : \alpha \sim_E \beta$. □

Proposition 4.26. *$\chi : \mathcal{K}^{\mathrm{ap}}(E) \to \mathcal{L}^{\mathrm{ap}}(E)$ is an isomorphism.*

Proof. By Proposition 4.22, it remains to show surjectivity. Let $\alpha \in \mathcal{C}(X)$ with sum 0 or ∞ on each E-class; we must find an $A \in \mathcal{B}(X)$ such that $\alpha \sim_E \chi_A$.

First, we claim that we may assume that α only takes values in $\{0\} \cup \{2^{-n} \mid n \in \mathbb{N}\}$. By compressibility of E, we may assume that $(X, E) = (Y \times 2 \times \mathbb{N}, F \times I_2 \times I_{\mathbb{N}})$ for some (Y, F) (where $I_2, I_{\mathbb{N}}$ are the indiscrete equivalence relations $2 \times 2, \mathbb{N} \times \mathbb{N}$). By Lemma 4.25, we may assume that α is supported on $Y \times \{0\} \times \{0\}$. Now for each $y \in Y$, "spread out" $\alpha(y, 0, 0)$ according to its binary expansion along $\{y\} \times 2 \times \mathbb{N}$ to get $\beta \in \mathcal{C}(X)$. That is, if $\alpha(y, 0, 0) = \infty$ then put $\beta(y, i, j) := 1$ for all i, j; otherwise, let $\alpha(y, 0, 0) = a.b_1 b_2 b_3 \cdots$ be the binary expansion, put $\beta(y, 0, i) := 1$ for $i < a$ and $\beta(y, 0, i) := 0$ for $i \geq a$, and put $\beta(y, 1, 0) := 0$ and $\beta(y, 1, i) := b_i 2^{-i}$ for $i > 0$. Then clearly $\alpha \sim_E \beta$ and β only takes values in $\{0\} \cup \{2^{-n} \mid n \in \mathbb{N}\}$, so we may replace α by β.

Now, the union $A \subseteq X$ of those E-classes C such that $C \cap \alpha^{-1}(2^{-n})$ is nonempty finite for some n is clearly smooth, and so we easily have $\chi_A \sim_E \alpha|A$ (e.g., because $\chi_A \sim_E \beta \sim_E \alpha|A$ where β is ∞ on a single point in each $E|A$-class). So we may assume that for each n, $E|\alpha^{-1}(2^{-n})$ is aperiodic. For each n, using Lemma 4.4, let F_n be a finite Borel subequivalence relation of $E|\alpha^{-1}(2^{-n})$ with all classes of size 2^n, and let $A_n \subseteq \alpha^{-1}(2^{-n})$ be a Borel transversal of F_n. Then it is easily seen that $\chi_{A_n} \sim_E \alpha|\alpha^{-1}(2^{-n})$, whence putting $A := \bigcup_n A_n$, we have $\chi_A \sim_E \alpha$. □

Using Propositions 4.24 and 4.26, we now transfer most of the properties of $\mathcal{K}(E)$ to $\mathcal{L}(E)$, yielding the analogue of Theorem 4.16 for $\mathcal{L}(E)$:

Theorem 4.27. *(i) $\mathcal{L}(E)$ is a cardinal algebra with finite meets, countable joins, and real multiples of all elements.*

(ii) The embedding $\chi : \mathcal{K}(E) \hookrightarrow \mathcal{L}(E)$ preserves finite meets and countable joins, and restricts to an isomorphism $\mathcal{K}^{\mathrm{ap}}(E) \cong \mathcal{L}^{\mathrm{ap}}(E)$. Furthermore, the closure of the image of χ under real multiples and countable sums is all of $\mathcal{L}(E)$.

(iii) We have canonical bijections (where $\bigvee^{\uparrow}$ denotes countable increasing joins)

$$\begin{aligned}\mathrm{INV}_E^* &\cong \{\textstyle\sum\text{-}\textit{homomorphisms } \mathcal{L}(E) \to \overline{\mathbb{R}^+}\} \\ &= \{(\textstyle\sum, \bigvee^{\uparrow}, \overline{\mathbb{R}^+})\text{-}\textit{homomorphisms } \mathcal{L}(E) \to \overline{\mathbb{R}^+}\}, \\ \mathrm{EINV}_E^* &\cong \{(\textstyle\sum, \wedge, \bigvee, \overline{\mathbb{R}^+})\text{-}\textit{homomorphisms } \mathcal{L}(E) \to \overline{\mathbb{R}^+}\}\end{aligned}$$

compatible with those for $\mathcal{K}(E)$ from Theorem 4.16.

(iv) We have an $(\sum, \wedge, \bigvee, \overline{\mathbb{R}^+})$-embedding

$$\begin{aligned}\iota : \mathcal{L}(E) &\longrightarrow \overline{\mathbb{R}^+}^{\mathrm{EINV}_E^\sigma} \\ \widetilde{\alpha} &\longmapsto (\mu \mapsto \mu(\alpha))\end{aligned}$$

extending $\eta : \mathcal{K}(E) \to \overline{\mathbb{R}^+}^{\mathrm{EINV}_E^\sigma}$. In particular, $\mathcal{L}(E)$ obeys all Horn axioms in the operations $(\sum, \wedge, \bigvee, \overline{\mathbb{R}^+})$ that hold in $\overline{\mathbb{R}^+}$.

Proof. (i): $\mathcal{L}(E)$ is a cardinal algebra by Proposition 4.20, and clearly has real multiples inherited from $\mathcal{C}(X)$ (by Lemma 4.8); it remains to construct finite meets. The greatest element of $\mathcal{L}(E)$ is $\infty := \widetilde{\infty}$ where $\infty \in \mathcal{C}(X)$ is the constantly ∞ function. To compute the meet of $\widetilde{\alpha}, \widetilde{\beta}$: let $A, B \subseteq X$ be the unions of the E-classes on which α, β respectively have finite sum, so that

$\alpha|A, \beta|B$ are finite while $\alpha|(X \setminus A), \beta|(X \setminus B)$ are aperiodic. Using Proposition 4.24, the meet of $[\alpha|(A \cap B)]_{\sim_E}, [\beta|(A \cap B)]_{\sim_E}$ is given by $\widetilde{\gamma}$ where $\sum_E \gamma = \sum_E \alpha|(A \cap B) \wedge \sum_E \beta|(A \cap B)$. Using Lemma 4.23, the meet of $[\alpha|(A \setminus B)]_{\sim_E}, [\beta|(A \setminus B)]_{\sim_E}$ is the former (since β has infinite sum on every $E|(A \setminus B)$-class), and similarly the meet of $[\alpha|(B \setminus A)]_{\sim_E}, [\beta|(B \setminus A)]_{\sim_E}$ is the latter. Using Proposition 4.26, the meet of $[\alpha|(X \setminus (A \cup B))]_{\sim_E}, [\beta|(X \setminus (A \cup B))]_{\sim_E}$ may be computed in $\mathcal{K}(E)$. The sum of these four meets is $\widetilde{\alpha} \wedge \widetilde{\beta}$.

Note that binary joins in $\mathcal{L}(E)$ may be computed in a similar manner.

(ii): It is easily verified that the above procedure for computing binary meets and joins in $\mathcal{L}(E)$ agrees, when $\alpha = \chi_A$ and $\beta = \chi_B$, with the computation in $\mathcal{K}(E)$. Using that E is compressible, the greatest element $\infty = \widetilde{\infty} \in \mathcal{L}(E)$ is equal to $\widetilde{1} = \chi(\widetilde{X}) = \chi(\infty)$. So χ preserves finite meets and (by Lemma 4.8) countable joins. That χ restricts to an isomorphism $\mathcal{K}^{\mathrm{ap}}(E) \cong \mathcal{L}^{\mathrm{ap}}(E)$ is Proposition 4.26. For every $\alpha \in \mathcal{C}(X)$, we can write α as a countable real linear combination $\sum_i r_i \chi_{A_i}$ of characteristic functions of $A_i \in \mathcal{B}(X)$, whence $\widetilde{\alpha} = \sum_i r_i \widetilde{\chi}_{A_i}$; thus the image of χ generates $\mathcal{L}(E)$ under real multiples and countable sums.

(iii): By Lemma 4.21 and the succeeding remarks, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{INV}^*_E \arrow[r] \arrow[dr, "\cong"'] & \{\textstyle\sum\text{-homomorphisms } \mathcal{L}(E) \to \overline{\mathbb{R}^+}\} \arrow[d, "(-)|\mathcal{K}(E)"] \\
& \{\textstyle\sum\text{-homomorphisms } \mathcal{K}(E) \to \overline{\mathbb{R}^+}\}
\end{tikzcd}$$

The vertical map is injective, since $\mathcal{K}(E)$ generates $\mathcal{L}(E)$ under countable sums and real multiples (by (ii)) and $\sum$-homomorphisms $\mathcal{L}(E) \to \overline{\mathbb{R}^+}$ preserve real multiples (by Lemma 4.8). It follows that the horizontal map is bijective, yielding the first bijection in (iii). For the second, a $(\sum, \wedge, \bigvee, \overline{\mathbb{R}^+})$-homomorphism $\mathcal{L}(E) \to \overline{\mathbb{R}^+}$ still preserves finite meets when restricted to $\mathcal{K}(E)$ by (ii), hence corresponds to an ergodic measure by Lemma 4.10; and conversely, it is easily seen from the computation of binary meets in (i) that an ergodic measure $\mu : \mathcal{L}(E) \to \overline{\mathbb{R}^+}$ preserves binary meets (hence countable joins and real multiples, by Lemmas 4.8 and 4.9).

(iv): It suffices to show that for $\widetilde{\alpha} \not\leq \widetilde{\beta} \in \mathcal{L}(E)$, there is $\mu \in \mathrm{EINV}^\sigma_E$ such that $\mu(\alpha) > \mu(\beta)$. By restricting E, we may assume that each of $\widetilde{\alpha}, \widetilde{\beta}$ is either finite or aperiodic. If both are aperiodic, apply Proposition 4.26 and Lemma 4.13. If $\widetilde{\alpha}$ is finite, since $\widetilde{\alpha} \not\leq \widetilde{\beta}$, by Lemma 4.23 there is an E-class C such that $\sum_{x \in C} \alpha(x) > \sum_{x \in C} \beta(x)$; let μ be an atomic measure on C. If $\widetilde{\alpha}$ is aperiodic while $\widetilde{\beta}$ is finite, since $\widetilde{\alpha} \not\leq \widetilde{\beta}$, there is an E-class C such that $\sum_{x \in C} \alpha(x) = \infty$; let μ be an atomic measure on C. $\square$

4.4 The duality theorem

Regarding $\mathcal{K}(E)$ and $\mathcal{L}(E)$ as algebras under the operations $\sum, \wedge, \bigvee, \overline{\mathbb{R}^+}$, Proposition 4.14 and Theorem 4.27(iv) say that these algebras admit enough homomorphisms to $\overline{\mathbb{R}^+}$ to separate points. It is thus natural to regard the space of all such homomorphisms, i.e., EINV^*_E, as the "dual" or "spectrum" of the algebra, and to ask whether we may recover the algebra as an "algebra of functions" on the space EINV^*_E equipped with suitable structure.

We give in this section a positive answer, subject to some technical caveats. First, since every element of $\overline{\mathbb{R}^+}$ is completely divisible, so will be every element of an "algebra of functions" with values in $\overline{\mathbb{R}^+}$; thus we can only hope to recover $\mathcal{L}(E)$, not $\mathcal{K}(E)$. Second, as mentioned previously, non-σ-finite measures in EINV^*_E are not so tractable; thus we will consider instead the subspace

$\mathrm{EINV}_E^\sigma \subseteq \mathrm{EINV}_E^*$ as the "dual" of $\mathcal{L}(E)$. Finally, there is the question of what kind of "space" EINV_E^σ is. It is not enough to regard it as a (nonstandard) Borel space, due to Remark 4.30 below.

A **σ-topology** on a set X is a collection of subsets of X (called **σ-open**), closed under countable unions and finite intersections; a **σ-topological space** is a set equipped with a σ-topology. A **σ-continuous map** between σ-topological spaces is a map such that the preimage of every σ-open set is σ-open. The notions of **product σ-topology** and **subspace σ-topology** are defined in the usual manner (i.e., the smallest σ-topology making the projection maps, respectively the inclusion, σ-continuous). Note that a σ-topology generates both a topology (by closing under arbitrary unions) and a σ-algebra (by closing under complements and countable unions), hence contains more information than both a topology and a Borel structure. In particular, every σ-continuous map is Borel with respect to the induced Borel structures.

We equip $\overline{\mathbb{R}^+} = [0,\infty]$ with the σ-topology whose nontrivial σ-open sets are $(r,\infty]$ for $r \in (0,\infty)$. We view EINV_E^σ as a σ-topological subspace of the product space $\overline{\mathbb{R}^+}^{\mathcal{B}(X)}$; thus, the σ-topology on EINV_E^σ is generated by the subbasic σ-open sets

$$U_{A,r} := \{\mu \in \mathrm{EINV}_E^\sigma \mid \mu(A) > r\}$$

for $A \in \mathcal{B}(X)$ and $r \in (0,\infty)$. In addition, we also equip EINV_E^σ with the multiplication action of the multiplicative monoid $(0,\infty)$.

Lemma 4.28. *For every $\widetilde{\alpha} \in \mathcal{L}(E)$, the map*

$$\begin{aligned} \iota(\widetilde{\alpha}) : \mathrm{EINV}_E^\sigma &\longrightarrow \overline{\mathbb{R}^+} \\ \mu &\longmapsto \mu(\alpha) \end{aligned}$$

is σ-continuous and $(0,\infty)$-equivariant.

Proof. $(0,\infty)$-equivariance is obvious. For σ-continuity, write $\alpha = \sum_i r_i \chi_{A_i}$ as a countable linear combination of positive real multiples of characteristic functions of $A_i \in \mathcal{B}(X)$, so that $\mu(\alpha) = \sum_i r_i \mu(A_i)$; σ-continuity of $\iota(\widetilde{\alpha})$ thus follows from σ-continuity of $r_i \cdot (-) : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$, which is obvious, and of $\sum : \overline{\mathbb{R}^+}^{\mathbb{N}} \to \overline{\mathbb{R}^+}$, which is straightforward (since $\sum_i r_i > s \iff \exists n\, \exists i_1, \dots, i_n \in \mathbb{N}\, \exists q_1, \dots, q_n \in \mathbb{Q}\, (q_1 + \cdots + q_n > s \;\&\; r_{i_1} > q_1 \;\&\; \cdots \;\&\; r_{i_n} > q_n)$). $\square$

In other words, the map $\iota : \mathcal{L}(E) \to \overline{\mathbb{R}^+}^{\mathrm{EINV}_E^\sigma}$ from Theorem 4.27(iv) lands in the subalgebra of σ-continuous, $(0,\infty)$-equivariant maps $\mathrm{EINV}_E^\sigma \to \overline{\mathbb{R}^+}$. We now have the following duality theorem:

Theorem 4.29. *The map*

$$\iota : \mathcal{L}(E) \longrightarrow \{\sigma\text{-continuous, } (0,\infty)\text{-equivariant maps } \mathrm{EINV}_E^\sigma \to \overline{\mathbb{R}^+}\}$$

is an $(\sum, \wedge, \bigvee, \overline{\mathbb{R}^+})$-isomorphism.

Remark 4.30. Theorem 4.29 fails if we replace "σ-continuous" with "Borel" (with the nonstandard Borel structure on EINV_E^σ generated by the maps $\mu \mapsto \mu(A)$ for $A \in \mathcal{B}(X)$). Indeed, let $(X, E) = (\mathbb{R}, E_v)$ where E_v is the Vitali equivalence relation, and let $f : \mathrm{EINV}_E^\sigma \to \overline{\mathbb{R}^+}$ be given by $f(\mu) := 0$ if $\mu((0,1)) < \infty$, else $f(\mu) := \infty$. Clearly f is Borel and $(0,\infty)$-equivariant. Suppose we had $f = \iota(\widetilde{\alpha})$ for some $\widetilde{\alpha} \in \mathcal{L}(E)$. For atomic $\mu \in \mathrm{EINV}_E^\sigma$, we have $\mu((0,1)) = \infty$, whence $\infty = f(\mu) = \iota(\widetilde{\alpha})(\mu) = \int \alpha\, d\mu$; so α must be nonzero on every coset of $\mathbb{Q}$. But then for Lebesgue measure μ, we have $\iota(\widetilde{\alpha})(\mu) = \int \alpha\, d\mu > 0 = f(\mu)$, a contradiction.

Recall from Proposition 3.6 that for a Borel set $A \in \mathcal{B}(X)$ and an $E|A$-ergodic invariant probability measure $\mu \in \mathrm{EINV}_{E|A}$,

$$[\mu]_E \in \mathrm{EINV}_E^\sigma$$

denotes the unique E-ergodic invariant extension of μ. From the definition of $[\mu]_E$ in Proposition 3.6, we clearly have

Lemma 4.31. *The map $[-]_E : \mathrm{EINV}_{E|A} \to \mathrm{EINV}_E^\sigma$ is Borel (where $\mathrm{EINV}_{E|A}$ has the usual standard Borel structure).* ☐

Proof of Theorem 4.29. By Theorem 4.27(iv), it remains only to show that ι is surjective. Let $f : \mathrm{EINV}_E^\sigma \to \overline{\mathbb{R}^+}$ be σ-continuous and $(0,\infty)$-equivariant. So for each rational $r \in (0,\infty)$, $f^{-1}((r,\infty]) \subseteq \mathrm{EINV}_E^\sigma$ is a countable union of finite intersections of subbasic σ-open sets $U_{A,r}$; let $\mathcal{A}$ denote all countably many $A \in \mathcal{B}(X)$ involved in these expressions (for all r). In particular, $f(\mu)$ can only depend on the values of $\mu(A)$ for $A \in \mathcal{A}$.

Let Γ be a countable group with a Borel action on X inducing E. Equip X with a Polish topology given by Theorem 3.2, making every $A \in \mathcal{A}$ open and making the Γ-action continuous, so that we have a topological ergodic decomposition $p : X \twoheadrightarrow X /\!\!/ E$ which is also a measure-theoretic ergodic decomposition. Let the Borel sets $R \subseteq X /\!\!/ E$ and $S \subseteq X$ and the Borel map $(C \mapsto \mu_C) : R \to \mathrm{EINV}_{E|S}$ be given by Theorem 3.2.

For $C \in X /\!\!/ E \setminus R$, let $\nu_C \in \mathrm{EINV}_E^\sigma$ be an E-ergodic invariant σ-finite measure supported on C, such that $C \mapsto \nu_C$ is Borel; for example, compose a Borel section of $p : X \twoheadrightarrow X /\!\!/ E$ (which exists by Proposition 2.1) with a Borel map $X \to \mathrm{EINV}_E^\sigma$ taking a point x to an atomic measure on $[x]_E$. Note that by definition of R, each ν_C is necessarily totally singular.

Note that for any totally singular $\mu \in \mathrm{EINV}_E^\sigma$, we have $\mu(A) = 2\mu(A)$ for all $A \in \mathcal{A}$, whence $f(\mu) = f(2\mu) = 2f(\mu)$ since f is $(0,\infty)$-equivariant, whence $f(\mu) \in \{0,\infty\}$.

Now define $\alpha \in \mathcal{C}(X)$ by

$$\alpha(x) := \begin{cases} f([\mu_{[\![x]\!]_E}]_E) & \text{if } x \in S, \\ 0 & \text{if } x \in p^{-1}(R) \setminus S, \\ f(\nu_{[\![x]\!]_E}) & \text{if } x \notin p^{-1}(R). \end{cases}$$

By Lemma 4.31, this is Borel. We claim that $f = \iota(\widetilde{\alpha})$, i.e., $f(\mu) = \iota(\widetilde{\alpha})(\mu) = \int \alpha \, d\mu$ for all $\mu \in \mathrm{EINV}_E^\sigma$. Note that by ergodicity, each μ is supported on some $C \in X /\!\!/ E$.

- Suppose μ is non-totally-singular, hence supported on some $C \in R$. Then (by Theorem 3.2(ii)) $\mu = r[\mu_C]_E$ for some $r \in (0,\infty)$. We have

$$\begin{aligned} \textstyle\int \alpha \, d\mu &= r \textstyle\int \alpha \, d[\mu_C]_E \\ &= r f([\mu_C]_E)[\mu_C]_E(S \cap C) && \text{by definition of } \alpha \\ &= f(r[\mu_C]_E) && \text{by } (0,\infty)\text{-equivariance of } f \\ &= f(\mu), \end{aligned}$$

  as desired.

- Suppose μ is totally singular and supported on some $C \in X /\!\!/ E \setminus R$. Since μ is totally singular, $f(\mu) \in \{0, \infty\}$. For each $A \in \mathcal{A}$, since $A \subseteq X$ is open (and so meets C iff it meets every E-class in C), we have $\mu(A) = \nu_C(A) = \infty$ if $A \cap C \neq \varnothing$, otherwise $\mu(A) = \nu_C(A) = 0$; thus $f(\mu) = f(\nu_C) \in \{0, \infty\}$. So by definition of α, $\int \alpha \, d\mu = f(\nu_C) \cdot \mu(C) = f(\nu_C) \cdot \infty = f(\mu)$.

- Finally, suppose μ is totally singular and supported on some $C \in R$. We again have $f(\mu) \in \{0, \infty\}$, while by definition of α,

$$\int \alpha \, d\mu = f([\mu_C])_E \cdot \mu(S) = \begin{cases} 0 & \text{if } f([\mu_C]_E) = 0, \\ \infty & \text{if } f([\mu_C]_E) > 0. \end{cases}$$

So we must show that $f(\mu) > 0$ iff $f([\mu_C]_E) > 0$. By definition of $\mathcal{A}$, we may write

$$f^{-1}((0, \infty]) = \bigcup_{i \in I} \bigcap_{j \in J_i} U_{A_{ij}, r_{ij}}$$

where I is countable, each J_i is finite, each $A_{ij} \in \mathcal{A}$, and each $r_{ij} \in (0, \infty)$. If $[\mu_C]_E \in U_{A_{ij}, r_{ij}}$ for some i, j, i.e., $[\mu_C]_E(A_{ij}) > r_{ij}$, then A_{ij} must meet C, whence $\mu(C) = \infty > r_{ij}$, whence $\mu \in U_{A_{ij}, r_{ij}}$; thus $f([\mu_C]_E) > 0$ implies $f(\mu) > 0$. Conversely, if $f(\mu) > 0$, then $\mu \in \bigcap_{j \in J_i} U_{A_{ij}, r_{ij}}$ for some i, whence each A_{ij} for $j \in J_i$ must meet C, whence $[\mu_C]_E(A_{ij}) > 0$ for each $j \in J_i$; since J_i is finite, there is some $r \in (0, \infty)$ such that $r[\mu_C]_E(A_{ij}) > r_{ij}$ for each $j \in J_i$, whence $r[\mu_C]_E \in \bigcap_{j \in J_i} U_{A_{ij}, r_{ij}}$, whence $rf([\mu_C]_E) = f(r[\mu_C]_E) > 0$, whence $f([\mu_C]_E) > 0$. Thus $f(\mu) > 0$ iff $f([\mu_C]_E) > 0$, as desired. □

4.5 Other algebraic operations

We conclude by briefly considering the (non)existence of other canonical algebraic operations on $\mathcal{K}(E)$ and $\mathcal{L}(E)$.

Remark 4.32. Countable decreasing meets do not exist in general; indeed, a countable decreasing sequence in $\mathcal{K}(E)$ need not have a meet in $\mathcal{L}(E)$. Consider the Vitali equivalence relation $(\mathbb{R}, E_v)$. For each $n \geq 1$, let $A_n := [0, 1/n) \in \mathcal{B}(\mathbb{R})$. Then for Lebesgue measure μ, we have $\mu(A_n) = 1/n$; so if $\bigwedge_n \widetilde{\chi}_{A_n}$ existed, we must have $\mu(\bigwedge_n \widetilde{\chi}_{A_n}) = 0$. But for each $x \in \mathbb{R}$, we clearly have $\mathbb{Q} + x \preceq_E A_n$ for all n, whence $\widetilde{\chi}_{\mathbb{Q}+x} \leq \bigwedge_n \widetilde{\chi}_{A_n}$. So $\bigwedge_n \widetilde{\chi}_{A_n}$ must be represented by a function in $\mathcal{C}(X)$ which is nonzero on each E_v-class, whence $\mu(\bigwedge_n \widetilde{\chi}_{A_n}) > 0$, a contradiction.

Remark 4.33. Given $\widetilde{A} \leq \widetilde{B} \in \mathcal{K}(E)$, there is by definition some $\widetilde{C}$ such that $\widetilde{A} + \widetilde{C} = \widetilde{B}$. However, there does not seem to be a canonical choice of such a $\widetilde{C}$ that works for all $\widetilde{A}, \widetilde{B}$. In other words, there does not seem to be a canonical way of defining a partial "difference" operation $\widetilde{B} - \widetilde{A}$ for all $\widetilde{A} \leq \widetilde{B}$, such that $\widetilde{A} + (\widetilde{B} - \widetilde{A}) = \widetilde{B}$. The same is true in $\mathcal{L}(E)$.

In particular, there is not always a smallest or largest $\widetilde{C}$. Consider the tail equivalence relation $(2^{\mathbb{N}}, E_t)$ and a Borel complete section $A \subseteq 2^{\mathbb{N}}$ with $E_t | A \cong E_0$ (where E_0 is equality modulo finite on $2^{\mathbb{N}}$; the existence of such a complete section A is standard, following for example from [DJK, 9.3]). Then there is no smallest $\widetilde{\alpha} \in \mathcal{L}(E_t)$ (or $\widetilde{A}' \in \mathcal{K}(E_t)$) such that $\widetilde{\chi}_A + \widetilde{\alpha} = \infty$ (or $\widetilde{A} + \widetilde{A}' = \infty$): the union B of the E_t-classes on which α is zero must be such that $E_t|(A \cap B)$ is compressible, or else we could not have $\widetilde{\chi}_A + \widetilde{\alpha} = \infty$; and given such α, we can always make α zero on a single class outside B to get a strictly smaller $\widetilde{\beta} < \widetilde{\alpha}$ with $\widetilde{\chi}_A + \widetilde{\beta} = \infty$. And there is no largest $\widetilde{\alpha} \in \mathcal{L}(E_t)$ (or $\widetilde{A}' \in \mathcal{K}(E_t)$) such that $\widetilde{\chi}_A + \widetilde{\alpha} = \widetilde{\chi}_A$ (or $\widetilde{A} + \widetilde{A}' = \widetilde{A}$): such α are precisely those for which the union C of the E_t-classes on which α is nonzero has $E_t|(A \cap C)$ compressible.

A Inverse limits of quasi-Polish spaces

We prove here some technical results regarding inverse limits of quasi-Polish spaces.

Proposition A.1. *Let $X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \cdots$ be a sequence of continuous maps between quasi-Polish spaces, such that each f_i has dense image. Then each projection map $p_i : \varprojlim_i X_i \to X_i$ has dense image.*

Proof. The **lax colimit** of the sequence is the space

$$X' := \bigsqcup_i X_i \sqcup \varprojlim_i X_i,$$

with topology given by the basic open sets

$$\Uparrow U := \bigcup_{j \geq i} (f_i \circ \cdots \circ f_{j-1})^{-1}(U) \cup p_i^{-1}(U) \qquad \text{for } i \in \mathbb{N} \text{ and open } U \subseteq X_i.$$

Lemma A.2. *X' is quasi-Polish.*

Proof. Given a quasi-Polish space X, the space

$$X_\perp := X \sqcup \{\perp\}$$

with open sets consisting of open sets in X together with all of $X_\perp$ is easily seen to be quasi-Polish; see e.g., [Ch, 3.4]. We claim that we have a homeomorphism

$$\begin{aligned} X' &\longrightarrow \{(x_i)_i \in \textstyle\prod_i (X_i)_\perp \mid x_0 \in X_0 \ \& \ \forall i \, (x_{i+1} \in X_{i+1} \implies x_i = f_i(x_{i+1}))\} \\ x \in X_i &\longmapsto ((f_0 \circ \cdots \circ f_{i-1})(x), \dots, f_{i-1}(x), x, \perp, \perp, \dots), \\ y \in \varprojlim_i X_i &\longmapsto (p_0(y), p_1(y), \dots). \end{aligned}$$

This is easily seen to be a bijection. A subbasic open set in $\prod_i (X_i)_\perp$ consists of all $(x_i)_i$ such that $x_i \in U$, for some $i \in \mathbb{N}$ and open $U \subseteq X_i$; the preimage of such a set is precisely $\Uparrow U$. ☐

Each $\Uparrow X_i = \bigcup_{j \geq i} X_j \cup \varprojlim_i X_i \subseteq X'$ is dense: given nonempty basic open $\Uparrow U \subseteq X'$, with $U \subseteq X_j$ open, from the definition of $\Uparrow U$ we must have $U \neq \varnothing$; then since $f_j, f_{j+1}, \dots$ have dense image, we must have $f_j^{-1}(U), f_{j+1}^{-1}(f_j^{-1}(U)), \dots \neq \varnothing$, whence for any $k \geq i, j$, there is some $x \in (f_j \circ \cdots \circ f_{k-1})^{-1}(U)$, whence $x \in \Uparrow U \cap \Uparrow X_i$. It follows by Baire category that $\varprojlim_i X_i = \bigcap_i \Uparrow X_i \subseteq X'$ is dense. Thus, for any nonempty open $U \subseteq X_i$, we have $p_i^{-1}(U) = \Uparrow U \cap \varprojlim_i X_i \neq \varnothing$, i.e., p_i has dense image. ☐

Proposition A.3. *Let*

$$\begin{tikzcd} X_0 \arrow[d, "h_0"'] & X_1 \arrow[l, "f_0"'] \arrow[d, "h_1"'] & X_2 \arrow[l, "f_1"'] \arrow[d, "h_2"'] & \cdots \arrow[l, "f_2"'] & X = \varprojlim_i X_i \arrow[l, "p_i"'] \arrow[d, "h = \varprojlim_i h_i"] \\ Y_0 & Y_1 \arrow[l, "g_0"] & Y_2 \arrow[l, "g_1"] & \cdots \arrow[l, "g_2"] & Y = \varprojlim_i Y_i \arrow[l, "q_i"] \end{tikzcd}$$

be a commutative diagram of quasi-Polish spaces, where the h_i are open, the $p_i : X \to X_i$ and $q_i : Y \to Y_i$ are the limit projections, and for all i and open $U \subseteq X_i$ we have the following ***Beck–Chevalley condition:***

$$h_{i+1}(f_i^{-1}(U)) = g_i^{-1}(h_i(U))$$

(the $\subseteq$ containment is automatic). Then h is open, and satisfies for open $U \subseteq X_i$

$$h(p_i^{-1}(U)) = q_i^{-1}(h_i(U)). \tag{$*$}$$

In particular, if h_0 is surjective, then so is h.

Proof. It is enough to prove that if h_0 is surjective, then so is h. Indeed, granting this, we may prove $(*)$ (in which only $\supseteq$ is nontrivial) by truncating part of the diagram if necessary and assuming $i = 0$, then replacing X_0 with U, Y_0 with $h_0(U)$, X_1 with $f_0^{-1}(U)$, Y_1 with $h_1(f_0^{-1}(U)) = g_0^{-1}(h_0(U))$ (by Beck–Chevalley), X_2 with $f_1^{-1}(f_0^{-1}(U))$, etc., so that X becomes $p_0^{-1}(U)$ and Y becomes $q_0^{-1}(h_0(U))$. Since $\varprojlim_i X_i$ has a basis of open sets consisting of $p_i^{-1}(U)$ for open $U \subseteq X_i$, it follows from $(*)$ that h is open.

So assume h_0 is surjective, and let $y \in Y$. Then $h_0^{-1}(q_0(y)) \neq \varnothing$. For each i, the restriction $f_i : h_{i+1}^{-1}(q_{i+1}(y)) \to h_i^{-1}(q_i(y))$ has dense image, since for open $U \subseteq X_i$ such that $U \cap h_i^{-1}(q_i(y)) \neq \varnothing$, i.e., $g_i(q_{i+1}(y)) = q_i(y) \in h_i(U)$, we have $q_{i+1}(y) \in g_i^{-1}(h_i(U)) = h_{i+1}(f_i^{-1}(U))$ by Beck–Chevalley, i.e., $f_i^{-1}(U) \cap h_{i+1}^{-1}(q_{i+1}(y)) \neq \varnothing$. Thus by Proposition A.1, $h^{-1}(y) = \varprojlim_i h_i^{-1}(q_i(y)) \neq \varnothing$. $\square$

Remark A.4. Simpler but more abstract proofs of these results may be given using the correspondence between quasi-Polish spaces and countably presented locales [Hec].

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