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Contravariant Connections on Poisson Manifolds

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Abstract. We consider the geometric concept of a contravariant connection on a Poisson manifold. This leads to the operational notion of a contravariant derivative, previously introduced by I. Vaisman. We sketch several key geometric concepts which should allow one to study both local and global properties of Poisson manifolds.

1. Introduction

Let M be a Poisson manifold, and suppose we require the existence of a linear affine connection on M, compatible with the Poisson bracket. Since parallel transport preserves the rank of the Poisson tensor, we see that the Poisson manifold must be regular in order for such connection to exist. Therefore, the usual notion of an affine connection is not appropriate for the study of Poisson manifolds since the most interesting examples of Poisson manifolds are non-regular. In fact, since for non-regular Poisson manifolds the symplectic foliation is singular and the dimension of the leaves vary, one can only hope to compare tangent spaces at different points of the *same* symplectic leaf.

One possible way around this difficulty would be to use families of connections parameterized by the leaves. However, there are examples showing that the symplectic foliation can be wild, so the space of leaves will not be a nice space, and hence not easy to parameterized. We propose a more efficient and direct approach, through the notion of a contravariant connection, a concept that mimics the usual notion of a covariant connection, for the case of Poisson manifolds.

In spite of its formal similarities with covariant geometry, there are striking differences in this contravariant geometry. For example, the holonomy of a point may be non-trivial, contravariant derivatives do not have functorial properties (in general, they cannot be pulled back or pushed forward), etc.

However, just like in ordinary affine geometry, linear contravariant connections are useful to study geometric properties of Poisson manifolds. Among its applications we mention:

a) Local properties of Poisson manifolds: local equivalence, linearization of Poisson tensors;
b) Global properties of Poisson manifolds: Poisson holonomy, Poisson and hamiltonian symmetric spaces, Poisson cohomology;
c) Quantization of Poisson manifolds: geometric and deformation quantization;

Some of these applications will be studied in a upcoming paper [4]. In this note we give a flavor of this new type of geometry. Finally, we also point out that contravariant connections can also be studied in the more general context of Lie algebroids.

2. Contravariant Connections

Suppose we are given a principal bundle over a manifold M:

$$\begin{array}{c} P \circlearrowleft G \\ \Big\downarrow p \\ M \end{array}$$

then a covariant connection Γ on this principal bundle is defined by a G-invariant horizontal distribution $u \mapsto H_u$ in P. Given a connection Γ, we have its horizontal lift: $h(u,v) \in T_uP$ is the horizontal lift of the vector v tangent to M at x, to the point u in the fiber over x. Conversely, the horizontal lift h defines the horizontal distribution $H_u = \left\{h(u,v) : v \in T_{p(u)}M\right\}$, so to give h is a completely equivalent way of defining the connection. We shall define a contravariant connection on a principal bundle over a Poisson manifold by defining analogously the horizontal lift of tangent covectors.

To formalize this notion, observe that h is defined precisely for pairs (u,v) in p^*TM, the pullback bundle by p of the tangent bundle over M. Denote by $\widehat{p}: p^*TM \to TM$ the induced bundle map so we have the commutative diagram

$$\begin{array}{ccc} p^*TM & \xrightarrow{\widehat{p}} & TM \\ \Big\downarrow \widehat{\pi} & & \Big\downarrow \pi \\ P & \xrightarrow[p]{} & M \end{array}$$

Then we can define a covariant connection to be a bundle map $h: p^*TM \to TP$, such that:

(CI) h is G-invariant: $h(ua,v) = (R_a)_* h(u,v)$, for all $a \in G$;

(CII) The following diagram commutes:

$$\begin{array}{ccc} p^*TM & \xrightarrow{h} & TP \\ \Big\downarrow \widehat{p} & & \Big\downarrow p_* \\ TM & \xrightarrow[\mathrm{id}]{} & TM \end{array}$$

Condition (CII) guarantees that h is "horizontal".

Assume now that M is a Poisson manifold. As is often the case in Poisson geometry, the cotangent bundle plays the role of the tangent bundle, so we replace in the diagrams above TM by T^*M. Thus we are lead to the notion of a *contravariant connection* on a Poisson manifold: this is a bundle map $h : p^*T^*M \to TP$, such that:

(CI*) h is G-invariant: $h(ua, \alpha) = (R_a)_* h(u, \alpha)$, for all $a \in G$;

(CII*) The following diagram commutes:

$$\begin{array}{ccc} p^*T^*M & \xrightarrow{h} & TP \\ \downarrow{\scriptstyle \widehat{p}} & & \downarrow{\scriptstyle p_*} \\ T^*M & \xrightarrow[\#]{} & TM \end{array}$$

(where $\# : T^*M \to TM$ is the bundle map induced by the Poisson tensor Π).

Given a point x in M and a covector $\alpha \in T^*_x M$, the vector $h(u, \alpha) \in T_u P$ will be called the *horizontal lift* of α to the point u in the fiber over x.

There are two important features that distinguish the contravariant case from the covariant case:

i) For a contravariant connection, the distribution $H_u = \{h(u, v) : v \in T_{p(u)} M\}$ does not determine the connection;

ii) Tangent vectors in TP do not split into a sum of horizontal and vertical vectors;

However, we claim that one can still define the analogue of the usual concepts of affine geometry: parallelism, curvature, holonomy, etc. Here we limit ourselves to explain briefly how one defines parallel transport (more details will appear in [4]).

3. Parallelism

Parallel displacement of fibers can be defined along curves *lying* on a symplectic leaf of M.

If $\gamma : [0, 1] \to M$ is a smooth curve lying on a symplectic leaf S, then γ is also smooth as map $\gamma : [0, 1] \to S$. This follows from the existence of "canonical coordinates" for M as given by the generalized Darboux's theorem (see [6], thm. 2.1). Also, by the same theorem, we can choose (not uniquely) a smooth family $t \mapsto \alpha(t) \in T^*M$ of covectors such that $\#\alpha(t) = \dot{\gamma}(t)$. Following [1], we shall call the pair $(\gamma(t), \alpha(t))$ a *cotangent curve*.

Proposition 3.1. *Let $(\gamma(t), \alpha(t))$ be a cotangent curve. For any u_0 in P with $p(u_0) = \gamma(0)$ there exists a unique horizontal lift $\tilde{\gamma} : [0, 1] \to P$, which satisfies the system*

$$\begin{cases} \dot{\tilde{\gamma}}(t) = h(\tilde{\gamma}(t), \alpha(t)), \\ \tilde{\gamma}(0) = u_0. \end{cases} \tag{3.1}$$

Proof. By standard results from the theory of o.d.e.'s with time dependent coefficients, system (3.1) has a unique maximal solution. We claim that this solution exists for all $t \in [0, 1]$.

By local triviality of the bundle we can find a curve $\bar{\gamma} : [0,1] \to P$ with $\bar{\gamma}(0) = u_0$ and $p(\bar{\gamma}(t)) = \gamma(t)$. We look for a curve $a(t) \in G$, such that $\tilde{\gamma}(t) = \bar{\gamma}(t)a(t)$ satisfies (3.1). Differentiating, we have

$$\dot{\tilde{\gamma}}(t) = \dot{\bar{\gamma}}(t)a(t) + \bar{\gamma}(t)\dot{a}(t).$$

We therefore require $a(t)$ to satisfy the equation

$$\dot{\bar{\gamma}}(t)a(t) + \bar{\gamma}(t)\dot{a}(t) = h(\bar{\gamma}(t)a(t), \alpha(t)),$$

or, equivalently,

$$\bar{\gamma}(t)\dot{a}(t)a^{-1}(t) = h(\bar{\gamma}(t), \alpha(t)) - \dot{\bar{\gamma}}(t).$$

The right hand side of this equation belongs to $G_{\bar{\gamma}(t)}$ since

$$p_*(h(\bar{\gamma}(t), \alpha(t)) - \dot{\bar{\gamma}}(t)) = \#\alpha(t) - \frac{d}{dt}p(\bar{\gamma}(t)) = \#\alpha(t) - \dot{\gamma}(t) = 0.$$

Therefore, there exists some curve $A(t) : [0,1] \to \mathfrak{g}$ such that

$$\bar{\gamma}(t)\dot{a}(t)a^{-1}(t) = \bar{\gamma}(t)A(t).$$

Since the initial value problem

$$\dot{a}(t)a^{-1}(t) = A(t), \qquad a(0) = e,$$

always has a solution, defined wherever $A(t)$ is defined, our claim follows. □

Using the proposition we can define parallel displacement of the fibers along a cotangent curve $(\gamma(t), \alpha(t))$ in the usual form: if $u_0 \in p^{-1}(\gamma(0))$ we define $\tau(u_0) = \tilde{\gamma}(1)$, where $\tilde{\gamma}(t)$ is the unique horizontal lift of $(\gamma(t), \alpha(t))$ starting at u_0, We obtain a map $\tau : p^{-1}(\gamma(0)) \to p^{-1}(\gamma(1))$, which will be called *parallel displacement* of the fibers along the cotangent curve $(\gamma(t), \alpha(t))$. It is clear, since horizontal curves are mapped by R_a to horizontal curves, that parallel displacement commutes with the action of G:

$$\tau \circ R_a = R_a \circ \tau. \tag{3.2}$$

Therefore, parallel displacement is an isomorphism between the fibers.

If $x \in M$ lies in the symplectic leaf S, let $C(x, S)$ be the loop space of S at x. Then for each cotangent loop (γ, α), with $\gamma \in C(x, S)$, parallel displacement along (γ, α) gives a an isomorphism of the fiber $p^{-1}(x)$ into itself. The set of all such isomorphisms forms the holonomy group of Γ, with reference point x, and is denoted $\Phi(x)$. Similarly, one has the restricted holonomy group, with reference point x, denoted $\Phi^0(x)$, defined by using cotangent loops in S which are homotopic to the zero.

If $u \in p^{-1}(x)$ then we can also define the holonomy groups $\Phi(u)$ and $\Phi^0(u)$. Just as in the covariant case, $\Phi(u)$ is the subgroup of G consisting of those elements $a \in G$ such that u and ua can be joined by an horizontal curve. We have that $\Phi(u)$ is a Lie subgroup of G, whose connected component of the identity is $\Phi^0(u)$, and we have isomorphisms $\Phi(u) \simeq \Phi(x)$ and $\Phi(u)^0 \simeq \Phi(x)^0$.

If $x, y \in M$ belong to the same symplectic leave then the holonomy groups $\Phi(x)$ and $\Phi(y)$ are isomorphic. This is because if $u, v \in P$ are points such that, for some $a \in G$, there exists an horizontal curve connecting ua and v, then $\Phi(v) = Ad(a^{-1})\Phi(u)$, so $\Phi(u)$ and $\Phi(v)$ are conjugate in G. However, if $x, y \in M$ belong to different leaves the holonomy groups $\Phi(x)$ and $\Phi(y)$ will be, in general, non-isomorphic.

4. Contravariant Connections on Vector Bundles

Let $P(M, G)$ be a principal bundle over a Poisson manifold M with a contravariant connection Γ. Suppose that G acts linearly on a vector space V, so on the associated vector bundle $E(M, V, G, P)$ we have the notion of parallel displacement of fibers along cotangent curves (γ, α).

Given a section ϕ of E defined along a cotangent curve (γ, α), we define the *contravariant derivative* $D_{(\gamma,\alpha)}\phi$ by setting:

$$D_{(\gamma,\alpha)}\phi(t) = \lim_{h\to 0} \frac{1}{h}\left[\tau_t^{t+h}(\phi(\gamma(t+h))) - \phi(\gamma(t))\right] \tag{4.1}$$

where $\tau_t^{t+h} : p_E^{-1}(\gamma(t+h)) \to p_E^{-1}(\gamma(t))$ denotes parallel transport of the fibers from $\gamma(t+h)$ to $\gamma(t)$ along the cotangent curve (γ, α).

Proposition 4.1. *Let ϕ and ψ be sections of E and f a function on M defined along γ. Then*

i) $D_{(\gamma,\alpha)}(\phi + \psi) = D_{(\gamma,\alpha)}\phi + D_{(\gamma,\alpha)}\psi$;

ii) $D_{(\gamma,\alpha)}(f\phi) = (f \circ \gamma)D_{(\gamma,\alpha)}\phi + \dot{\gamma}(f)(\phi \circ \gamma)$;

Proof. i) is obvious from the definition. On the other hand, we have

$$\tau_t^{t+h}(f(\gamma(t+h))\phi(\gamma(t+h))) = f(\gamma(t+h))\tau_t^{t+h}(\phi(\gamma(t+h))),$$

and ii) follows by the Leibniz rule. □

Now let $\alpha \in T_x^*M$ be a covector and ϕ a cross section of E defined in a neighborhood of x. The contravariant derivative $D_\alpha\phi$ of ϕ in the direction of α is defined as follows: choose a cotangent curve $(\gamma(t), \alpha(t))$ defined for $t \in (-\varepsilon, \varepsilon)$, and such that $\gamma(0) = x$ and $\alpha(0) = \alpha$. Then we set:

$$D_\alpha\phi = D_{(\gamma,\alpha)}\phi(0). \tag{4.2}$$

It is easy to see that $D_\alpha\phi$ is independent of the choice of cotangent curve. Clearly, a cross section ϕ of E defined on an open set $U \subset M$ is flat iff $D_\alpha\phi = 0$ for all $\alpha \in T_xM$, $x \in M$.

Proposition 4.2. *Let $\alpha, \beta \in T_x^*M$, ϕ and ψ cross sections of E defined in a neighborhood U of x. Then*

i) $D_{\alpha+\beta}\phi = D_\alpha\phi + D_\beta\phi$;

ii) $D_\alpha(\phi+\psi) = D_\alpha\phi + D_\alpha\psi$;

iii) $D_{c\alpha}\phi = cD_\alpha\phi$, *for any scalar* c;

iv) $D_\alpha(f\phi) = f(x)D_\alpha\phi + \#\alpha(f)\phi(x)$, *for any function* $f \in C^\infty(U)$;

Proof. iii) is obvious, while ii) and iv) follow from proposition 4.1. To prove i) observe that any section ϕ of E, defined in a open set U, can be identified with a function $F : p^{-1}(U) \to V$ by letting

$$F(u) = u^{-1}(\phi(p(u))), \qquad u \in p^{-1}(U),$$

where we view $u \in P$ as a linear isomorphism $u : V \to p^{-1}(u)$. Then, as in the covariant case, we find

$$D_\alpha\phi = u(h(u,\alpha)\cdot F).$$

From this expression for the contravariant derivative, i) follows immediately. □

Now let $\alpha \in \Omega^1(M)$ be a 1-form and ϕ a section of E. We define the contravariant derivative $D_\alpha\phi$ to be the section of E given by:

$$D_\alpha\phi(x) = D_{\alpha_x}\phi. \tag{4.3}$$

Proposition 4.3. *Let* $\alpha, \beta \in \Omega^1(M)$, ϕ *and* ψ *cross sections of* E, *and* $f \in C^\infty(M)$. *Then*

i) $D_{\alpha+\beta}\phi = D_\alpha\phi + D_\beta\phi$;

ii) $D_\alpha(\phi+\psi) = D_\alpha\phi + D_\alpha\psi$;

iii) $D_{f\alpha}\phi = fD_\alpha\phi$;

iv) $D_\alpha(f\phi) = fD_\alpha\phi + \#\alpha(f)\phi$;

Proof. From proposition 4.2 we obtain immediately that i)-iv) hold. □

Conversely, one can show that every such operator is induced by a contravariant connection on E (details in [4]). In [8] Vaisman introduces the notion of contravariant derivative using i)-iv) as axioms.

A *linear contravariant connection* is a contravariant connection on the coframe bundle $P = F^*(M)$ over M, so $G = GL(n)$ where $n = \dim M$. For the associated bundle $T^*(M)$ we get a contravariant derivative operator D which to a pair of 1-forms α and β associates a another 1-form $D_\alpha\beta$. In this case, one defines the *torsion tensor field* T and the *curvature tensor field* R, respectively, to be the tensor fields of types $(2,1)$ and $(3,1)$ given by

$$T(\alpha,\beta) = D_\alpha\beta - D_\beta\alpha - [\alpha,\beta], \tag{4.4}$$
$$R(\alpha,\beta)\gamma = D_\alpha D_\beta\gamma - D_\beta D_\alpha\gamma - D_{[\alpha,\beta]}\gamma, \tag{4.5}$$

for any 1-forms $\alpha, \beta, \gamma \in \Omega^1(M)$.

5. Geodesics

Because, for contravariant connections, parallel transport can only be defined along curves lying in symplectic leaves of M, the same restriction applies to geodesics:

Definition 5.1. Let M be a Poisson manifold with a linear contravariant connection. A cotangent curve $(\gamma(t), \alpha(t))$ on M is called a GEODESIC if:

$$(D_\alpha \alpha)_{\gamma(t)} = 0. \tag{5.1}$$

Let $(x^1, \dots, x^n)$ be local coordinates on a neighborhood U in M. Then one defines Christoffel symbols Γ_k^{ij} for a linear contravariant connection by the usual formula

$$D_{dx^i} dx^j = \Gamma_k^{ij} dx^k. \tag{5.2}$$

In local coordinates, a curve $(\gamma(t), \alpha(t)) = (x^1(t), \dots, x^n(t), \alpha_1(t), \dots, \alpha_n(t))$ is a geodesic iff it satisfies the following system of ode's

$$\begin{cases} \frac{dx^i(t)}{dt} = \pi^{ji}(x^1(t), \dots, x^n(t))\alpha_j(t), \\ \frac{d\alpha_i(t)}{dt} = -\Gamma_i^{jk}((x^1(t), \dots, x^n(t))\alpha_j\alpha_k. \end{cases} \quad (i = 1, \dots, n) \tag{5.3}$$

where $\Pi = \sum_{i<j} \pi^{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ is the Poisson tensor. From this we have:

Proposition 5.2. *Let M be a Poisson manifold, with a contravariant connection Γ, and $p \in M$. Given $\alpha_p \in T_p^*M$, there is a unique maximal geodesic $t \mapsto (\gamma(t), \alpha(t))$, starting at $p \in M$, with $\alpha(0) = \alpha_p$.*

Proof. Choose a systems of coordinates $(x^1, \dots, x^n)$ centered at p. By standard uniqueness and existence results for ode's, system (5.3) has a unique solution with initial condition $(x^1(0), \dots, x^n(0), \alpha_1(0), \dots, \alpha_n(0)) = (0, \dots, 0, \alpha_{p1}, \dots, \alpha_{pn})$. □

The geodesic given by this proposition is called the geodesic through p with tangent covector α_p. Note that if S is the symplectic leave through p and $v \in T_pM$ is a tangent vector, there can be several geodesics with this tangent vector at p.

The following result is the analogue of a well known result in affine geometry:

Proposition 5.3. *Let Γ be a linear contravariant connection on M. There exists a unique contravariant connection on M with the same geodesics and zero torsion.*

Proof. Choose local coordinates on M so D has symbols Γ_k^{ij}, and consider the set of functions

$$^*\Gamma_k^{ij} = \frac{1}{2}\left(\Gamma_k^{ij} + \Gamma_k^{ji} + \frac{\partial \pi^{ij}}{\partial x^k}\right) \tag{5.4}$$

One checks that this indeed gives a well defined contravariant connection D^* on M, by a standard argument involving change of coordinates. From expressions (4.4) and (5.3) for the torsion and the geodesics, we see that D^* has zero torsion and the same geodesics as D.

For uniqueness, let D and D^* be two connections with the same geodesics and torsion 0. We let

$$S(\alpha,\beta) = D_\alpha\beta - D^*_\alpha\beta, \qquad \alpha,\beta\in\Omega^1(M). \tag{5.5}$$

Then S is C^∞-linear, so it is a tensor. Since the connections have 0 torsion, we have:

$$\begin{aligned} S(\alpha,\beta) - S(\beta,\alpha) &= (D_\alpha\beta - D_\beta\alpha) - (D^*_\alpha\beta - D^*_\beta\alpha) \\ &= [\alpha,\beta] - [\alpha,\beta] = 0. \end{aligned} \tag{5.6}$$

so S is a symmetric tensor. Now if $\alpha_p \in T^*_pM$, we can choose the geodesic (for D and D^*) with tangent covector α_p and associated 1-form α along γ. We have

$$S(\alpha,\alpha) = D_\alpha\alpha - D^*_\alpha\alpha = 0, \tag{5.7}$$

so $S = 0$ and $D = D^*$. □

6. Relationship to Ordinary Connections

Let M be a symplectic manifold and Γ a contravariant connection on $P(M,G)$ with horizontal lift $h : T^*M \to TP$. Then we have a bundle map $\tilde{h} : p^*TM \to TP$ defined by

$$\tilde{h}(u,v) = h(u, \#^{-1}v), \qquad (u,v)\in p^*TM.$$

This map is obviously G-invariant and makes the following diagram commute

$$\begin{array}{ccc} p^*TM & \xrightarrow{\tilde{h}} & TP \\ {\scriptstyle \hat{p}}\downarrow & & \downarrow{\scriptstyle p_*} \\ TM & \xrightarrow[\text{id}]{} & TM \end{array}$$

It follows that $\tilde{h}$ is the horizontal lift of a covariant connection on M. This construction shows that there are always contravariant connections on any principal bundle $P(M,G)$ over a Poisson manifold M. In the case of linear connections the contravariant and covariant derivatives are related by:

$$D_\alpha = \nabla_{\#\alpha}.$$

For a general Poisson manifold with a contravariant connection Γ on $P(M,G)$ and horizontal lift $h : T^*M \to TP$, we say that Γ is *induced by a covariant connection* if

$$h(u,\alpha) = \tilde{h}(u,\#\alpha), \qquad (u,\alpha)\in p^*T^*M,$$

where $\tilde{h} : p^*TM \to TP$ is the horizontal lift of some covariant connection on M. Note that in this case the lift h satisfies:

$$\#\alpha = 0 \quad\Longrightarrow\quad h(u,\alpha) = 0, \qquad (u,\alpha)\in p^*T^*M. \tag{6.1}$$

In the case of a linear contravariant connection, this condition says

$$\#\alpha = 0 \quad\Longrightarrow\quad D_\alpha = 0, \qquad \alpha\in\Omega^1(M). \tag{6.2}$$

Definition 6.1. A contravariant connection Γ on a principal bundle $P(M,G)$ is called a $\mathcal{F}$-connection if its horizontal lift satisfies condition (6.1)

Assume we have a contravariant $\mathcal{F}$ connection Γ on $P(M,G)$. If $i : S \hookrightarrow M$ is a symplectic leave, then on the pull-back bundle $\tilde{p} : i^*P \to M$ we have an induced connection Γ_S: on the total space $i^*P = \{(y,u) \in S \times P : i(y) = p(u)\}$ we define the horizontal lift $h_S : p_S^* T^*S \to T(i^*P)$ by setting

$$h_S((s,u),\alpha) = (p_* h(u,\beta), h(u,\beta)), \qquad (s,u) \in i^*P, (u,\alpha) \in p^*T^*M, \tag{6.3}$$

where $\beta \in T^*_{i(s)}M$ is such that $(d_s i)^*\beta = \alpha$, and we are considering the canonical identification $T(i^*P) = \{(v,w) \in TS \times TP : v = p_* w\}$. If $(d_s i)^*\beta' = (d_s i)^*\beta$, then $\#\beta' = \#\beta$, so we get the same result in (6.3) and so Γ is well defined. S being symplectic, the connection Γ_S is induced by a covariant connection on i^*P. It follows that a contravariant $\mathcal{F}$-connection in P can be thought of as a *family* of ordinary connections over the symplectic leaves of M. In fact, it is easy to see that contravariant $\mathcal{F}$-connections can be defined in terms of an horizontal, G-invariant, (generalized) distribution in P, which projects to the leaves of M, i. e., they are just partial connections along the leaves.

For a $\mathcal{F}$-connection, horizontal lifts of cotangent curves (γ,α) depend only on γ. Therefore, one has a well determined notion of horizontal lift of a curve lying on a symplectic leaf. It follows that for these connections, parallel displacement can also be defined by first reducing to the pull-back bundle over a symplectic leaf and then parallel displace the fibers. Hence, the holonomy groups $\Phi(x)$ and $\Phi^0(x)$ coincide with the usual holonomy groups of the pull-back connection on the symplectic leaf S through x. From this one obtains an analogue of the holonomy theorem (see [3]), which we state here only for linear connections:

Theorem 6.2. *(Holonomy Theorem) Let Γ be a linear contravariant $\mathcal{F}$-connection. The Lie algebra of the holonomy group $\Phi(x) \subset GL(n,\mathbb{R})$ is the ideal of $\mathfrak{gl}(n,\mathbb{R})$ spanned by all elements of the form $R(\alpha,\beta)_x$, where $\alpha,\beta \in \Omega^1(M)$.*

A natural question is whether or not this result holds for general contravariant connections.

We finish with a natural example of a contravariant connection which is not an $\mathcal{F}$-connection.

Example 6.3. Recall that a Poisson Lie group is a Lie group G with a Poisson bracket such that the group product $G \times G \to G$ is a Poisson map (see [7] a discussion of Poisson Lie groups). A theorem of A. Weinstein ([5]) and M. Karasev ([2]) states that the left invariant 1-forms $\Omega^1_{\text{inv}}(G)$ on a Poisson Lie group form a subalgebra of $\Omega^1(G)$ with respect to the bracket [,] on 1-forms. Therefore, we can define a linear contravariant connection in G by setting

$$D_\alpha\beta = \frac{1}{2}[\alpha,\beta], \quad \alpha,\beta \in \Omega^1_{\text{inv}}(G)$$

This connection is the analogue of the natural left invariant, torsion free, covariant connection, on a Lie group. In fact, this connection has zero torsion and its curvature is parallel:

$$T(\alpha,\beta) = 0, \qquad D_\gamma R(\alpha,\beta) = 0.$$

However, in general, this connection is not left invariant. This is because, in general (if the Poisson tensor is not trivial), left multiplication in G is not a Poisson map. For the same reason, in general, we have $D\pi \neq 0$.

To see that these connection are not, in general, $\mathcal{F}$-connections, we consider a simple example. Let $G = (\mathbb{R}^2, +)$ with coordinates (x, y) and bracket

$$\{x, y\} = x.$$

Then G is a Poisson-Lie group, and for the natural connection in G we have:

$$D_{dx}dx = D_{dy}dy = 0, D_{dx}dy = \frac{1}{2}dx = -D_{dy}dx.$$

Note that the Poisson tensor vanishes at the origin, but there are contravariant derivatives which do not vanish at the origin, so D is not a $\mathcal{F}$-connection.

This example also suggests that one should study Poisson (locally) symmetric spaces, i. e., Poisson manifolds with a linear contravariant connection such that

$$T = 0 \quad \text{and} \quad DR = 0.$$

We refer the reader to [4].

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