Lie Algebroids, Holonomy and Characteristic Classes

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We extend the notion of connection in order to study singular geometric structures, namely, we consider a notion of connection on a Lie algebroid which is a natural extension of the usual concept of a covariant connection. It allows us to define holonomy of the orbit foliation of a Lie algebroid and prove a Stability Theorem. We also introduce secondary or exotic characteristic classes, thus providing invariants which generalize the modular class of a Lie algebroid.

Key Words: Lie algebroid, connection, holonomy, characteristic classes

0. INTRODUCTION AND BASIC DEFINITIONS

The theory of connections is a classical topic in differential geometry. They provide an extremely important tool to study geometric structures on manifolds and, as such, they have been applied with great success in many different settings.

However, the use of connections has been very limited whenever singular behavior is present. The reason is that if some geometric structure admits a compatible connection then parallel transport will preserve any algebraic invariant of the structure, and that prevents the presence of singular behavior. For example, a Poisson manifold admitting a connection compatible with the Poisson tensor must have constant rank and hence is a regular Poisson manifold (see [28]). In this work we explain how one can extend the notion of connection in order to include geometric structures that may exhibit singular behavior.

One of the basic ideas underlying our construction of connections is that one should replace the tangent bundle of the manifold M by a new bundle which reflects more faithfully the (possible singular) geometric structure

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on M. In this paper we take the point of view that every such geometric structure has an underlying *Lie algebroid* structure, which plays the role of the tangent bundle. Many common geometric structures, some of which we recall below, admit such a description. We believe that Lie algebroids provide the appropriate setting to develop a complete geometric theory of connections as well as other key concepts of differential geometry for singular geometric structures.

To every Lie algebroid there is associated a foliation of M, which in general will be singular. Conversely, it is not known if every singular foliation is associated with a Lie algebroid. There is some evidence that this is actually the case, and in many ways this work is inspired by the theory of regular foliations. Several new results to be presented here are extensions to singular foliations associated with Lie algebroids of well-known results in foliation theory.

Since the notion of a Lie algebroid is still not part of mainstream differential geometry, we start by recalling its definition (for an introduction to the theory see the recent monograph [1], and also the survey article [24]):

Definition 0.1. A Lie algebroid A over a smooth manifold M is a vector bundle $\pi : A \to M$ together with a Lie algebra structure $[\ ,\ ]$ on the space of sections $\Gamma(A)$ and a bundle map $\# : A \to TM$, called the *anchor*, such that:

i) the induced map $\# : \Gamma(A) \to \mathcal{X}^1(M)$ is a Lie algebra homomorphism([1]);

(ii) for any sections $\alpha, \beta \in \Gamma(A)$ and smooth function $f \in C^\infty(M)$ we have the Leibniz identity:

$$[\alpha, f\beta] = f[\alpha, \beta] + \#\alpha(f)\beta. \tag{1}$$

The image of # defines a smooth generalized distribution in M, in the sense of Sussman [27], which is integrable (this follows, for example, from the Local Splitting Theorem to be presented below). The integrable leaves are called *orbits* of A and they form the *orbit foliation* of the Lie algebroid. We call A a *regular Lie algebroid* if the rank of # is locally constant, so the orbit foliation is not singular. We call A a *transitive Lie algebroid* if # is surjective, so the leaves are the connected components of M.

The definition of a morphism of Lie algebroids, not necessarily over the same base manifold, is rather subtle and there are actually two distinct

[1] We denote by $\Omega^r(M)$ and $\mathcal{X}^r(M)$, respectively, the spaces of differential r-forms and r-multivector fields on a manifold M.If E is a bundle over M, $\Gamma(E)$ will denote the space of global sections.

notions (see e.g. [24]). We will be dealing mostly with isomorphisms, and for these the two definitions coincide. To introduce the definition we will be using, first observe that any bundle map $\phi : A_2^* \to A_1^*$ induces a map $\Phi : \Gamma(A_1) \to \Gamma(A_2)$ which assigns to each section $\alpha \in \Gamma(A_1)$ the section $\Phi(\alpha) \in \Gamma(A_2)$ given by

$$\Phi(\alpha)(y) \equiv \phi^* \alpha(\phi_0(y)), \qquad \forall y \in M_2,$$

where we have denoted by $\phi_0 : M_2 \to M_1$ the map induced by ϕ on the base manifolds and by $\phi^* : (A_1)_{\phi_0(y)} \to (A_2)_y$ the fiberwise transpose of ϕ.

Definition 0.2. Let $A_1 \to M_1$ and $A_2 \to M_2$ be Lie algebroids. A morphism of Lie algebroids from A_1 to A_2 is a bundle map $\phi : A_2^* \to A_1^*$ such that:

(i) the induced map $\Phi : \Gamma(A_1) \to \Gamma(A_2)$ preserves brackets

$$[\Phi(\alpha), \Phi(\beta)]_2 = \Phi([\alpha, \beta]_1), \qquad \alpha, \beta \in \Gamma(A_1); \tag{2}$$

(ii) the vector fields $\#_1\Phi(\alpha)$ and $\#_2\alpha$ are ϕ_0-related:

$$\#_2\alpha = (\phi_0)_* \#_1 \Phi(\alpha), \quad \alpha, \beta \in \Gamma(A_1). \tag{3}$$

We shall denote such a Lie algebroid homomorphism by $\Phi : A_1 \to A_2$.

To study global properties of Lie algebroids one needs to consider connections that are adapted to the orbit foliation. In this paper, following the approach in [10] for the special case of Poisson manifolds, we introduce *Lie algebroid connections* based on the notion of *horizontal lift*. The basic observation in [10], which also applies in the present setting, is that one should lift the appropriate geometric objects rather than tangent vectors as one does in the ordinary theory of connections.

Definition 0.3. Let $\pi : A \to M$ be a Lie algebroid with anchor $\# : A \to TM$ and Lie bracket $[\ ,\ ]$. An A-connection on a fiber bundle $p : E \to M$ over M is a bundle map $h : p^*A \to TE$, where $p^*A \to E$ is the pullback bundle of A by p, such that the following diagram commutes:

$$\begin{array}{ccc} p^*A & \xrightarrow{h} & TE \\ {\scriptstyle \widehat{p}}\downarrow & & \downarrow{\scriptstyle p_*} \\ A & \xrightarrow[\#]{} & TM \end{array}$$

If $(u,\alpha)\in p^*A$, where $u\in E$ and $\alpha\in A_x$ with $x=p(u)$, we call $h(u,\alpha)$ the *horizontal lift* of α to the point u in the fiber over x.

Depending on the bundle structure, we may require some additional conditions on the lift h:

(i) If $E=P(M,G)$ is a principal bundle with structure group G, then we require h to be G-invariant:

$$h(u\cdot g,\alpha)=(R_g)_*h(u,\alpha),\qquad \forall g\in G;$$

(ii) If E is a vector bundle, then we require $h(u,\cdot)$ to be linear:

$$h(u,\alpha+\beta)=h(u,\alpha)+h(u,\beta);$$

We now recall some basic classes of Lie algebroids:

Tangent Lie algebroid. Let M be a manifold. The tangent bundle TM becomes a transitive Lie algebroid when we take as bracket the usual Lie bracket on vector fields and as anchor map the identity map. This example is important for us so we can compare new concepts we shall introduced for Lie algebroids with the standard ones in ordinary differential geometry.

If $E\to M$ is a fiber bundle, a TM-connection on E is just a connection in the usual sense. The bundle map h is the horizontal lift in the usual theory of connections and $\mathcal{H}_u=\{h(u,\alpha):\alpha\in TM\}$ is the horizontal distribution (as in [20]). In this way, we may say that the theory of A-connections is a generalization of the usual theory of connections. Henceforth, we shall refer to TM-connections as *covariant connections*.

Regular foliations. Let $A\subset TM$ be an integrable subbundle defining a regular foliation $\mathcal{F}$ of M. A section of A is a vector field in M which is tangent to $\mathcal{F}$. If X and Y are vector fields tangent to $\mathcal{F}$, their Lie bracket $[X,Y]$ is also a vector field tangent to $\mathcal{F}$. In this way we have a Lie bracket defined on $\Gamma(A)$ and if we let $\#:A\to TM$ be the inclusion, we obtain a Lie algebroid. For this Lie algebroid the anchor $\#$ is *injective* and the orbit foliation is $\mathcal{F}$. Conversely, every Lie algebroid with anchor map $\#$ injective has a regular foliation $\mathcal{F}$ and is canonically isomorphic with the Lie algebroid of $\mathcal{F}$.

If A is the Lie algebroid of a regular foliation $\mathcal{F}$, an A-connection on a fiber bundle $E\to M$ is sometimes called a *partial connection* along the leaves of $\mathcal{F}$ (see, e.g., [18]). This is because the horizontal lift is only defined for tangent vectors that are tangent to leaves. Partial connections were used by Kubarski in [22] to study regular Lie algebroids.

Later we shall see that one can use A-connections to define the A-holonomy of the orbit foliation of an *arbitrary* Lie algebroid A. In case $\#$ is injective this holonomy coincides with the usual holonomy of the theory of regular foliations.

Poisson manifolds. Let M be a Poisson manifold with Poisson tensor $\Pi \in \mathcal{X}^2(M)$. Then Π determines a bundle map $\# : T^*M \to TM$ as well as a Lie bracket on the space of differential 1-forms $\Omega^1(M)$, which may be defined by

$$[\alpha, \beta] \equiv \mathcal{L}_{\#\alpha}\beta - \mathcal{L}_{\#\beta}\alpha - \Pi(\alpha, \beta).$$

It is well known that $(T^*M, [\ ,\ ], \#)$ is a Lie algebroid. Some of the results to be presented in this paper generalize corresponding results for Poisson manifolds presented in [10]. A T^*M-connection on a fiber bundle $E \to M$ is just a *contravariant connection* on E in the terminology of [10]. This example can be extended in two distinct directions, namely to Dirac manifolds ([4]) and to Jacobi manifolds ([19]).

Transformation Lie algebroids. Let $\rho : \mathfrak{g} \to \mathcal{X}^1(M)$ be an infinitesimal (right) action of a Lie algebra on the manifold M. The associated *transformation Lie algebroid* is the trivial bundle $M \times \mathfrak{g} \to M$ with anchor map $\# : M \times \mathfrak{g} \to TM$ defined by

$$\#(x, v) \equiv \rho(v)_x,$$

and with Lie bracket

$$[v, w](x) = [v(x), w(x)] + (\rho(v(x)) \cdot w)|_x - (\rho(w(x)) \cdot v)|_x,$$

where we identify a section v of $M \times \mathfrak{g} \to M$ with a $\mathfrak{g}$-valued function $v : M \to \mathfrak{g}$. If ρ can be integrated to a Lie group action, the orbit foliation of M coincides with the orbits of the action. Therefore, connections on transformations Lie algebroids allows one to study the singular foliations associated with Lie group actions. For example, the holonomy (see below) of an orbit is useful in the study of orbit types.

Bundles of Lie algebras. Let A be a Lie algebroid with $\# \equiv 0$. For each $x \in M$ one can define a Lie algebra structure in A_x as follows: if $\alpha, \beta \in A_x$ are in the fiber over x choose sections $\tilde{\alpha}$ and $\tilde{\beta}$ such that $\tilde{\alpha}(x) = \alpha$ and $\tilde{\beta}(x) = \beta$. Then $[\alpha, \beta] \equiv [\tilde{\alpha}, \tilde{\beta}](x)$. Using the Leibniz identity one checks that this definition does not depend on the choice of sections. Thus we see that such a Lie algebroid is a vector bundle with varying Lie algebra structure on the fibers. Conversely, any bundle of Lie algebras determines a Lie algebroid with trivial anchor.

A special case is when $M = \{*\}$, so a Lie algebra can be consider as a Lie algebroid over a point. It is easy to see that a linear flat connection is just a representation of a Lie algebra.

Connections are specially useful to compare geometric structures at different points of M. For non-regular Lie algebroids the orbit foliation is singular and the dimension of the leaves varies, so one can only hope to compare spaces at different points of the same orbit. For that one needs the following fundamental notion of path which was discovered independently by several authors (e.g., in [32] they are called *admissible paths*, and in [11] they are called *cotangent paths* in the case $A = T^*M$).

Definition 0.4. An A-path is a piecewise smooth path $\alpha : [0,1] \to A$, such that:

$$\#\alpha(t) = \frac{d}{dt}\pi(\alpha(t)), \qquad t \in [0,1]. \tag{4}$$

The curve $\gamma : [0,1] \to M$ given by $\gamma(t) \equiv \pi(\alpha(t))$ will be called the *base path* of α.

Notice that the base path of an A-path lies on a fixed leaf of the Lie algebroid. We shall show that given an A-connection one can define *parallel transport* along any A-path. Once the notion of parallelism is available, one can then proceed to develop a theory of connections where standard concepts such as curvature, holonomy, geodesic, etc, make sense. In particular we show that a linear A-connection on a vector bundle $p : E \to M$ gives, in a way entirely analogous to the ordinary case, an *A-derivative* operator $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$ which satisfies:

(i) $\nabla_{\alpha+\beta}\phi = \nabla_\alpha\phi + \nabla_\beta\phi$;
(ii) $\nabla_\alpha(\phi+\psi) = \nabla_\alpha\phi + \nabla_\alpha\psi$;
(iii) $\nabla_{f\alpha} = f\nabla_\alpha\phi$;
(iv) $\nabla_\alpha(f\phi) = f\nabla_\alpha\phi + \#\alpha(f)\phi$;

where $\alpha, \beta \in \Gamma(A)$, ϕ, ψ are sections of E, and $f \in C^\infty(M)$. Conversely, every such operator is induced by a linear A-connection. Connections from this operational point of view were first introduced in the case of Poisson manifolds by Vaisman ([29]). Flat linear A-connections on a vector bundle E is an important special case which has also been studied by several authors usually under the name of *representations of Lie algebroids* ([9, 15, 34, 21]).

In spite of its formal similarities with ordinary connections, there are many striking differences in Lie algebroid connection theory: parallel transport does not depend only on the base path, the holonomy of a flat A-connection may be non-discrete, etc.

However, just like in ordinary geometry, A-connections are useful to study global properties of Lie algebroids. Using Lie algebroid connections we show that we have a notion of *holonomy* of the associated foliation to a Lie algebroid. For the case of a regular foliation it coincides with the usual notion of holonomy. We show below that the transversal geometry to a leaf of a Lie algebroid is described by a germ of an algebroid, so we have a notion of *transverse Lie algebroid structure*. The holonomy map is by automorphisms of this transversal algebroid germ.

In general, holonomy is not homotopy invariant, but factoring out the inner Lie algebroid automorphisms one obtains a notion of *reduced holonomy* which is invariant by homotopy, and we can prove the following analogue of the Reeb Stability Theorem:

THEOREM 0.1. *Let L be a compact, transversally stable leaf, with finite reduced holonomy. Then L is stable, i.e., L has arbitrarily small neighborhoods which are invariant under all inner automorphisms. Moreover, each leaf near L is a bundle over L with fiber a finite union of leaves of the transverse Lie algebroid structure.*

Linear holonomy of a Lie algebroid is obtained simply by linearizing the holonomy homomorphism. In the case of Poisson manifolds, it was studied by Ginzburg and Golubev in [11]. It can also be discussed more efficiently from the point of view of linear Lie algebroid connections and, for each leaf, there is a notion of *Bott A-connection*.

As it was shown in [9], for a non-regular Lie algebroid there is a natural vector bundle playing the role of the normal bundle (over the whole of M) to the singular foliation, and which allows us to introduce the notion of a *basic connection*: these are linear A-connections which preserve the Lie algebroid structure and restrict in each leaf to the Bott A-connection. Comparing a basic connection to a riemannian connection, as in the regular theory of foliations, one is lead to *exotic* or *secondary Lie algebroid characteristic classes*. These are A-cohomology classes which give information on both the geometry of the Lie algebroid and the topology of the associated foliation of M. In degree 1, this class actually coincides with the *modular class* of the Lie algebroid, introduced by Weinstein in [31].

The remainder of the paper is organized as follows. In Section 1, we we describe some elementary properties of Lie algebroids and their differential geometry, including local splitting and transverse structure which we could not find in the literature. In Section 2, we sketch the theory of Lie algebroid connections. In Section 3, we introduce holonomy of a leaf of a Lie algebroid foliation and we prove the Stability Theorem. In the fourth and final section, we introduce characteristic classes for Lie algebroids and construct the invariants we have mentioned above. We also give explicit computations of these invariants for some classes of Lie algebroids.

Finally, we remark that several authors have considered connections on Lie algebroids in order to study its global properties (see for example [9, 17, 22, 25]). Until the work of Evans, Lu and Weinstein [9], all these authors considered regular, or even transitive, Lie algebroids. In [9] connections on non-regular Lie algebroids are used for the first time. There, the authors consider zero curvature A-connections on vector bundles, which they call *representations of Lie algebroids*, to construct the modular class. These results were extended by Xu [34] to so called BV-algebras, and Huebschmann in [15, 16] developed a complete algebraic theory.

Remark. In the final stages of preparation of this work, I learn of a preprint by Marius Crainic [5] where an approach to secondary characteristic classes for representations of Lie algebroids is proposed (see also the remark at the end of section 4.3). The discussions I have had with him after the present paper was submitted, were extremely influential in shaping my view of the subject. The relationship between the two approaches is explained in [6]. Our discussions eventually led to a solution to the problem of integrating Lie algebroids to Lie groupoids (see [7]), where we make use of some of the results presented here. In the preprint [12], Ginzburg proposes a K-theory for Poisson manifolds and Lie algebroids which is also related to the present work.

1. THE LOCAL STRUCTURE OF LIE ALGEBROIDS

1.1. The Dual Lie-Poisson Bracket

From now on we fix a Lie algebroid $\pi : A \to M$ over M with anchor map $\#$ and Lie bracket $[\ ,\ ]$. We let m denote the dimension of M and we let r

denote the rank of A. We start by recalling the construction of a canonical Poisson bracket on the dual bundle A^* (see [1], sect. 16.5).

If one fixes local coordinates $(x^1, \dots, x^m)$ over a trivializing neighborhood U of M where A admits a basis of local sections $\{\alpha^1, \dots, \alpha^r\}$ over U, we have *structure functions* $b^{is}, c_u^{st} \in C^\infty(U)$ defined by

$$\#\alpha^s = \sum_{i=1}^m b^{si} \frac{\partial}{\partial x^i}, \qquad (s = 1, \dots, r), \tag{5}$$

$$[\alpha^s, \alpha^t] = \sum_{u=1}^r c_u^{st} \alpha^u, \qquad (s, t = 1, \dots, r). \tag{6}$$

The defining relations for a Lie algebroid translate into certain p.d.e's involving the structure functions.

One defines a Poisson structure on A^* as follows. Let $(\xi^1, \dots, \xi^r)$ denote the linear coordinates on the fibers of A^* associated with the basis of sections $\{\alpha^1, \dots, \alpha^r\}$. The Poisson bracket $\{\ ,\ \}_A$ on $C^\infty(A^*)$ is defined by:

$$\begin{aligned} \{x^i, x^j\} &= 0, \\ \{x^i, \xi^s\} &= -b^{si}, \\ \{\xi^s, \xi^t\} &= \sum_u c_u^{st} \xi^u. \end{aligned} \tag{7}$$

One checks that this bracket is independent of the choice of local coordinates and basis. Because this bracket is linear on the fibers, one also calls it the *dual Poisson-Lie bracket* of A.

Let α be a section of A. Then α defines in a natural way a function $f_\alpha : A^* \to \mathbb{R}$ which is linear in the fibers. One has the following properties of the dual Poisson-Lie bracket.

PROPOSITION 1.1. *The assignment $\alpha \mapsto f_\alpha$ defines a Lie algebra homomorphism $(\Gamma(A), [\ ,\ ]) \to (C^\infty(A^*), \{\ ,\ \}_A)$. Moreover, if X_{f_α} denotes the hamiltonian vector field associated with f_α, then X_{f_α} is π-related to $\#\alpha$:*

$$\pi_* X_{f_\alpha} = \#\alpha,$$

where $\pi : A^* \to M$ is the natural projection.

Proof. We use local coordinates. If $\alpha = \sum_s a_s(x)\alpha^s$ then $f_\alpha(x,\xi) = \sum_s a_s(x)\xi^s$ and the associated hamiltonian vector field is

$$X_{f_\alpha} = \sum_{s,i} a_s b^{si}\frac{\partial}{\partial x^i} + \sum_{t,u}\left(\sum_s a_s c_u^{st} - \sum_i \frac{\partial a_u}{\partial x^i} b^{ti}\right)\xi^u \frac{\partial}{\partial \xi^t}. \qquad (8)$$

This expression shows that X_{f_α} projects to $\#\alpha$,

On the other hand, if $\beta = \sum_t b_t(x)\alpha^t$, one computes:

$$\begin{aligned}\{f_\alpha, f_\beta\} &= \left\{\sum_s a_s(x)\xi^s, \sum_t b_t(x)\xi^t\right\}\\ &= \sum_t \#\alpha(b_t)\xi^t - \sum_s \#\beta(a_s)\xi^s + \sum_{s,t,u} a_s b_t c_u^{st}\xi^u = f_{[\alpha,\beta]},\end{aligned}$$

so the result follows. ▮

The dual Lie-Poisson structure of a Lie algebroid codifies all the information regarding the Lie algebroid structure. In fact, the category of vector bundles with Poisson brackets linear on the fibers is equivalent to the category of Lie algebroids. For example, a morphism $\Phi : A_1 \to A_2$ of Lie algebroids, as defined above, is just a bundle map $\phi : A_2^* \to A_1^*$ which is a Poisson map.

Let $\alpha \in \Gamma(A)$ be a section, so we have the associated hamiltonian vector field X_{f_α} in A^*. For each t, the flow ϕ_t^α defines a Poisson automorphism of A^* (wherever defined). From (8) we see that X_{f_α} is linear along the fibers, so in fact $\phi_t^\alpha : A^* \to A^*$ is a bundle map. It follows that each section determines a 1-parameter family of (local) Lie algebroid morphisms $\Phi_t^\alpha : A \to A$. If $A = \mathfrak{g}$ is a Lie algebra, considered as a Lie algebroid over a point, $\Phi_t^\alpha : A \to A$ (resp., $\phi_t^\alpha : A^* \to A^*$) is just $Ad(\exp(t\alpha))$ (resp., $Ad^*(\exp(t\alpha))$, so this construction generalizes the usual adjoint action, and will be refer to as *integration of sections*. We remark that we can also integrate time-dependent sections α_t.

Let us denote by $\mathrm{Aut}\,(A)$ the group of automorphisms of the Lie algebroid A, and by $\mathrm{Aut}^0(A)$ its connected component of the identity: given $\Phi \in \mathrm{Aut}^0(A)$ there exists a smooth family $\Phi_t \in \mathrm{Aut}\,(A)$, $t \in [0,1]$, such that $\Phi_0 =$id, $\Phi_1 = \Phi$. An element $\Phi \in \mathrm{Aut}^0(A)$ is called a *inner automorphism* if there exists some smooth family of sections $\alpha_t \in \Gamma(A)$ which can be integrated to a 1-parameter family of Lie algebroid automorphisms $\Phi_t^{\alpha_t}$ with $\Phi_1^{\alpha_t} = \Phi$.

PROPOSITION 1.2. *The set* $\mathrm{Inn}\,(A) \subset \mathrm{Aut}\,(A)$ *of inner Lie algebroid automorphisms is a normal subgroup.*

We define the group of *outer Lie algebroid automorphisms* to be the quotient $\mathrm{Out}\,(A) \equiv \mathrm{Aut}\,(A)/\mathrm{Inn}\,(A)$.

1.2. Local Splitting

By choosing appropriate coordinates and sections one can simplify the expressions of the structure functions, and we obtain the following analogue of the Weinstein Splitting Theorem for Poisson manifolds ([33], Thm. 2.1).

THEOREM 1.1 (Local Splitting). *Let* $x_0 \in M$ *be a point where* $\#_{x_0}$ *has rank* q. *There exist coordinates* (x^i, y^j), $(i = 1, \dots, q, j = q+1, \dots, m)$, *valid in a neighborhood* U *of* x_0, *and a basis of sections* $\{\alpha^1, \dots, \alpha^r\}$, *of* A *over* U, *such that:*

$$\#\alpha^i = \frac{\partial}{\partial x^i}, \qquad (i = 1, \dots, q), \tag{9}$$

$$\#\alpha^s = \sum_j b^{sj} \frac{\partial}{\partial y^j}, \qquad (s = q+1, \dots, r), \tag{10}$$

where $b^{sj} \in C^\infty(U)$ *are smooth functions depending only on the* y*'s and vanishing at* x_0: $b^{sj} = b^{sj}(y^j)$, $b^{sj}(0) = 0$. *Moreover,*

$$[\alpha^s, \alpha^t] = \sum_u c_u^{st} \alpha^u, \tag{11}$$

where $c_u^{st} \in C^\infty(U)$ *vanish if* $u \le q$ *and satisfy*

$$\sum_{u>q} \frac{\partial c_u^{st}}{\partial x^i} b^{uj} = 0. \tag{12}$$

Proof. If the rank of $\#$ at x_0 is $q = 0$ we are done, so we can assume $q \ge 1$. If $q \ge 1$ we proceed, by induction, straightening out vector fields of the form $\#\alpha$. So let $0 \le k < q$ and assume we have constructed coordinates

$$(x^i, \tilde{y}^j), \text{ where } i \le k, \; k < j \le m,$$

valid on a domain U, and a basis of sections for A over U,

$$\left\{\alpha^i, \tilde{\alpha}^s\right\}, \text{ where } i \le k, \; k < s \le r,$$

such that

$$\#\alpha^i = \frac{\partial}{\partial x^i}, \qquad (i \leq k),$$
$$\#\tilde{\alpha}^s = \sum_j b^{sj} \frac{\partial}{\partial \tilde{y}^j}, \qquad (k < s \leq r),$$

where $b^{sj} \in C^\infty(U)$ depend only on the $\tilde{y}$'s. Since $q > k$, there exists an s such that the vector field $\#\tilde{\alpha}^s$ does not vanish at x_0. By relabeling, we can assume that $s = k+1$ and we set $\alpha^{k+1} = \tilde{\alpha}^{k+1}$.

By straightening out $\#\alpha^{k+1}$, we can perform a change of coordinates

$$x^{k+1} = x^{k+1}(\tilde{y}^{k+1}, \dots, \tilde{y}^m),$$
$$y^j = y^j(\tilde{y}^{k+1}, \dots, \tilde{y}^m), \qquad j = k+2, \dots, m,$$

such that

$$\#\alpha^{k+1} = \frac{\partial}{\partial x^{k+1}},$$
$$\#\tilde{\alpha}^s = b^{s,k+1} \frac{\partial}{\partial x^{k+1}} + \cdots$$

Replacing $\tilde{\alpha}^s$ by $\tilde{\alpha}^s - b^{s,k+1}\alpha^{k+1}$, we see that we can assume $b^{s,k+1} = 0$. Therefore,

$$\#\tilde{\alpha}^s = \sum_j b^{sj} \frac{\partial}{\partial y^j},$$

where $b^{sj} = b^{sj}(x^{k+1}, y^{k+2}, \cdots, y^m)$.

Using $\#[\alpha^{k+1}, \tilde{\alpha}^s] = [\#\alpha^{k+1}, \#\tilde{\alpha}^s]$ for $s > k+1$, we see that

$$[\alpha^{k+1}, \tilde{\alpha}^s] = \sum_{t > k+1} c_t^{s,k+1} \tilde{\alpha}^t,$$

where the structure functions are related by

$$\frac{\partial b^{sj}}{\partial x^{k+1}} = \sum_{u > k+2} c_u^{s,k+1} b^{uj}.$$

We can think of this equation as a time-dependent linear o.d.e. for b^{sj} in the variable x^{k+1}. Let us denote by $X(x^{k+1})$ the fundamental matrix of solutions such that $X(0) = I$, and by $Y(x^{k+1})$ its inverse. We consider new sections

$$\alpha^s = \sum_{t > k+2} Y_t^s(x^{k+1}) \tilde{\alpha}^t.$$

Then we find

$$\begin{aligned}\#\alpha^s &= \sum_j \sum_{t>k+2} Y_t^s(x^{k+1}) b^{tj}(x^{k+1}, y^{k+2}, \cdots, y^m) \frac{\partial}{\partial y^j} \\ &= \sum_j b^{sj}(0, y^{k+2}, \cdots, y^m) \frac{\partial}{\partial y^j}.\end{aligned}$$

We conclude that there exist coordinates (x^i, y^j) and sections $\{\alpha^s\}$, as in the statement of the theorem, such that (9) and (10) hold, for some smooth functions $b^{sj} \in C^\infty(U)$ depending only on the y's. Since at x_0 the bundle map $\#$ has rank q, we must have $b^{sj}(0) = 0$.

Comparing coefficients of $\frac{\partial}{\partial x^i}$ in $\#[\alpha^s, \alpha^t] = [\#\alpha^s, \#\alpha^t]$ we check easily that the structure functions $c_u^{st} \in C^\infty(U)$ must vanish for $u \leq q$. Using the Jacobi identity, we find for $i \leq q$ and $q < s, t \leq r$,

$$\#[\alpha^i, [\alpha^s, \alpha^t]] = [[\#\alpha^i, \#\alpha^s], \#\alpha^t]] + [\#\alpha^s, [\#\alpha^i, \#\alpha^t]] = 0.$$

On the other hand,

$$\begin{aligned}\#[\alpha^i, [\alpha^s, \alpha^t]] &= \#[\alpha^i, \sum_{u>q} c_u^{st} \alpha^u] \\ &= [\#\alpha^i, \sum_{u>q} c_u^{st} \#\alpha^u] = \sum_j \sum_{u>q} \frac{\partial c_u^{st}}{\partial x^i} b^{uj} \frac{\partial}{\partial x^j},\end{aligned}$$

so (12) follows. ▮

In general, the structure functions that appear in relations (11) will depend both on the x's and y's variables, subject to (12). For special classes of Lie algebroids one might have extra information that leads to further simplification of the structure functions. For example, in the case of a Poisson manifold, one always has the relationship:

$$c_k^{ij} = \frac{\partial b^{ij}}{\partial x^k}.$$

Then, all structure functions in (11) depend only on the y's variables, and one obtains the Weinstein Splitting Theorem.

Note that Theorem 1.1 *is not* the Weinstein Splitting Theorem for the Lie-Poisson structure on A^*. The reason is that we are only allowed to make changes of coordinate of the base manifold M and of sections of A. These lead to changes of coordinates of A^* which are linear in the fiber variables:

$$y^i = y^i(x^1, \dots, x^m), \qquad \eta^s = \sum_t a_t^s(x^1, \dots, x^m)\xi^t.$$

These changes of coordinate are usually not sufficient to obtain the Weinstein splitting for A^*.

As we mentioned above, a simple consequence of the Local Splitting Theorem is that the generalized distribution Im# is integrable.

1.3. Transverse Structure

The local splitting of a Lie algebroid can be used to define a transverse Lie algebroid structure, similar to the case of a Poisson manifold. We first give a more invariant description, and later come back to the local coordinate approach.

First we observe that every (embedded) submanifold $N \subset M$ which is transverse to the orbit foliation

$$TN + \mathrm{Im}(\#) = TM,$$

has a natural induced Lie algebroid structure $A_N \to N$. We take for A_N the vector bundle over N with fibers

$$(A_N)_x = \{\alpha \in A_x : \#\alpha \in T_x N\}.$$

Because of the transversality assumption, this is indeed a subbundle of A. The anchor map $\# : A_N \to TN$ is obtained simply by restriction of #. Also, every section α in $\Gamma(A_N)$ extends to a section $\tilde{\alpha}$ of A defined in an open set containing N, and given two sections $\alpha, \beta \in \Gamma(A_N)$, we set

$$[\alpha, \beta]_{A_N}(x) \equiv [\tilde{\alpha}, \tilde{\beta}](x).$$

One checks that (i) this bracket does not depend on the extensions considered, and (ii) that it defines a section of A_N. It follows that $(A_N, \#, [\ ,\ ]_{A_N})$ is a Lie algebroid over N.

The notion of *transverse Lie algebroid structure* is based on the following result, also inspired in Poisson geometry (cf. [33], Section 2).

THEOREM 1.2 (Transverse Structure). *Let L be a leaf of the orbit foliation of A, and suppose N_0 and N_1 are submanifolds of M of complementary dimension to L and intersecting L transversally on a single point. Then there exists an automorphism of A which maps a neighborhood V_0 of*

$N_0 \cap L$ in N_0 onto a neighborhood V_1 of $N_1 \cap L$ in N_1, and which induces an isomorphism of the induced Lie algebroid structures on the neighborhoods.

Proof. If $x_0 = L \cap N_0$ and $x_1 = L \cap N_1$, there exists a piece-wise smooth path made of orbits of vector fields of the form $\#\alpha$, with α a section of A. Integrating sections we can map x_0 to x_1, so we may assume that these points of intersection are actually the same.

Around $x_0 = x_1$ we choose coordinates (x^i, y^j) and sections $\{\alpha^s\}$ as in the Local Splitting Theorem. We interpolate between N_0 and N_1 by a family of manifolds N_t defined by equations of the form

$$x^i = X^i(y^1, \dots, y^{m-q}, t), \qquad (i = 1, \dots, q).$$

Then we look for a time-dependent section α_t which, by integration, gives a Lie algebroid automorphism $\Phi_t : A \to A$, covering a diffeomorphism $\phi_t : M \to M$, which maps a neighborhood of $x_0 = x_1$ in N_0 onto a neighborhood of N_t.

Let us write $\alpha_t = \sum_s a_s(x^i, y^j, t)\alpha^s$. In order for the ϕ_t to track the N_t we must have the equations

$$a_i = \sum_{j,s} \frac{\partial X^i}{\partial y^j} b^{sj} a_s + \frac{\partial X^i}{\partial t}, \qquad (i = 1, \dots, q)$$

satisfied along N_t. It is clear than one can choose a_s such that this equations holds. Integration of α_t gives a Lie algebroid automorphism

$$\Phi_t^{\alpha_t} : A \to A$$

which induces a Lie algebroid isomorphism between A_{N_0} and A_{N_t}. ∎

The transversal geometry to a Lie algebroid, around a point, is described by a *transversal algebroid germ*, that is to say a germ of a Lie algebroid for which the anchor vanishes at the base point. For all cross sections N to L the induced Lie algebroid structures A_N are locally isomorphic, but there is no natural choice for this transverse structure. In other words,, we have a well defined notion of *transverse Lie algebroid structure* along a leaf L.

In the local splitting coordinates (x^i, y^j) and sections $\{\alpha^s\}$ given by Theorem 1.1, the transverse Lie algebroid structure $A_N \to N$, has coordinates (y^j) in the base, sections $\left\{\bar{\alpha}^s(y) \equiv \alpha^s(0, y^j) : \ q < s \leq r\right\}$, anchor map

$$\#\bar{\alpha}^s = \sum_{j>q} b^{sj}(y) \frac{\partial}{\partial y^j}, \qquad (s > q)$$

and Lie algebra structure

$$[\bar{\alpha}^s, \bar{\alpha}^t] = \sum_{u>q} c_u^{st}(0,y)\bar{\alpha}^u.$$

Now the Local Splitting Theorem can be stated as follows: Let x_0 be any point in a Lie algebroid A, and denote by L the leaf through x_0 and by N a cross-section to L at x_0. Then, locally, A is an extension of A_N by TL, i.e., we have Lie algebroids morphisms

$$A_N \to A \to TL$$

such that the corresponding bundle maps form a short exact sequence:

$$0 \to T^*L \to A^* \to A_N^* \to 0.$$

1.4. Linear Approximation to a Lie Algebroid

By a *linear Lie algebroid* we shall mean a Lie algebroid $\pi : A \to M$ satisfying the following properties:

(i) The base $M = V$ is a vector space so $\pi : A \to M$ is a trivial bundle;
(ii) For any trivialization, the bracket of any constant sections α and β is a constant section $[\alpha, \beta]$;
(iii) For any trivialization, the vector field $\#\alpha$ is linear whenever α is a constant section;

Conditions (i) and (ii) mean that A is a transformation Lie algebroid: $A = \mathfrak{g} \times V$ and $\# : \mathfrak{g} \to \mathcal{X}^1(V)$ is an action of the Lie algebra $\mathfrak{g}$ on V. Condition (iii) means that this action is linear. So a linear Lie algebroid is just a representation of a Lie algebra.

Now let A be any Lie algebroid and fix $x_0 \in M$. The normal space $N_{x_0} = T_{x_0}M/\mathrm{Im}\#_{x_0}$ carries a natural linear action of the Lie algebra $\mathfrak{g} = Ker\#_{x_0}$, and the associated transformation Lie algebroid is called the *linearization of A at x_0*. This linearization can be seen in two different ways:

(a) We take $\mathfrak{g} = Ker\#_{x_0}$ with the Lie algebra structure induced from A, and we define a linear action of $\mathfrak{g}$ on N_{x_0} by

$$\rho(z) = \delta_{x_0}(\#\alpha), \qquad z \in \mathfrak{g}$$

where α is such that $\alpha_{x_0} = z$ and $\delta_{x_0}X$ is the linearization of the vector field X at x_0. Since the flow of $\#\alpha$ preserves the orbit foliation this actually defines a linear endomorphism of $N_{x_0} = T_{x_0}M/\mathrm{Im}\#_{x_0}$. If α' is another section with $\alpha'_{x_0} = z$, one checks that $\delta_{x_0}(\#\alpha)$ and $\delta_{x_0}(\#\alpha')$ induce the same endomorphism of N_{x_0}. The associated transformation Lie algebroid $\mathfrak{g} \times N_{x_0}$ is the linearization of A.

(b) Again we take the trivial vector bundle $\mathfrak{g} \times N_{x_0} \to N_{x_0}$ and we define the anchor $\tilde{\#} : \mathfrak{g} \times N_{x_0} \to TN_{x_0}$ to be the intrinsic derivative at x_0 of the bundle map $\# : A \to TM$. Then we define the bracket on constant sections to be the pointwise bracket, and we extend to any section by requiring the Leibniz identity to hold.

If we pick splitting coordinates (x^i, y^j) and basis of sections $\{\alpha^s\}$ around x_0, then $\{e^s \equiv \alpha^s(x_0) : s > q\}$ give a basis for $\mathfrak{g} = \mathrm{Ker}\,\#_{x_0}$, with structure constants $c_u^{st}(x_0)$. The tangent vectors $v_j = \left.\frac{\partial}{\partial y^j}\right|_{x_0}$ induce a basis for N_{x_0}, determining linear coordinates (w^j) and relative to these coordinates the anchor map is given by:

$$\tilde{\#}(e^s) = \sum_{j,k} \frac{\partial b^{sj}}{\partial y^k}(0) w^k \frac{\partial}{\partial w^j},$$

where we view the $\{e^s\}$ as constant sections of $\mathfrak{g} \times N_{x_0}$.

One should notice that the transverse Lie algebroid structure is an equivalence class of isomorphic structures, for which there is no natural choice of representative, while the linearization at x_0 lives on a well-defined bundle over the normal space N_{x_0}. The problem of linearizing a Lie algebroid is discussed in [30].

1.5. Lie Algebroid Cohomology

The existence of a Lie bracket on the space of sections of a Lie algebroid leads to a calculus on its sections analogous to the usual Cartan calculus on differential forms. In this paragraph we give only the relevant formulas for Lie algebroid cohomology we shall need later, and refer the reader to the monograph [25] for further details.

One defines the exterior differential $d_A : \Gamma(\wedge^\bullet A^*) \longrightarrow \Gamma(\wedge^{\bullet+1} A^*)$ by:

$$\begin{aligned} d_A Q(\alpha_0, \dots, \alpha_r) &= \frac{1}{r+1} \sum_{k=0}^{r+1} (-1)^k \#\alpha_k (Q(\alpha_0, \dots, \widehat{\alpha}_k, \dots, \alpha_r) \\ &+ \frac{1}{r+1} \sum_{k<l} (-1)^{k+l+1} Q([\alpha_k, \alpha_l], \alpha_0, \dots, \widehat{\alpha}_k, \dots, \widehat{\alpha}_l, \dots, \alpha_r). \quad (13) \end{aligned}$$

where $\alpha_0, \dots, \alpha_r$ are sections of A. This differential satisfies:

$$d_A^2(Q) = 0, \tag{14}$$
$$d_A(Q_1 \wedge Q_2) = d_A Q_1 \wedge Q_2 + (-1)^{\deg Q_1} Q_1 \wedge d_A Q_2. \tag{15}$$

The cohomology associated with d_A is called the *Lie algebroid cohomology* of A (with trivial coefficients) and is denoted by $H^\bullet(A)$.

Define a homomorphism of exterior algebras $\#^* : \Omega^\bullet(M) \to \Gamma(\wedge^\bullet A^*)$ by setting:

$$\#^*\omega(\alpha_1, \dots, \alpha_r) = (-1)^r \omega(\#\alpha_1, \dots, \#\alpha_r).$$

We compute

$$d_A \#^* \omega = \#^* d\omega. \tag{16}$$

so there is a ring homomorphism

$$\#^* : H^\bullet_{\text{de Rham}}(M) \to H^\bullet(A).$$

For the examples we have mentioned above these cohomology groups are well known. Special cases to be used later are:

(i) $A = TM$, where $H^\bullet(A) = H^\bullet_{\text{de Rham}}(M)$ is the *de Rham cohomology*;

(ii) $A = T^*M$ with (M, Π) a Poisson manifold, where one obtains the *Poisson cohomology* of M denoted $H^\bullet_\Pi(M)$;

(iii) $A = T\mathcal{F} \subset TM$ an integrable subbundle associated with a regular foliation $\mathcal{F}$, where one gets the *tangential cohomology* denoted $H^\bullet_{\mathcal{F}}(M)$;

(iv) $A = \mathfrak{g} \times V$ the Lie algebroid associated with a Lie algebra representation $\rho : \mathfrak{g} \to \text{Aut}\,(V)$, where one gets the *Chevalley-Eilenberg cohomology* $H^\bullet(\mathfrak{g}, \rho)$;

The Lie algebroid cohomology, in general, will not be homotopy invariant and hence it may be hard to compute (to say the least). This is intimately related with the singular behaviour of the orbit foliation, one of the main topics to be discussed here.

Finally, we note, for later reference, that if $\phi : M \to N$ is a smooth map its *A-differential* is the bundle map $d_A\phi : A \to TN$ defined by:

$$d_A\phi(\alpha_x) = d_x\phi \cdot \#\alpha_x, \qquad \alpha_x \in A_x. \tag{17}$$

If $N = \mathbb{R}$ this notation is consistent with the A-differential introduced above, if we think of sections in $\Gamma(\wedge^0 A^*)$ as functions on M.

2. LIE ALGEBROID CONNECTIONS

2.1. Connections on Principal Bundles

Let $P(M, G)$ be a smooth principal bundle over the manifold M with structure group G. We let $p : P \to M$ be the projection, and for each $u \in P$ we denote by $G_u \subset T_u(P)$ the subspace consisting of vectors tangent to the fiber through u. If we let $\sigma : \mathfrak{g} \to \mathcal{X}^1(P)$ be the infinitesimal G-action on P, we have $G_u = \{\sigma(B)_u |\ B \in \mathfrak{g}\}$.

Now let p^*A denote the pullback bundle of A by p. There is a bundle map $\widehat{p} : p^*A \to A$ which makes the following diagram commutative

$$\begin{array}{ccc} p^*A & \xrightarrow{\ \widehat{p}\ } & A \\ {\scriptstyle \widehat{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ P & \xrightarrow[\ p\ ]{} & M \end{array}$$

where on the vertical arrows we have the canonical projections. Recalling that $p^*A = \{(u, \alpha) \in P \times A : p(u) = \pi(\alpha)\}$, we see that we have a natural right G-action on p^*A defined by $(u, \alpha) \cdot a \equiv (ua, \alpha)$, if $a \in G$. Our basic definition is then the following:

Definition 2.1. An A-connection in the principal bundle $P(M, G)$ is a smooth bundle map $h : p^*A \to TP$, such that:

(CI) h is horizontal, i.e., the following diagram commutes:

$$\begin{array}{ccc} p^*A & \xrightarrow{\ h\ } & TP \\ {\scriptstyle \widehat{p}}\big\downarrow & & \big\downarrow{\scriptstyle p_*} \\ A & \xrightarrow[\ \#\ ]{} & TM \end{array}$$

(CII) h is G-invariant, i.e., we have

$$h(ua, \alpha) = (R_a)_* h(u, \alpha), \qquad \text{for all } a \in G;$$

The subspace of T_uP formed by all horizontal lifts $h(u, \alpha) \in T_uP$, where $(u, \alpha) \in p^*A$, is denoted by $\mathcal{H}_u$. The assignment $u \mapsto \mathcal{H}_u$ is a smooth distribution on P called the *horizontal distribution* of the connection([2]).

[2] In this paper "smooth distributions" are always in the sense of Sussman [27], so that for each point $u_0 \in P$ there exists a neighborhood $u_0 \in U \subset P$ and smooth vector fields $X_1, \ldots, X_r$ in U, such that $\mathcal{H}_u = \text{span}\,\{X_1|_u, \ldots, X_r|_u\}$ for all $u \in U$.

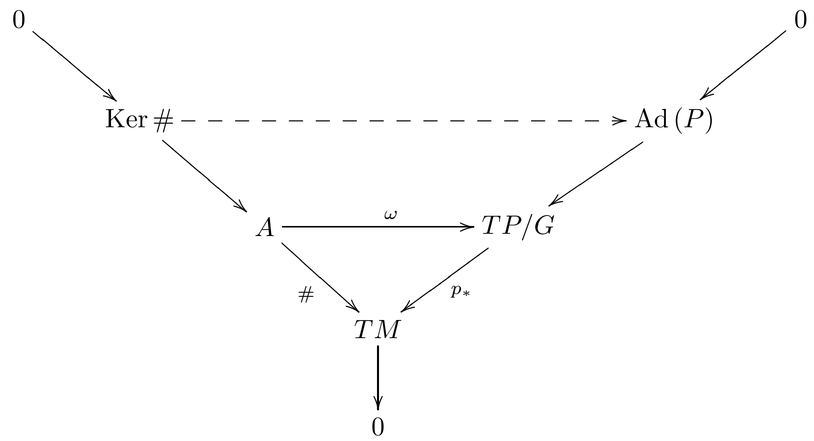
Note that, unlike the ordinary case, the rank of the horizontal distribution will vary, and that this distribution does not define the connection uniquely.

It follows from (CI) in the definition of an A-connection, that the horizontal spaces $\mathcal{H}_u$ project onto the tangent space T_xL to the orbit leaf L through $x = p(u)$. In general, we have neither $T_uP = G_u + \mathcal{H}_u$ nor $G_u \cap \mathcal{H}_u = \{0\}$. As usual, a vector $X \in T_uP$ will be called *vertical* (resp. *horizontal*), if it lies in G_u (resp. $\mathcal{H}_u$). Since, in general, a tangent vector to P does not split into a sum of an horizontal and a vertical component, the usual definitions of lift of curves, connection form, etc., do not make sense in this context. We will show below how to define these notions appropriately.

Recall that the *Atiyah sequence* of the principal bundle $P(M, G)$ is the short exact sequence of vector bundles

$$0 \longrightarrow \mathrm{Ad}\,(P) \xrightarrow{j} TP/G \xrightarrow{p_*} TM \longrightarrow 0\ ,$$

where $Ad(P) = \frac{P\times\mathfrak{g}}{G}$ is the associated bundle to P obtained from the adjoint representation of G on $\mathfrak{g}$ (in fact, this is a short exact sequence of Lie algebroids for the obvious Lie algebroid structures). Now, if $h : p^*A \to TP$ is a connection, the G-invariance implies that h induces a bundle map $\omega : A \simeq p^*A/G \to TP/G$. The following commutative diagram, which was suggested to me by Alan Weinstein, is helpful in understanding the relationship between the different geometric objects associated with an A-connection:



Note, however, that in this diagram Ker # is not a vector bundle, except if A is a regular Lie algebroid. The dash arrow will be explained later.

In the usual theory of covariant connections, one has $A = TM$ and # is the identity map, so a connection can be thought as a splitting of the Atiyah sequence. This is the approach taken by Mackenzie in [25] and which lead him to the introduction of a connection for a *transitive Lie algebroid* as a splitting of the analogous short exact sequence:

$$0 \longrightarrow \operatorname{Ker}\# \longrightarrow A \xrightarrow{\#} TM \longrightarrow 0\,.$$

This approach was also followed by Kubarski in his theory of characteristic classes for regular Lie algebroids ([22, 22]).

2.2. Connection 1-section and Curvature 2-section

Let $E \to M$ be any vector bundle. In the theory of Lie algebroids, elements of $\Gamma^\bullet(A^*, E) \equiv \Gamma(\wedge^\bullet A^*) \otimes \Gamma(E)$ play the role of (E-valued) differential forms. We shall refer to an element in $\Gamma^r(A^*, E)$ as an E-valued r-section, or simply as an *r-section* if it is clear from the context what E is. In case E is also a Lie algebroid, we have an induced (super) Lie bracket on $\Gamma^\bullet(A^*, E)$ by setting:

$$[P, Q](\alpha_1, \dots, \alpha_{p+q}) = \frac{1}{(p+q)!}\sum_\sigma (-1)^\sigma [P(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(p)}), Q(\alpha_{\sigma(p+1)}, \dots, \alpha_{\sigma(p+q)})], \quad (18)$$

where the sum is over all permutations σ of $p+q$ elements.

To deal with connections we let $E = TP/G$. As we remarked above, a connection h defines a bundle map $\omega : A \simeq p^*A/G \to TP/G$, i.e., an element in $\Gamma^1(A^*, TP/G)$, and we call ω the *connection 1-section*. We define the *exterior A-derivative*

$$D : \Gamma^\bullet(A^*, TP/G) \to \Gamma^{\bullet+1}(A^*, TP/G)$$

by setting:

$$\begin{aligned} DQ(\alpha_0, \dots, \alpha_q) &= \frac{1}{q+1}\sum_{k=0}^{q} (-1)^k [\omega(\alpha_k), Q(\alpha_0, \dots, \widehat{\alpha}_k, \dots, \alpha_q)] \\ &+ \frac{1}{q+1}\sum_{k<l} (-1)^{k+l} Q([\alpha_k, \alpha_l], \alpha_0, \dots, \widehat{\alpha}_k, \dots, \widehat{\alpha}_l, \dots, \alpha_q). \quad (19) \end{aligned}$$

Now we introduce the *curvature 2-section* $\Omega \in \Gamma^2(A^*, TP/G)$ of the connection by setting:

$$\Omega(\alpha, \beta) \equiv \frac{1}{2}\left([\omega(\alpha), \omega(\beta)] - \omega([\alpha, \beta])\right), \tag{20}$$

for $\alpha, \beta \in \Gamma(A)$. The curvature 2-section measures to which extent the horizontal distribution fails to be integrable.

Proposition 2.1. *The curvature 2-section satisfies the structure equation:*

$$\Omega = D\omega - \frac{1}{2}[\omega, \omega]. \tag{21}$$

Proof. We compute using (18) and (19):

$$\begin{aligned} D\omega(\alpha, \beta) &= [\omega(\alpha), \omega(\beta)] - \frac{1}{2}\omega([\alpha, \beta]), \\ [\omega, \omega](\alpha, \beta) &= [\omega(\alpha), \omega(\beta)] \end{aligned}$$

so we have

$$\Omega(\alpha, \beta) = D\omega(\alpha, \beta) - \frac{1}{2}[\omega, \omega](\alpha, \beta) = \frac{1}{2}\left([\omega(\alpha), \omega(\beta)] - \omega([\alpha, \beta])\right),$$

which shows that (21) is satisfied. ▮

The horizontal distribution in general will have non-constant rank. Still, if we call a *flat A-connection* an A-connection for which the horizontal distribution is integrable, we have:

Corollary 2.1. *An A-connection is flat iff its curvature 2-section vanishes.*

Proof. By a result of Hermann [14], a generalized distribution associated with a vector subspace $\mathcal{D} \subset \mathcal{X}^1(M)$ is integrable iff it is involutive and rank invariant. Taking $\mathcal{D} = \{h(\alpha) : \alpha \in \Gamma(A)\}$, so that $\mathcal{H}_u = \{X(u) : X \in \mathcal{D}\}$, (20) shows that $\mathcal{D}$ is involutive iff the curvature 2-section vanishes. Hence, all it remains to show is that if the curvature vanishes and $\gamma(t)$ is an integral curve of $h(\alpha)$ then $\dim \mathcal{H}_{\gamma(t)}$ is constant, for all small enough t.

Fix $\alpha \in \Gamma(A)$ and let ϕ_t^α be the flow X_{f_α}, let ψ_t^α be the flow of $\#\alpha$ and let $\tilde{\psi}_t^\alpha$ be the flow of $h(\alpha)$. We have $\psi_t^\alpha = p \circ \tilde{\psi}_t^\alpha = \pi \circ \phi_t^\alpha$ (see prop. 1.1).

If $\beta \in \Gamma(A)$ we claim that

$$(\tilde{\psi}_t^\alpha)_* h(\beta) = h(\phi_t^\alpha \beta),$$

for small enough t. In fact, the infinitesimal version of this relation is

$$[h(\alpha), h(\beta)] = h([\alpha, \beta]),$$

which holds, since we are assuming that the curvature vanishes.

Therefore, the flow $\tilde{\psi}_t^\alpha$ gives an isomorphism between $\mathcal{H}_{\gamma(0)}$ and $\mathcal{H}_{\gamma(t)}$, for small enough t, so $\mathcal{D}$ is rank invariant. ▮

We also have an analogue of the usual Bianchi's identity:

PROPOSITION 2.2. *The curvature 2-section Ω satisfies the Bianchi identity:*

$$D\Omega = 0. \tag{22}$$

Proof. From expression (20) for the curvature and the definition (19) of the exterior A-derivative, we compute:

$$\begin{aligned} D\Omega(\alpha,\beta,\gamma) = \bigodot_{\alpha,\beta,\gamma} \frac{1}{2} \left([\omega(\alpha),[\omega(\beta),\omega(\gamma)]] - [\omega([\alpha,\beta]),\omega(\gamma)]\right) \\ - \bigodot_{\alpha,\beta,\gamma} \frac{1}{2} \left([\omega(\alpha),\omega([\alpha,\beta])] + \omega([[\alpha,\beta],\gamma])\right), \end{aligned}$$

where the symbol $\bigodot$ denotes cyclic sum over the subscripts. The first and fourth term vanish because of Jacobi's identity, while the two middle terms cancel out. ▮

Let $s_j : U_j \to P$ be a local section of $P(M,G)$ defined over an open set $U_j \subset M$. Then we have a trivializing isomorphism $\psi_j : p^{-1}(U_j) \to U_j \times G$ such that $s_j(x) = \psi_j^{-1}(x,e)$, where $e \in G$ is the identity. If $\mathfrak{g}$ denotes the Lie algebra of G, we also have an isomorphism $T_{U_j}P/G \simeq TU_j \times \mathfrak{g}$. Given an A-connection h, the corresponding connection 1-section ω trivializes over U_j as

$$\omega(\alpha) \simeq (\#\alpha, \omega_j(\alpha)),$$

for some $\omega_j \in \Gamma(A^*, \mathfrak{g})$. We also have the following alternative description of ω_j: if $\alpha \in \Gamma(A)$, $x \in U_j$, and $u = s_j(x)$, then

$$X_u = (s_j)_* \#\alpha_x - h(s_j(x), \alpha_x) \in T_u P$$

is a vertical vector since, by (CI), we have:

$$p_* X_u = p_* \cdot (s_j)_* \# \alpha_x - p_* h(s_j(x), \alpha_x) = \#\alpha_x - \#\alpha_x = 0.$$

Then $\omega_j(\alpha)_x$ is the unique element $B \in \mathfrak{g}$ such that $X_u = \sigma(B)_u$, which exists by (CII). The $\{\omega_j\}$ will be called the *local connection 1-sections* of the A-connection.

If $s_k : U_k \to P$ is another local section with $U_j \cap U_k \neq \emptyset$, we denote by $\psi_{jk} : U_j \cap U_k \to G$ the corresponding transition function. The following proposition gives the transformation rule for the local connection 1-sections. The proof is similar to the proof for the Poisson case ([10], Prop. 1.3.1) and so it will be omitted.

Proposition 2.3. *The local connection 1-sections $\{\omega_j\}$ transform by*

$$\omega_k = Ad(\psi_{jk}^{-1})\omega_j + \psi_{jk}^{-1} d_A \psi_{jk}, \quad \text{on } U_j \cap U_k. \tag{23}$$

Conversely, given a family of $\mathfrak{g}$-valued 1-sections $\{\omega_j\}$, each defined in U_j and satisfying relations (23), there is a unique A-connection in $P(M,G)$ which gives rise to the $\{\omega_j\}$.

For the local description of the curvature we observe that the 2-section Ω is vertical: in fact, by (20), for $\alpha, \beta \in \Gamma(A)$ we have

$$\begin{aligned} p_*\Omega(\alpha,\beta) &= \frac{1}{2}\left([p_*\omega(\alpha), p_*\omega(\beta)] - p_*\omega([\alpha,\beta])\right) \\ &= \frac{1}{2}\left([\#\alpha, \#\beta] - \#[\alpha,\beta]\right) = 0. \end{aligned}$$

Hence, over each trivializing neighborhood U_j, the curvature 2-section trivializes as

$$\Omega(\alpha,\beta) \simeq (0, \Omega_j(\alpha,\beta)),$$

for some $\Omega_j \in \Gamma^2(A^*, \mathfrak{g})$.

The *local curvature 2-sections* $\{\Omega_j\}$ satisfy local versions of the structure equation (21) and Bianchi identity (22). Again, the proof is similar to the Poisson case ([10], Prop. 1.4.1) and will be omitted.

Proposition 2.4. *The local curvature 2-sections of a connection transform by*

$$\Omega_k = Ad(\psi_{jk}^{-1})\Omega_j, \quad \text{on } U_j \cap U_k. \tag{24}$$

Moreover, they are related to the local 1-sections by the first structure equation

$$\Omega_j = d_A\omega_j + \frac{1}{2}[\omega_j, \omega_j]. \tag{25}$$

and they satisfy the Bianchi identity:

$$d_A\Omega_j + [\omega_j, \Omega_j] = 0. \tag{26}$$

Note that since the curvature is a vertical 2-section, given $\alpha, \beta \in \Gamma(A)$ we can identify $\Omega(\alpha, \beta)$ with a $\mathfrak{g}$-valued map in P. Under this identification relation (20) can be written as:

$$[h(\alpha), h(\beta)] - h([\alpha, \beta]) = -2\sigma(\Omega(\alpha, \beta)), \tag{27}$$

and we have

$$\Omega(\alpha, \beta)_{s_j(x)} = \Omega_j(\alpha, \beta)_x.$$

Later we shall use this identification without further notice.

2.3. Parallelism and Holonomy

If $\gamma : [0, 1] \to M$ is a smooth curve lying on a leaf L of the Lie algebroid A, then γ is also smooth as map $\gamma : [0, 1] \to L$. This follows from the existence of "canonical coordinates" for M given by the Local Splitting Theorem. Also, by the same theorem, we can choose (not uniquely) a piecewise smooth family $t \mapsto \alpha(t) \in A$ such that $\#\alpha(t) = \dot{\gamma}(t)$. Recalling definition 0.4, this means that any path that lies on a leaf is the base path of some A-path. Clearly, if $\#$ is not injective, different A-paths can have the same base path.

Let $\alpha(t)$ be an A-path with base path $\gamma(t)$. For any $u_0 \in P$ with $p(u_0) = \gamma(0)$ one can show, using the G-invariance of h, that there exists a unique horizontal lift $\tilde{\gamma} : [0, 1] \to P$, which satisfies the system

$$\begin{cases} \dot{\tilde{\gamma}}(t) = h(\tilde{\gamma}(t), \alpha(t)), \\ \tilde{\gamma}(0) = u_0. \end{cases} \tag{28}$$

Hence, we can define parallel displacement of the fibers along an A-path $\alpha(t)$ in the usual way: if $u_0 \in p^{-1}(\gamma(0))$ we define $\tau(u_0) = \tilde{\gamma}(1)$, where $\tilde{\gamma}(t)$ is the unique horizontal lift of $\alpha(t)$ starting at u_0. We obtain a map $\tau : p^{-1}(\gamma(0)) \to p^{-1}(\gamma(1))$, which will be called *parallel displacement* of the fibers along the A-path $\alpha(t)$. It is clear, since horizontal curves are mapped by R_a to horizontal curves, that parallel displacement commutes with the action of G:

$$\tau \circ R_a = R_a \circ \tau. \tag{29}$$

Therefore, parallel displacement is an isomorphism between the fibers.

If $x \in M$ belongs to the leaf L, let $\Omega(L, x)$ be the loop space of L at x. An A-path $\alpha(t)$ for which the base path is a loop $\gamma \in \Omega(L, x)$ will be called an A-*loop* in L based at x. Parallel displacement along such an A-loop $\alpha(t)$ gives a an isomorphism of the fiber $p^{-1}(x)$ into itself. The set of all such isomorphisms forms the holonomy group of the A-connection, with reference point x, and is denoted $\Phi(x)$. Similarly, one has the restricted holonomy group, with reference point x, denoted $\Phi^0(x)$, which is defined by using A-loops in L with base paths homotopic to the constant path.

For any $u \in p^{-1}(x)$ we can also define the holonomy groups $\Phi(u)$ and $\Phi^0(u)$. Just as in the covariant case, $\Phi(u)$ is the subgroup of G consisting of those elements $a \in G$ such that u and ua can be joined by an horizontal curve. We have that $\Phi(u)$ is a Lie subgroup of G, with connected component of the identity $\Phi^0(u)$, and we have isomorphisms $\Phi(u) \simeq \Phi(x)$ and $\Phi(u)^0 \simeq \Phi(x)^0$.

If $x, y \in M$ belong to the same leaf then the holonomy groups $\Phi(x)$ and $\Phi(y)$ are isomorphic. This is because if $u, v \in P$ are points such that, for some $a \in G$, there exists an horizontal curve connecting ua and v, then $\Phi(v) = Ad(a^{-1})\Phi(u)$, so $\Phi(u)$ and $\Phi(v)$ are conjugate in G. However, if $x, y \in M$ belong to different leaves the holonomy groups $\Phi(x)$ and $\Phi(y)$ will be, in general, non-isomorphic.

The holonomy groups can be given an infinitesimal description as in the Ambrose-Singer Holonomy Theorem. For that, suppose that $\gamma \in A_x$ satisfies $\#\gamma = 0$. If $u \in p^{-1}(x)$ we set:

$$\Lambda(\gamma)_u \equiv \omega_j(\gamma)_x,$$

where j is such that $s_j(x) = u$. It follows from the transformation rule (23) that the previous formula gives a well defined map $\Lambda : \Omega(L, x) \to \mathfrak{g}$ (the dashed arrow in the diagram above). Also, denote by $P(u_0)$ the set of points $u \in P$ that can be joined to u_0 by an horizontal curve. We have:

THEOREM 2.1 (Holonomy Theorem). *Given any A-connection in the principal bundle $P(M, G)$ and $u_0 \in P$, the Lie algebra of the holonomy group $\Phi(u_0) \subset G$ is the ideal of $\mathfrak{g}$ spanned by all elements $\Omega(\alpha, \beta)_u$ and $\Lambda(\gamma)_u$, where $u \in P(u_0)$ and $\alpha, \beta, \gamma \in A_{p(u)}$, with $\#\gamma = 0$.*

The proof is also analogous to the Poisson case ([10], Thm. 1.5.2) and so it will be omitted.

Note the presence of the extra term Λ in the Holonomy Theorem. This means that a Lie algebroid connection can be flat and still have non-discrete holonomy, a phenomenon that is not present in the covariant case or whenever $\#$ is injective.

2.4. Relationship to Ordinary Connections

Consider the tangent Lie algebroid $A = TM$ and a TM-connection in $P(M,G)$, i.e., a covariant connection. Its horizontal lift $\bar{h} : p^*TM \to TP$ is completely determined by the horizontal distribution $\mathcal{H}$. For a Lie algebroid $A \to M$, the formula $h(u,\alpha) \equiv \bar{h}(u,\#\alpha)$ defines an A-connection in $P(M,G)$ which is said to be *induced by the covariant connection.* Note that in this case the lift h satisfies:

$$\#\alpha = 0 \Longrightarrow h(u,\alpha) = 0, \qquad (u,\alpha) \in p^*A. \tag{30}$$

This construction shows that there are always A-connections on any principal bundle $P(M,G)$ over a Lie algebroid $A \to M$.

Let $\bar{\omega}$ be the connection 1-form and let $\bar{\Omega}$ be the curvature 2-form of the covariant connection $\bar{h}$. Then it is clear from the definitions given above that the connection 1-section ω and the curvature 2-section Ω of the induced A-connection h are given by:

$$\omega = \#^*\bar{\omega}, \qquad \Omega = \#^*\bar{\Omega}. \tag{31}$$

Also, given trivialization isomorphisms $\{\psi_j\}$, inducing local sections $\{s_j\}$, we see that the associated local connection 1-sections and curvature 2-sections are related by:

$$\omega_j = \#^*\bar{\omega}_j, \qquad \Omega_j = \#^*\bar{\Omega}_j. \tag{32}$$

However, in general, a connection will not satisfy property (30) and we set:

Definition 2.2. An A-connection on a principal bundle $P(M,G)$ is called an $\mathcal{F}$-connection if its horizontal lift satisfies condition (30)

Let us fix one such $\mathcal{F}$-connection on $P(M,G)$. Then, on the pull-back bundle $p_L : i^*P \to L$, we have a naturally induced TL-connection, i.e., a covariant connection. On the total space

$$i^*P = \{(x,u) \in L \times P : i(x) = p(u)\}$$

we define the horizontal lift $\bar{h}_L : p_L^* TL \to T(i^*P)$ by setting

$$\bar{h}_L((x,u),v) = (v, h(u,\alpha)), \qquad (x,u) \in i^*P, v \in T_xL, \tag{33}$$

where we choose any $\alpha \in A_x$ such that $\#\beta = v$, and we are identifying $T(i^*P) = \{(v,w) \in TS \times TP : v = p_*w\}$. Note that if $\#\beta' = \#\beta = v$ we get the same result in (33) since h is an $\mathcal{F}$-connection.

PROPOSITION 2.5. *Let Ω and ω be the connection and curvature sections for an $\mathcal{F}$-connection in $P(M,G)$. For a leaf $i : L \hookrightarrow M$ denote by $\bar{\omega}^L$ and $\bar{\Omega}^L$ the connection 1-form and the curvature 2-form for the induced connection on $i^*P(M,G)$. Then ω and Ω are i-related to $\#^*\bar{\omega}^L$ and $\#^*\bar{\Omega}^L$:*

$$i_*\#^*\bar{\omega}^L = \omega, \qquad i_*\#^*\bar{\Omega}^L = \Omega. \tag{34}$$

Therefore, an $\mathcal{F}$-connection can be thought of as a *family* of ordinary connections over the leaves of M. The connection 1-section ω and the curvature 2-sections Ω are obtained by gluing together the connection 1-sections $\#^*\bar{\omega}^L$ and the curvature 2-sections $\#^*\bar{\Omega}^L$ of the connections on the leaves of M.

For an $\mathcal{F}$-connection, horizontal lifts of A-paths $\alpha(t)$ depend only on the base path $\gamma(t)$. Therefore, one has a well determined notion of horizontal lift of a curve lying on a leaf. It follows that for these connections, parallel displacement can also be defined by first reducing to the pull-back bundle over a leaf and then parallel displace the fibers. Hence, the holonomy groups $\Phi(x)$ and $\Phi^0(x)$ coincide with the usual holonomy groups of the pull-back connection on the leaf L through x.

2.5. Connections on a Vector Bundle

If G acts on the left on a manifold F we denote by $p_E : E(M,F,G,P) \to M$ the fiber bundle associated with $P(M,G)$ with standard fiber F.

Given an A-connection in $P(M,G)$ with horizontal lift $h : p^*A \to TP$, we define the induced horizontal lift $h_E : p_E^*A \to TE$ as follows: given $w \in E$ choose $(u,\xi) \in P \times F$ which is mapped to w, and set

$$h_E(w,\alpha) \equiv \xi_* h(u,\alpha), \tag{35}$$

where we are identifying ξ with the map $P \to E$ which sends an element $u \in P$ to the equivalence class $[u, \xi] \in E$. One can check easily that this definition does not depend on the choice of (u, ξ), so we obtain a well defined bundle map $h_E : p_E^* A \to TE$ which makes the following diagram commute:

$$\begin{array}{ccc} p_E^* A & \xrightarrow{h_E} & TE \\ {\scriptstyle \widehat{p}_E}\downarrow & & \downarrow{\scriptstyle p_{E*}} \\ A & \xrightarrow[\#]{} & TM \end{array} \tag{36}$$

As before, we can define horizontal and vertical vectors in TE, horizontal lifts to E of curves lying on leaves of the orbit foliation, and parallel displacement of fibers of E. We shall call a cross section σ of E over an open set $U \subset M$ *parallel* if $\sigma_*(v)$ is horizontal for all tangent vectors $v \in T_U M$.

Now assume that G acts linearly on a vector space V. On the associated vector bundle $E(M, V, G, P)$ we obtain an horizontal lift $h_E : p_E^* A \to TE$ which has the distinguish property of being *linear*:

$$h_E(w, a_1\alpha_1 + a_2\alpha_2) = a_1 h_E(w, \alpha_1) + a_2 h_E(w, \alpha_2). \tag{37}$$

Conversely, given a bundle map $h_E : p_E^* A \to TE$ satisfying (37) and making the diagram (36) commute, it is associated with some A-connection on the principal bundle $P(M, G)$ through relation (35).

For an A-connection on a vector bundle E we have the notion of A-*derivative* of sections of E along A-paths, analogous to the notion of covariant derivative of sections for covariant connections. Given a section ϕ of E defined along an A-path $\alpha(t)$, its A-*derivative* $\nabla_\alpha \phi$ is the section defined by

$$\nabla_\alpha \phi(t) \equiv \lim_{h \to 0} \frac{1}{h} \left[\tau_t^{t+h}(\phi(\gamma(t+h))) - \phi(\gamma(t)) \right], \tag{38}$$

where $\gamma(t)$ denotes the base path of $\alpha(t)$ and $\tau_t^{t+h} : p_E^{-1}(\gamma(t+h)) \to p_E^{-1}(\gamma(t))$ denotes parallel transport of the fibers from $\gamma(t+h)$ to $\gamma(t)$ along the A-path. The proof of the following proposition is elementary.

PROPOSITION 2.6. *Let ϕ and ψ be sections of E and f a function on M defined along γ. Then*

(i)$\nabla_\alpha(\phi + \psi) = \nabla_\alpha \phi + \nabla_\alpha \psi$;
(ii)$\nabla_\alpha(f\phi) = (f \circ \gamma)\nabla_\alpha \phi + \dot{\gamma}(f)(\phi \circ \gamma)$;

Now let $\alpha \in A_x$ and let ϕ be a cross section of E defined in a neighborhood of x. The A-derivative $\nabla_\alpha \phi$ of ϕ in the direction of α is defined as follows: choose an A-path $\alpha(t)$, with base path $\gamma(t)$, defined for $t \in (-\varepsilon, \varepsilon)$ and such that $\gamma(0) = x$, $\alpha(0) = \alpha$. Then we set:

$$\nabla_\alpha \phi \equiv \nabla_{\alpha(t)} \phi(0). \tag{39}$$

It is easy to see that $\nabla_\alpha \phi$ is independent of the choice of A-path. Clearly, a cross section ϕ of E defined on an open set $U \subset M$ is flat iff $\nabla_\alpha \phi = 0$ for all $\alpha \in A_x$, $x \in M$.

Finally, given $\alpha \in \Gamma(A)$ and ϕ a section of E, we can also define the A-derivative $\nabla_\alpha \phi$ to be the section of E given by:

$$\nabla_\alpha \phi(x) = \nabla_{\alpha_x} \phi. \tag{40}$$

Moreover, we have the following properties of the A-derivative:

Proposition 2.7. *The A-derivative ∇ is a map $\Gamma(A) \times \Gamma(E) \to \Gamma(E)$ that satisfies*

(i)$\nabla_{\alpha+\beta}\phi = \nabla_\alpha \phi + D_\beta \phi$;
(ii)$\nabla_\alpha(\phi + \psi) = \nabla_\alpha \phi + \nabla_\alpha \psi$;
(iii)$\nabla_{f\alpha} = f \nabla_\alpha \phi$;
(iv)$\nabla_\alpha(f\phi) = f \nabla_\alpha \phi + \#\alpha(f)\phi$;

for all $\alpha, \beta \in \Gamma(A)$, $\phi, \psi \in \Gamma(E)$, and $f \in C^\infty(M)$.

It is also true that the A-derivative uniquely determines the connection: Given a map $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$ satisfying properties (i) to (iv) of Proposition 2.7, there exists a unique A-connection on the associated principal bundle $P(M, G)$ for which the induced A-derivative on E is ∇.

For an A-connection in a vector bundle E we define the *curvature section* R to be the section of $\wedge^2 A^* \otimes \mathrm{End}(E)$ given by

$$R(\alpha, \beta)\gamma \equiv s_j(x) \left[\Omega_j(\alpha, \beta) \cdot s_j^{-1}(x)(\gamma) \right]. \tag{41}$$

where $x \in U_j$, $\alpha, \beta \in A_x$ and $\gamma \in E_x$ (here we view $u \in P(M, G)$ as an isomorphism $u : V \to E_{p(u)}$). Note that if $x \in U_j \cap U_k$ and $s_k(x) = \psi_{jk}(x) s_k(x)$ we obtain the same values in formula (41), so this really defines a global section on M. This section can be easily expressed in terms of A-derivatives as

$$R(\alpha,\beta)\gamma = \nabla_\alpha\nabla_\beta\gamma - \nabla_\beta\nabla_\alpha\gamma - \nabla_{[\alpha,\beta]}\gamma. \tag{42}$$

Moreover, Bianchi's identity (26) in this notation reads

$$\bigodot_{\alpha_1,\alpha_2,\alpha_3} \nabla_{\alpha_1}(R(\alpha_2,\alpha_3)) - \bigodot_{\alpha_1,\alpha_2,\alpha_3} R([\alpha_1,\alpha_2],\alpha_3) = 0. \tag{43}$$

If the A-connection h is induced by a covariant connection $\bar{h}$, the A-derivative ∇ and the covariant derivative $\bar{\nabla}$ are related by

$$\nabla_\alpha = \bar{\nabla}_{\#\alpha}. \tag{44}$$

On the other hand, $\mathcal{F}$-connections can be characterized by the condition:

$$\#\alpha = 0 \implies \nabla_\alpha = 0, \qquad \forall\alpha \in \Gamma(A). \tag{45}$$

Moreover, by Proposition 2.5, for an $\mathcal{F}$-connection, on each leaf $i : L \hookrightarrow M$ there is a covariant connection on the pullback bundle i^*P, inducing a covariant derivative ∇^L on i^*E, with the following property: if ψ is any cross section of E, then

$$i^*\nabla_\alpha\psi = \nabla^L_{\#i^*\alpha} i^*\psi, \tag{46}$$

where $i^*\psi$ denotes the section of the pullback bundle i^*E induced by ψ.

Remark. A flat A-connection on a vector bundle E is sometimes called a *Lie algebroid representation* of A or an A-module (see e.g. [9, 16]). In fact, if we set $\alpha \cdot s \equiv \nabla_\alpha s$ we get a bilinear product $\Gamma(A) \times \Gamma(E) \to \Gamma(E)$ and the axioms for ∇ to be a flat A-connection are translated into

$$(f\alpha)\cdot s = f(\alpha\cdot s), \tag{47}$$
$$\alpha\cdot(fs) = f(\alpha\cdot s) + \#\alpha(f)s, \tag{48}$$
$$[\alpha,\beta]\cdot s = \alpha\cdot(\beta\cdot s) - \beta\cdot(\alpha\cdot s). \tag{49}$$

One reason for this terminology is that in the case of a Lie algebra $\mathfrak{g}$, viewed as a Lie algebroid over a one point space, E is just a vector space and these are the requirements for E to be a $\mathfrak{g}$-module.

2.6. Linear A-Connections

A *linear A-connection* is a A-connection on the frame bundle $P = GL(A)$, so $G = GL(r)$ where $r = \operatorname{rank} A$. If $u = (\alpha_1, \ldots, \alpha_r) \in GL(A)$ is a frame, we can view u as a linear isomorphism $u : \mathbb{R}^r \to A_{p(u)}$ by setting

$$u(v_1, \ldots, v_r) = \sum_s v_s \alpha_s, \qquad (v_1, \ldots, v_r) \in \mathbb{R}^r.$$

We define the *canonical 1-sections* $\theta_j \in \Gamma(A^*)$ on an open set U_j, with trivializing isomorphism $\psi_j : p^{-1}(U_j) \to U_j \times G$, and associated section $s_j(x) = \psi_j^{-1}(x, e)$, to be the $\mathbb{R}^r$-valued 1-sections defined by

$$\theta_j(\alpha)_x = s_j(x)^{-1}(\alpha), \qquad x \in U_j. \tag{50}$$

Given an A-connection these allow us to define the *torsion 2-sections* $\Theta_j \in \Gamma(\wedge^2 A^*)$ to be the $\mathbb{R}^r$-valued 2-sections given by

$$\Theta_j(\alpha, \beta) = d_A\theta_j(\alpha, \beta) + \omega_j(\alpha) \cdot \theta(\beta) - \omega_j(\beta) \cdot \theta_j(\alpha). \tag{51}$$

We state the main properties of the torsion:

Proposition 2.8. *The canonical 1-sections and the torsion 2-sections of a linear A-connection are related by*

$$\theta_k = \psi_{jk}^{-1} \cdot \theta_j, \tag{52}$$

$$\Theta_k = \psi_{jk}^{-1} \cdot \Theta_j. \tag{53}$$

Moreover, they satisfy the Bianchi identity

$$d_A\Theta_j(\alpha, \beta, \gamma) = \bigodot_{\alpha,\beta,\gamma} d_A\omega_j(\alpha, \beta) \cdot \theta_j(\gamma) - \bigodot_{\alpha,\beta,\gamma} \omega_j(\alpha) \cdot d_A\theta_j(\beta, \gamma). \tag{54}$$

The vector bundle A is associated with the principal bundle $GL(A)$ of frames of A. Therefore, as it was explained in the previous paragraph, any linear A-connection determines an A-derivative operator $\nabla : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ such that:

$$\nabla_{f_1\alpha_1 + f_2\alpha_2} = f_1\nabla_{\alpha_1} + f_1\nabla_{\alpha_1}, \qquad \text{for all } f_i \in C^\infty(M),\ \alpha_i \in \Gamma(A), \tag{55}$$

$$\nabla_\alpha(f\beta) = f\nabla_\alpha\beta + \#\alpha(f)\beta, \qquad \text{for all } f \in C^\infty(M),\ \alpha, \beta \in \Gamma(A). \tag{56}$$

One can also consider other associated vector bundles to $GL(A)$ which lead, just us in the covariant case, to A-derivatives of any r-sections over A. For example, if X is a section of A^*, then $\nabla_\alpha X$, the *A-derivative of X* along $\alpha \in \Gamma(A)$, is completely characterized by the relation

$$\langle \nabla_\alpha X, \beta \rangle = \#\alpha(\langle X, \beta \rangle) - \langle X, \nabla_\alpha \beta \rangle \,, \tag{57}$$

which must hold for every section $\beta \in \Gamma(A)$.

For a linear A-connection we define the *torsion section* T to be the section of $A \otimes A^* \otimes A^*$ given by

$$T(\alpha, \beta) \equiv s_j(x)(\Theta_j(\alpha, \beta)), \tag{58}$$

where $x \in U_j$, $\alpha, \beta, \gamma \in A_x$. Note that if $x \in U_j \cap U_k$ and $s_k(x) = \psi_{jk}(x) s_k(x)$ we obtain the same values in formula (58), so this really defines a global section of M. In terms of A-derivatives, the torsion is given by

$$T(\alpha, \beta) = \nabla_\alpha \beta - \nabla_\beta \alpha - [\alpha, \beta]. \tag{59}$$

Moreover, the Bianchi identities (43) and (54) can also be expressed as

$$\bigodot_{\alpha,\beta,\gamma} (\nabla_\alpha R(\beta, \gamma) + R(T(\alpha, \beta), \gamma)) = 0, \tag{60}$$

$$\bigodot_{\alpha,\beta,\gamma} (R(\alpha, \beta)\gamma - T(T(\alpha, \beta), \gamma) - \nabla_\alpha T(\beta, \gamma)) = 0. \tag{61}$$

If it happens that the A-connection is related to some covariant connection by:

$$\#\nabla_\alpha \beta = \bar{\nabla}_{\#\alpha} \#\beta,$$

the torsion and curvature sections of the A-connections are transformed by the musical homomorphism to the usual torsion and tensor fields of ∇:

$$\bar{T}(\#\alpha, \#\beta) = \#T(\alpha, \beta),$$
$$\bar{R}(\#\alpha, \#\beta)\#\gamma = \#R(\alpha, \beta)\gamma.$$

Local coordinate expressions for linear A-connections can be obtained in a way similar to the covariant case. Let $(x^1, \ldots, x^m)$ be local coordinates and $(\alpha^1, \ldots, \alpha^r)$ local sections for A in a trivializing neighborhood $U \subset M$. Then we can define *Christoffel symbols* Γ^{st}_u by

$$\nabla_{\alpha^s} \alpha^t = \Gamma^{st}_u \alpha^u, \tag{62}$$

where we are using the repeated index sum convention. It is easy to see that under a change of coordinates and a change of basis of the form

$$y^j = y^j(x^1, \dots, x^m), \qquad \tilde{\alpha}^{s'} = a_s^{s'}(x^1, \dots, x^m)\alpha^s,$$

these symbols transform according to

$$\tilde{\Gamma}_{u'}^{s't'} = a_s^{s'} a_t^{t'} \tilde{a}_{u'}^u \Gamma_u^{st} + a_s^{s'} b^{si} \frac{\partial a_t^{t'}}{\partial x^i} \tilde{a}_{u'}^t \tag{63}$$

where b^{si} are the structure functions of $\#$ in the original coordinates, and $(\tilde{a}_{s'}^s)$ denotes the inverse of $(a_s^{s'})$. Conversely, given a family of symbols that transform according to this rule under a change of coordinates and basis of sections, we obtain a well defined A-derivative/connection on A.

Let us call an element

$$K \in \Gamma(\underbrace{A \otimes \cdots \otimes A}_{k \text{ times}} \otimes \underbrace{A^* \otimes \cdots \otimes A^*}_{l \text{ times}})$$

a section of type (k, l), or simply a (k, l)-section. Using these symbols, it is easy to get the local coordinates expressions for the A-derivatives of a such a (k, l)-section: Let $\alpha = \sum_s c_s \alpha^s$ be a section of A and write K as

$$K = K_{s_1 \dots s_k}^{t_1 \dots t_l} \alpha^{s_1} \otimes \cdots \otimes \alpha^{s_k} \otimes \check{\alpha}_{t_1} \otimes \cdots \otimes \check{\alpha}_{t_l}$$

where $\{\check{\alpha}_1, \dots, \check{\alpha}_r\}$ is the basis of local sections of A^* dual to $\{\alpha^1, \dots \alpha^r\}$. Then the A-derivative of K along α is the (k, l)-section

$$(\nabla_\alpha K)_{t_1 \dots t_k}^{s_1 \dots s_l} = b^{sj} \alpha_s \frac{\partial K_{t_1 \dots t_k}^{s_1 \dots s_l}}{\partial x^j} - \sum_{a=1}^{l} \left(\Gamma_u^{s s_a} \alpha_s K_{t_1 \dots t_s}^{s_1 \dots u \dots s_l} \right) + \sum_{b=1}^{k} \left(\Gamma_{t_b}^{su} \alpha_s K_{t_1 \dots u \dots t_s}^{s_1 \dots s_l} \right). \tag{64}$$

Given a section K of type (k, l) we shall write, as in the covariant case, ∇K for the unique section of type $(k, l+1)$ that satisfies

$$(\nabla K)_{t_1 \dots t_k}^{s_1 \dots s_l, s} = (\nabla_{\alpha^s} K)_{t_1 \dots t_k}^{s_1 \dots s_l}. \tag{65}$$

A tensor field K on M is *parallel* iff $\nabla K = 0$.

From formulas (59) and (42), we obtain immediately the following local coordinates expressions for the torsion and curvature in terms of Christoffel symbols and structure functions:

$$T^{st}_u = \Gamma^{st}_u - \Gamma^{st}_u - c^{st}_u, \tag{66}$$

$$R^{stu}_v = \Gamma^{sa}_v \Gamma^{tu}_a - \Gamma^{ta}_v \Gamma^{su}_a + b^{si}\frac{\partial \Gamma^{tu}_v}{\partial x^i} - b^{ti}\frac{\partial \Gamma^{su}_v}{\partial x^i} - c^{st}_a \Gamma^{au}_v. \tag{67}$$

2.7. Connections Compatible with the Lie Algebroid Structure

In Poisson geometry, linear connections for which the Poisson tensor is parallel play an important role. We recall that for a Poisson manifold (M, Π) a contravariant connection is just a linear connection ∇ on T^*M. It induces a T^*M-connection $\check{\nabla}$ on TM in the usual way:

$$\langle \check{\nabla}_\alpha X, \beta \rangle = \#\alpha(\langle X, \beta \rangle) - \langle X, \nabla_\alpha \beta \rangle, \qquad \forall X \in \mathcal{X}^1(M), \alpha, \beta \in \Omega^1(M).$$

The connection ∇ is a Poisson connection, in the sense that $\nabla \Pi = 0$, iff

$$\check{\nabla} \# = \# \nabla.$$

For a general Lie algebroid we do not have a relationship between A and TM analogue to the duality between T^*M and TM that occurs in the Poisson case. The notion of a Poisson connection is replaced by the following:

DEFINITION 2.3. A linear connection on a Lie algebroid A, with associated A-derivative ∇, is said to be COMPATIBLE WITH THE LIE ALGEBROID STRUCTURE of A if there exists an A-connection in TM, with associated A-derivative $\check{\nabla}$, such that

$$\check{\nabla} \# = \# \nabla.$$

In [10], Prop. 2.5.1, a simple argument shows that Poisson manifolds always admit Poisson connections. This argument extends to Lie algebroids.

PROPOSITION 2.9. *Every Lie algebroid admits compatible linear connections.*

Proof. Let U_a be a domain of a chart $(x^1, \ldots, x^m)$ where there exists a basis of trivializing sections $\left\{\alpha^1, \ldots, \alpha^r\right\}$. On U_a, we define a linear A-connection by

$$\nabla^{(a)}_{\alpha^s} \alpha^t = c^{st}_u \alpha^u,$$

and an A-connection on TU_a by

$$\check{\nabla}^{(a)}_{\alpha^s}\frac{\partial}{\partial x^i}=\frac{\partial b^{sj}}{\partial x^i}\frac{\partial}{\partial x^j},$$

where b^{sj} and c^{st}_u denote, as usual, the structure functions for this choice of coordinates and basis. A straight forward computation shows that the relation $\#[\alpha^s,\alpha^t]=[\#\alpha^s,\#\alpha^t]$ implies that

$$\check{\nabla}^{(a)}\#=\#\nabla^{(a)},$$

so $\nabla^{(a)}$ is a linear connection in U_a compatible with the Lie algebroid structure.

If we take an open cover of M by such chart domains and if $\sum_a\phi^{(a)}=1$ is a partition of unity subordinated to this cover, then $\nabla\equiv\sum_a\phi^{(a)}\nabla^{(a)}$ and $\check{\nabla}\equiv\sum_a\phi^{(a)}\check{\nabla}^{(a)}$ define A-connections that satisfy $\check{\nabla}\#=\#\nabla$, i.e., ∇ is a connection in M compatible with the Lie algebroid structure. ∎

An alternative approach to compatible connections is the following. Let us write $E=A\ominus T^*M$ for the vector bundle over M which is the "formal difference" of the vector bundles A and T^*M. If we are given a linear A-connection compatible with the Lie algebroid structure, we obtain an A-connection on E by setting:

$$\tilde{\nabla}_\gamma(\alpha+\omega)\equiv\nabla_\gamma\alpha+\check{\nabla}_\gamma\omega,\qquad\gamma,\alpha\in\Gamma(A),\omega\in\Omega^1(M).\tag{68}$$

Also on E we have a *canonical skew-symmetric bilinear form* $(\cdot,\cdot)$ given by the formula:

$$(\alpha+\omega,\beta+\eta)\equiv\eta(\#\alpha)-\omega(\#\beta),\tag{69}$$

and which, in general, will be degenerate.

PROPOSITION 2.10. *∇ is compatible with the Lie algebroid structure iff the skew-symmetric form $(\cdot,\cdot)$ is parallel with respect to $\tilde{\nabla}$, i.e.,*

$$\#\gamma((\zeta,\xi))=(\tilde{\nabla}_\gamma\zeta,\xi)+(\zeta,\tilde{\nabla}_\gamma\xi),\qquad\gamma\in\Gamma(A),\ \zeta,\xi\in\Gamma(E).$$

Proof. A straightforward computation. ∎

3. HOLONOMY

For a regular foliation the topological behaviour close to a given leaf is controlled by the holonomy of the leaf. For singular foliations the situation is more complex (see e.g. [8], where holonomy is defined for transversally stable leaves).

In this section, we will show that for any Lie algebroid it is possible to introduce a notion of holonomy. This holonomy can be defined as a map between the transversal algebroid germs that describe the transversal geometry of the Lie algebroid (cf. Section 1.3). In this theory of holonomy A-connections play a crucial role.

Later in the section we consider linear holonomy which will take us to the concept of a basic connection. Basic connections will be used in the next section to define secondary characteristic classes for the orbit foliation of a Lie algebroid.

3.1. Holonomy of a Leaf

Throughout this discussion we will consider a fixed leaf $i : L \hookrightarrow M$ of the Lie algebroid $\pi : A \to M$. We denote by $\nu(L) = T_L M/TL$ the normal bundle to L and by $p : \nu(L) \to L$ the natural projection. By the Tubular Neighborhood Theorem, there exists a smooth immersion $\tilde{i} : \nu(L) \to M$ satisfying the following properties:

(i) $\tilde{i}|_Z = i$, where we identify the zero section Z of $\nu(L)$ with L;

(ii) $\tilde{i}$ maps the fibers of $\nu(L)$ transversally to the foliation of M;

Assume that we have fixed such an immersion, and let $x \in L$. Each fiber $F_x = p^{-1}(x)$ is a submanifold of M transverse to the foliation so we have (see Section 1.3) the transverse Lie algebroid structure $A_{F_x} \to F_x$. Because F_x is a linear space we can choose a trivialization and identify the fibers $(A_{F_x})_u$ for different $u \in p^{-1}(x)$. Finally, we choose a complementary vector subbundle $E \subset A$ to A_{F_x}:

$$A_u = E_u \oplus (A_{F_x})_u. \tag{70}$$

Note that, by construction, the anchor $\# : A \to TM$ maps A_{F_x} onto TF_x, its restriction to E is injective, and vectors in $\#E$ are tangent to the orbit foliation.

Let $\alpha \in A_x$. We decompose α according to (70):

$$\alpha = \alpha^{\parallel} + \alpha^{\perp}, \text{where } \alpha^{\parallel} \in E_x, \quad \alpha^{\perp} \in (A_{F_x})_x.$$

For each $u \in F_x = p^{-1}(x)$, we denote by $\tilde{\alpha}_u^{\|} \in E_u$ the unique element such that $d_u p \cdot \#\tilde{\alpha}_u^{\|} = \#\alpha^{\|}$, and by $\tilde{\alpha}_u^{\perp} \in (A_{F_x})_u$ the element corresponding to $\alpha^{\perp}$ under the identification $(A_{F_x})_u \simeq (A_{F_x})_x$. We also set $\tilde{\alpha} \equiv \tilde{\alpha}^{\|} + \tilde{\alpha}^{\perp}$.

Given $\alpha \in A_x$, $x \in L$, and $u \in F_x$, define the *horizontal lift* to $\nu(L)$ by

$$h(u, \alpha) = \#\tilde{\alpha}_u \in T_u \nu(L).$$

By construction, we have the defining property of an A-connection:

$$p_* h(u, \alpha) = \#\alpha, \quad u \in p^{-1}(x).$$

Note that h depends on several choices made: tubular neighborhood, trivialization of A_{F_x} and complementary vector bundle E.

Let $\alpha(t)$, $t \in [0, 1]$, be an A-path with base path $\gamma(t)$ lying in the leaf L. If $u \in F_{\gamma(0)} = \nu(L)|_{\gamma(0)}$ is a point in the fiber over $\gamma(0)$, there exists an $\varepsilon > 0$ and a horizontal curve $\tilde{\gamma}(t)$ in $\nu(L)$, defined for $t \in [0, \varepsilon)$, which satisfies:

$$\begin{cases} \frac{d}{dt}\tilde{\gamma}(t) = h(\tilde{\gamma}(t), \alpha(t)), & t \in [0, \varepsilon), \\ \tilde{\gamma}(0) = u. \end{cases}$$

If we take $u = 0$ the lift $\tilde{\gamma}(t)$ coincides with $\gamma(t)$, and so is defined for $t \in [0, 1]$. It follows that we can choose a neighborhood U_γ of 0 in $F_{\gamma(0)} = \nu(L)|_{\gamma(0)}$, such that for each $u \in U_\gamma$ the lift $\tilde{\gamma}(t)$ with initial point u is defined for all $t \in [0, 1]$. Moreover, by passing from initial to end point, this lift gives a diffeomorphism $H_L(\alpha)_0$ of U_γ onto a neighborhood V_γ of 0 in $F_{\gamma(1)} = \nu(L)|_{\gamma(1)}$, with the property that 0 is mapped to 0.

Proposition 3.1. *Let α be an A-path in a leaf $L \subset M$. The isomorphism $H_L(\alpha)_0$ is covered by (a germ of) a Lie algebroid isomorphism $H_L(\alpha)$ from $A_{F_{\gamma(0)}}$ to $A_{F_{\gamma(1)}}$. If α' is another A-path in L such that $\gamma(1) = \gamma'(0)$ we have*

$$H_L(\alpha \cdot \alpha') = H_L(\alpha) \circ H_L(\alpha'), \tag{71}$$

where the dot denotes concatenation of A-paths.

Proof. Let $\alpha(t)$ be an A-path in L. We can find a time-dependent section α_t of A over L such that $\alpha_t(\gamma(t)) = \alpha(t)$. Using the notation above, we define a time-dependent section $\tilde{\alpha}_t$ over the tubular neighborhood such that for $x \in L$ and $u \in p^{-1}(x)$

$$\tilde{\alpha}_t = \tilde{\alpha}_t^{\|} + \tilde{\alpha}_t^{\perp}, \quad \text{where } \tilde{\alpha}_t^{\|} \in E_u,\ \tilde{\alpha}_t^{\perp} \in (A_{F_x})_u.$$

The lifts $\tilde{\gamma}$ are the integral curves of the vector field X_t defined by

$$X_t = \#\tilde{\alpha}_t,$$

so $H_L(\alpha)_0$ is the map induced by the time-1 flow of X_t on F_{x_0}.

The flow of X_t is induced by the 1-parameter family of Lie algebroid homomorphisms $\Phi_t^{\alpha_t}$ of A obtained by integrating the family $\tilde{\alpha}_t$ (see Section 1.1). The homomorphisms $\Phi_1^{\alpha_t}$ gives a Lie algebroid isomorphism

$$H_L(\alpha) : A_{F_{\gamma(0)}} \to A_{F_{\gamma(1)}},$$

which covers $H_L(\alpha)_0$. Relation (71) follows since we have just shown that $H_L(\alpha)$ is the time-1 map of some flow. ∎

We call $H_L(\alpha)$ the *A-holonomy* of the A-path $\alpha(t)$. One extends the definition of H_L for piecewise smooth A-paths in the obvious way.

Denote by $\mathfrak{Aut}(A_{F_x})$ the group of germs at 0 of Lie algebroid automorphisms of A_{F_x} which map 0 to 0, and by $\Omega_A(L, x_0)$ the group of piecewise smooth *A-loops* based at x_0.

Definition 3.1. The A-holonomy of the leaf L with base point x_0 is the map

$$H_L : \Omega_A(L, x_0) \to \mathfrak{Aut}(A_{F_{x_0}}).$$

Notice that the holonomy of a leaf L depends on the tubular neighborhood $\tilde{i} : \nu(L) \to M$, on the choice of trivialization, and on the choice of complementary bundles. However, two different choices lead to conjugate homomorphisms.

Example 3.1. Suppose $\#$ is injective, so we have a regular foliation $\mathcal{F}$ and A can be identified with $T\mathcal{F}$. Then A_{F_x} is a trivial Lie algebroid so $\mathfrak{Aut}(A_{F_{x_0}})$ can be identified with $\mathfrak{Aut}(F_{x_0})$, the group of germs of diffeomorphisms of F_{x_0} which map 0 to 0. Also, $E = A$ so the horizontal lift $h(u, \alpha)$ is the unique tangent vector to the leaf through u which projects to $\#\alpha$. We conclude that for a regular foliation, the Lie algebroid holonomy coincides with the usual holonomy.

Example 3.2. Let $A = T^*M$ be the Lie algebroid of a Poisson manifold (M, Π). In this case there is a natural choice for the complementary subbundle E, namely $E_u = (T_uF_x)^0$. It follows from the results in [10] that, for this choice, each automorphism $H_L(\alpha)$ covers a Poisson automorphism, and is in fact determined by the Poisson automorphism that it covers. Therefore, in this case, the Lie algebroid holonomy homomorphism is essentially the same as the Poisson holonomy defined in [10].

EXAMPLE 3.3. Let $A = M \times \mathfrak{g}$ be the transformation Lie algebroid associated with some infinitesimal action $\rho : \mathfrak{g} \to \mathcal{X}^1(M)$. Given $\alpha \in \mathfrak{g}$ we can identify it with a constant section of A. Let $x_0 \in M$ be a fixed point for the action and take $L = \{x_0\}$. Then $F_{x_0} = M$ and $A_{F_0} = A$, so E is the trivial bundle over M. The horizontal lift is given by $h(u, \alpha) = \#_u\alpha = \rho(\alpha) \cdot u$. If $\Psi : G \times M \to M$ denotes some local Lie group action that integrates ρ, we find for the constant A-path $\alpha(t) = \alpha$

$$H_L(\alpha)(x, v) = (\Psi(\exp(\alpha), x), Ad(\exp(\alpha)) \cdot v),$$

which is a Lie algebroid automorphism of $A_{F_{x_0}} \simeq M \times \mathfrak{g}$. Note that an A-loop with base path homotopic to a constant path might have non-trivial holonomy.

3.2. Reduced Holonomy

The Lie algebroid holonomy defined in the previous sections is not homotopy invariant (Example 3.3). Following the constructions given in [10] and [11] for the Poisson case, we can give a notion of *reduced holonomy* which is homotopy invariant.

Recall that $\mathfrak{Aut}(A_{F_x})$ denotes the group of germs at 0 of Lie algebroid automorphisms of A_{F_x} which map 0 to 0. We shall denote by $\mathfrak{Out}(A_{F_x})$ the corresponding group of germs of outer Lie algebroid automorphisms (see the end of Section 1.1).

PROPOSITION 3.2. *Let $x \in L \subset M$ be a leaf of A with associated A-holonomy $H_L : \Omega_A(L, x) \to \mathfrak{Aut}(F_x)$. If $\alpha_1(t)$ and $\alpha_2(t)$ are A-loops based at x with base paths $\gamma_1 \sim \gamma_2$ homotopic then $H_L(\alpha_1)$ and $H_L(\alpha_2)$ represent the same equivalence class in $\mathfrak{Out}(F_x)$.*

Proof. Recall that any piecewise smooth path $\gamma \subset L$ can be made into an A-path. By Proposition 3.1, property (71), it is enough to show that for every $x \in L$ there exists a neighborhood U of x in L such that if $\gamma(t) \subset U$ is a piecewise smooth loop based at x and $\alpha(t) \in A$ is a piecewise smooth family with $\#\alpha = \dot{\gamma}$ then $H_L(\alpha)$ is a inner automorphism of A_{F_x}.

We use the same notation as in the proof of Proposition 3.1, so we construct a time-dependent section $\tilde{\alpha}_t$ in a tubular neighborhood of L which decomposes as $\tilde{\alpha}_t = \tilde{\alpha}_t^{\|} + \tilde{\alpha}_t^{\perp}$, and $H_L(\alpha)$ is obtained by integrating this section up to time 1.

It is clear that the parallel component $\#\tilde{\alpha}_t^{\|}$ has no effect on the holonomy. Hence we can assume that $L = \{x\}$, $F_x = M$, γ is a constant path and $\tilde{\alpha}_t = \tilde{\alpha}_t^{\perp}$. But then $\Phi_t^{\tilde{\alpha}_t}$ is a 1-parameter family of automorphisms of A_{F_x} with $\Phi_1^{\tilde{\alpha}_t} = H_L(\alpha)$, so we conclude that $H_L(\alpha)$ is an inner automorphism of A_{F_x}. ∎

Given a loop γ in a leaf L we shall denote by $\bar{H}_L(\gamma) \in \mathfrak{Out}(A_{F_x})$ the equivalence class of $H_L(\alpha)$ for some piece-wise smooth family $\alpha(t)$ with $\#\alpha(t) = \gamma(t)$. The map $\bar{H}_L : \Omega(L, x) \to \mathfrak{Out}(A_{F_x})$ will be called the *reduced holonomy homomorphism* of L. This maps extends to continuous loops and, by a standard argument, it induces a homomorphism $\bar{H}_L : \pi_1(L, x) \to \mathfrak{Out}(A_{F_x})$ where $\pi_1(L, x)$ is the fundamental group of L (the use of the same letter to denote both these maps should not cause any confusion).

3.3. Stability

Recall that, for a foliation $\mathcal{F}$ of a manifold M, a *saturated set* is a set $S \subset M$ which is a union of leaves of $\mathcal{F}$. A leaf L is called *stable* if it has arbitrarily small saturated neighborhoods. In the case of the orbit foliation of a Lie algebroid a set is saturated iff it is invariant under all inner automorphisms. Hence, a leaf is stable iff it is has arbitrarily small neighborhoods which are invariant under all inner automorphisms.

We shall call a leaf L *transversally stable* if $N \cap L$ is a stable leaf for the transverse Lie algebroid structure A_N, i.e., if N has arbitrarily small neighborhoods of $N \cap L$ which are invariant under all inner automorphisms of A_N.

The following result is a generalization of the Reeb Stability Theorem for regular foliations.

Theorem 3.1 (Stability Theorem). *Let L be a compact, transversally stable leaf, with finite reduced holonomy. Then L is stable, i.e., L has arbitrarily small neighborhoods which are invariant under all inner automorphisms. Moreover, each leaf near L is a bundle over L with fiber a finite union of leaves of the transverse Lie algebroid structure.*

Proof. Assume first that L has trivial reduced holonomy and fix a base point $x_0 \in L$. We choose an embedding of $p : \nu(L) \to L$ in M, a complementary subbundle E and trivialization so we can define the holonomy map H_L. Also, we choose a Riemannian metric on L.

By compactness of L, there exists a number $c > 0$ such that every point $x \in L$ can be connected to x_0 by a smooth A-path of length $< c$. For some inner product on F_{x_0}, let D_ε be the disk of radius ε centered at 0. For each $\varepsilon > 0$, we can choose a neighborhood $U \subset D_\varepsilon$ such that:

(i) for any piecewise-smooth A-path in L, starting at x_0, with length $\leq 2c$ and for any $u \in U$, there exists a lifting with initial point u;

(ii) the lifting of any A-loop based at x_0 with initial point $u \in U$ has end point in U;

(iii) U is invariant under all inner automorphisms of $A_{F_{x_0}}$;

In fact, let $\alpha_1, \dots, \alpha_k$ be A-loops such that their base loops $\gamma_1, \dots, \gamma_k$ are generators of $\pi_1(L, x_0)$, and let Φ_i be Lie algebroid automorphisms which represent the germs $H_L(\alpha_i)$. Since the reduced holonomy is trivial, there is a neighborhood U' of 0 in F_{x_0} such that $U \subset \mathrm{domain}(\phi_1) \cap \cdots \cap \mathrm{domain}(\phi_k)$, and $\Phi_i|U' \in \mathrm{Inn}\,(A_{F_{x_0}})$, for all i. Since L is transversally stable, we can choose a smaller neighborhood $U \subset U'$ invariant under all inner automorphisms.

Given $x \in L$ and an A-path $\alpha(t)$ connecting x_0 to x, let us denote by $\sigma_\alpha : U \to F_x$ the diffeomorphism defined by lifting. It follows from i) and ii) above that if $\alpha'(t)$ is an A-path homotopic to $\alpha(t)$ then $\sigma_\alpha(U) = \sigma_{\alpha'}(U)$. It follows from iii) that $\sigma_\alpha(U)$ is also invariant under all inner automorphisms.

Let V be a neighborhood of L in M. There exists $\varepsilon(x) > 0$ such that for the corresponding $U_x \subset D_{\varepsilon(x)}$ we have $\sigma_\alpha(U_x) \subset V \cap F_x$. By compactness of L, we can choose $\varepsilon > 0$ (independent of $x \in L$) such that for the corresponding $U \subset D_\varepsilon$ we have

$$\sigma_\alpha(U) \subset V \cap F_x$$

Set

$$V_0 = \bigcup_\alpha \sigma_\alpha(U).$$

Then $V_0 \subset V$ is a open neighborhood of L which is invariant under all inner automorphisms of M. Therefore, L is stable.

If $u, u' \in V_0$ are two points in the same leaf of A such that $p(u) = p(u') = x$, then there is a path $\tilde{\gamma}$ in this leaf connecting these two points. We can choose a loop $\alpha(t)$ in L based at x such that $\tilde{\gamma}$ is a horizontal lift of this loop. Thus u' is the image of u by $H_L(\alpha)$ which is a inner automorphism of $V_0 \cap F_x$. Therefore, u and u' lie in the same leaf of $V_0 \cap F_x$. We conclude that each leaf of M near L is a bundle over L with fiber a leaf of the transverse Lie algebroid structure.

Assume now that L has finite reduced holonomy. We let $q : \tilde{L} \to L$ be a finite covering space such that $q_* \pi_1(\tilde{L}) = \operatorname{Ker} \bar{H}_L \subset \pi_1(L)$. If we embed $\nu(L)$ into M as above, and let $\nu(\tilde{L})$ be the pull back bundle of $\nu(L)$ over $\tilde{L}$, we have a unique Lie algebroid structure $\tilde{A}$ over $\nu(\tilde{L})$ and a Lie algebroid homomorphism $\Phi : A \to \tilde{A}$ which covers the natural map $\nu(\tilde{L}) \to \nu(L)$. Moreover, the reduced holonomy of $\tilde{A} \to \nu(\tilde{L})$ along $\tilde{L}$ is trivial, so we can apply the above argument to $\nu(\tilde{L})$ and the theorem follows. ▮

Remark. If a leaf L is transversally stable and $x \in L$, let N denote a stable neighborhood of F_x. For each A-loop α, the holonomy $H_L(\alpha)$ induces a homeomorphism of the orbit space of N, for the transverse Lie algebroid structure, mapping zero to zero. If $\alpha_1(t)$ and $\alpha_2(t)$ are A-loops such that $H_L(\alpha_1)$ and $H_L(\alpha_2)$ represent the same class in $\mathfrak{Out}(F_x)$, then they induce the same germ of homeomorphism of the orbit space mapping zero to zero. In [8] holonomy of a general, transversally stable, foliation is defined using germs of homeomorphisms of the orbit space, which in the case of a foliation defined by a Lie algebroid coincide with these homeomorphisms.

3.4. Linear Holonomy

Let $\pi : A \to M$ be a Lie algebroid and $i : L \hookrightarrow M$ a leaf of M with holonomy $H_L : \Omega_A(L,x) \to \mathfrak{Aut}(F_x)$ (once appropriate data has been fixed). Over $T_0 F_x \simeq F_x$ we consider the Lie algebroid $A^{\mathrm{Lin}}_{F_x} \simeq \mathfrak{g} \times F_x$, where $\mathfrak{g} = \operatorname{Ker} \#_x$ which is the linear approximation at 0 to the transverse Lie algebroid structure A_{F_x}. Also, we denote by $\operatorname{Aut}(A^{\mathrm{Lin}}_{F_x})$ the set of linear Lie algebroid automorphisms of $A^{\mathrm{Lin}}_{F_x}$. There is a map $d : \mathfrak{Aut}(A_{F_x}) \to \operatorname{Aut}(A^{\mathrm{Lin}}_{F_x})$ which assigns to a germ of a Lie algebroid automorphism of A_{F_x}, mapping zero to zero, its linear approximation at 0. Obviously, we can identify such a linear map with a pair (ϕ, ψ) where $\phi \in \mathrm{GL}(F_x)$ and ψ is a Lie algebra automorphism of $\operatorname{Ker} \#_x = \mathfrak{g}$.

Definition 3.2. The linear A-holonomy of the leaf L with base point x_0 is the map

$$H^{\mathrm{Lin}}_L \equiv dH_L : \Omega_A(L,x) \to \operatorname{Aut}(\mathfrak{g}) \times \mathrm{GL}(F_x).$$

One can also define the *reduced linear A-holonomy homomorphism* of a leaf L to be the class of $H_L^{\mathrm{Lin}}(\gamma,\alpha)$ in $\mathrm{Out}(\mathfrak{g}) \times \mathrm{GL}(F_x)$ where $\mathrm{Out}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$. The reduced holonomy is homotopy invariant. For Poisson manifolds, linear Poisson holonomy was first introduced by Ginzburg and Golubev in [11].

There is an alternative approach to linear holonomy using a linear A-connection which generalizes the Bott connection of ordinary foliation theory. In this differential operator formulation linear holonomy arises as the holonomy of a Lie algebroid connection.

We consider first the vector bundle $\mathrm{Ker}\,\#|_L$ over the leaf L where we have a *Bott A-connection* defined as follows: Given an element $\alpha \in A_x$, where $x \in L$, and a section β of $\mathrm{Ker}\,\#|_L$, we take (local) sections $\tilde{\alpha}, \tilde{\beta} \in \Gamma(A)$ such that $\tilde{\alpha}_x = \alpha$, $\tilde{\beta}|_L = \beta$, and we set:

$$\nabla^L_\alpha \beta \equiv [\tilde{\alpha}, \tilde{\beta}]|_x. \tag{72}$$

Lemma 3.1. *∇^L associates to each section α of A along L a linear operator $\nabla_\alpha : \Gamma(\mathrm{Ker}\,\#|_L) \to \Gamma(\mathrm{Ker}\,\#|_L)$.*

Proof. To check that expression (72) is independent of the extensions considered, we fix a local basis of sections $\{\gamma^1, \dots, \gamma^r\}$ for A in a neighborhood of x. If we write

$$\tilde{\alpha} = \sum_s \tilde{a}_s \gamma^s, \quad \tilde{\beta} = \sum_t \tilde{b}_t \gamma^t,$$

for some functions $\tilde{a}_s$ and $\tilde{b}_t$, we compute

$$\nabla^L_\alpha \beta = \sum_{s,t} \left(\tilde{a}_s(x)\tilde{b}_t(x)[\gamma^s, \gamma^t]|_x + \tilde{a}_s(x)\#\gamma^s(\tilde{b}_t)|_x \gamma^t|_x \right).$$

This expression shows that $\nabla^L_\alpha \beta$ only depends on the value of $\tilde{\alpha}$ at x and the values of $\tilde{\beta}$ along L, i.e., α and β.

Relation (72) also shows that $\nabla^L_\alpha \beta$ is in the kernel of $\#$ and so is a section of $\mathrm{Ker}\,\#|_L$. ∎

Next we consider the conormal bundle $\nu^*(L) = \{\omega \in T^*_L M : \omega | TL = 0\}$ over the leaf L where we also have a *Bott A-connection* defined as follows: Given an element $\alpha \in A_x$, where $x \in L$, and a section ω of $\nu^*(L)$, we take a section $\tilde{\alpha} \in \Gamma(A)$ and a 1-form $\tilde{\omega} \in \Omega^1(M)$ such that $\tilde{\alpha}_x = \alpha$, $\tilde{\omega}|_L = \omega$, and we set:

$$\check{\nabla}^L_\alpha \omega \equiv \mathcal{L}_{\#\tilde{\alpha}}\tilde{\omega}|_x. \tag{73}$$

A proof similar to the lemma above shows that:

LEMMA 3.2. *$\check{\nabla}^L$ associates to each section α of A along L a linear operator $\check{\nabla}_\alpha : \Gamma(\nu^*(L)) \to \Gamma(\nu^*(L))$.*

It is also easy to check that ∇^L and $\check{\nabla}^L$ satisfy the analogue of properties (i) to (iv) of Proposition 2.7. Note however that, in general, ∇^L and $\check{\nabla}^L$ *do not give* genuine Lie algebroid connections since they are only defined for sections of A along L.

It is convenient to consider the connections ∇^L and $\check{\nabla}^L$ all together, rather than leaf by leaf, so we set:

DEFINITION 3.3. A linear connection ∇ on A is called a BASIC CONNECTION if

(i) ∇ is compatible with the Lie algebroid structure, i.e., there exists a linear A-connection $\check{\nabla}$ on TM such that

$$\check{\nabla}\# = \#\nabla.$$

(ii) ∇ restricts to ∇^L on each leaf L, i.e., if $\alpha, \beta \in \Gamma(A)$ with $\#\beta|_L = 0$ then

$$\nabla_\alpha \beta|_L = \nabla^L_\alpha \beta.$$

(iii) $\check{\nabla}$ restricts to $\check{\nabla}^L$ on each leaf L, i.e., if $\alpha \in \Gamma(A)$ and $\omega \in \Omega^1(M)$ with $\omega|_{TL} = 0$ then

$$\nabla_\alpha \omega|_L = \check{\nabla}^L_\alpha \omega.$$

The holonomy along a leaf L of a basic connection ∇ coincides with the linear holonomy of L introduced above: the holonomy of the basic connection ∇ determines endomorphisms of the fiber A_x which map $\ker \#_x$ isomorphically into itself, and these are the linear holonomy maps.

PROPOSITION 3.3. *Every Lie algebroid has a basic connection.*

Proof. In fact, let us see that the compatible connection ∇ constructed in the proof of Proposition 2.9 is a basic connection. We use the same notation as in that proof, so if L is a leaf of M and $\#\beta|_L = 0$, we write $\beta = \sum_t b_t\alpha^t$ and we have

$$\begin{aligned}\nabla^{(a)}_{\alpha^s}\beta|_L &= \sum_t \left(b_t\nabla^{(a)}_{\alpha^s}\alpha^t + \#\alpha^s(b_t)\alpha^t\right)\\ &= \sum_t\left(\sum_u b_t c^{st}_u\alpha^u + \sum_j b^{sj}\frac{\partial b_t}{\partial x^j}\alpha^t\right) = [\alpha^s,\beta]|_L\end{aligned}$$

Therefore, for any 1-form $\alpha = \sum_s a_s\alpha^s$, we get

$$\begin{aligned}\nabla^{(a)}_{\alpha}\beta|_L &= \sum_s a_s\nabla^{(a)}_{\alpha^s}\beta|_L\\ &= \sum_s a_s[\alpha^s,\beta]|_L = [\alpha,\beta]|_L,\end{aligned}$$

since $\#\beta|_L = 0$. It follows that for any 1-form α we have

$$\nabla_\alpha\beta|_L = \sum_a \phi_a\nabla^{(a)}_\alpha\beta|_L = \nabla^L_\alpha\beta.$$

Similarly, for the connection $\check{\nabla}$, we have

$$\check{\nabla}^{(a)}_{\alpha^s}dx^i = \sum_j \frac{\partial b^{si}}{\partial x^j}dx^j = \mathcal{L}_{\#\alpha^s}dx^i,$$

so if $\omega|_{TL} = 0$ we find $\check{\nabla}^{(a)}_\alpha\omega|_L = \mathcal{L}_{\#\alpha}\omega$, and it follows that

$$\check{\nabla}_\alpha\omega|_L = \sum_a \phi_a\check{\nabla}^{(a)}_\alpha\omega|_L = \check{\nabla}^L_\alpha\omega.$$

Since $\nabla\# = \#\check{\nabla}$ we conclude that ∇ defines a basic connection. ∎

In the theory of regular foliations basic connections arise as connections in the normal bundle of the foliation (or equivalently in the conormal bundle). In the case of a Lie algebroid, as was first argued in [9], the "formal difference" $E = A \ominus T^*M$ plays the role of the (co)normal bundle. Now, a basic connection ∇ in A gives an A-connection $\tilde{\nabla}$ in the bundle E (see Section 2.7):

$$\tilde{\nabla}_\alpha(\beta+\omega) = \nabla_\alpha\beta + \check{\nabla}_\alpha\omega, \qquad \alpha,\beta\in\Gamma(A),\ \omega\in\Omega^1(M).$$

We shall see later that this connection plays a key role in defining *secondary characteristic classes* for Lie algebroids.

The following result shows that the curvature $\tilde{R}$ of $\tilde{\nabla}$ vanishes along a leaf.

PROPOSITION 3.4. *Let ∇ be a basic connection and L a leaf of A. Denote by R and $\check{R}$ the curvature of the connections ∇ and $\check{\nabla}$. If γ is a section of A such that $\#\gamma|_L = 0$, then*

$$R(\alpha,\beta)\gamma|_L = 0.$$

Similarly, if $\omega \in \Omega^1(M)$ is a differential form such that $\omega|_{TL} = 0$ then

$$\check{R}(\alpha,\beta)\omega|_L = 0.$$

Proof. If ∇ is any basic connection and $\#\gamma|_L = 0$, we have $\nabla_\alpha\gamma|_L = [\alpha,\gamma]|_L$, so expression (42) for the curvature tensor, gives

$$R(\alpha,\beta)\gamma|_L = [\alpha,[\beta,\gamma]]|_L - [\beta,[\alpha,\gamma]]|_L - [[\alpha,\beta],\gamma]|_L.$$

But the right hand side is zero, because of the Jacobi identity.

Similarly, if $\omega \in \Omega^1(M)$ is a differential form such that $\omega|_{TL} = 0$, we have $\check{\nabla}_\alpha\omega|_L = \mathcal{L}_{\#\alpha}\gamma|_L$. Hence, using $\#[\alpha,\beta] = [\#\alpha,\#\beta]$ and the well known formula for the Lie derivative of the Lie bracket of vector fields, we find

$$R(\alpha,\beta)\omega|_L = \mathcal{L}_{\#\alpha}(\mathcal{L}_{\#\beta}\gamma)|_L - \mathcal{L}_{\#\beta}(\mathcal{L}_{\#\alpha}\gamma)|_L - \mathcal{L}_{[\#\alpha,\#\beta]}\gamma|_L = 0,$$

so the second relation also holds. ∎

Remark. Although the curvature of a basic connection vanishes along $\ker\#$, the holonomy along $\#$ need not be discrete (this is because of the presence of an extra term in the Holonomy Theorem 2.1). Hence, in general, linear holonomy is not discrete and also not homotopy invariant (cf. Example 3.3). However, if one can find a basic $\mathcal{F}$*-connection* then one gets discrete holonomy. Such is the case whenever $\#$ is injective, so the orbit foliation is regular, and (linear) holonomy coincides with standard (linear) holonomy of a regular foliation.

4. CHARACTERISTIC CLASSES

4.1. Chern-Weil Homomorphism

The usual Chern-Weil theory for characteristic classes extends to A-connections. For contravariant connections this was already discussed in [10, 29]. For general Lie algebroids the theory is similar and only a short account will be given as we shall need it later in the section when we discuss secondary characteristic classes.

We consider a principal G-bundle $p : P \to M$ furnished with an A-connection. Given any symmetric, $\mathrm{Ad}\,(G)$-invariant, k-multilinear function

$$P : \mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathbb{R}$$

we can define a $2k$-section $\lambda(P)$ of A as follows. If U_j is a trivializing neighborhood, and $\alpha_1, \ldots, \alpha_{2k}$ are sections of A over U_j then we set

$$\lambda(P)(\alpha_1, \ldots, \alpha_{2k}) = \sum_{\sigma \in S_{2k}} (-1)^\sigma P(\Omega_j(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}), \ldots, \Omega_j(\alpha_{\sigma(2k-1)}, \alpha_{\sigma(2k)})), \quad (74)$$

where $\{\Omega_j\}$ are local curvature 2-sections. By the transformation rule for the local curvature 2-sections, this formula actually defines a $2k$-section $\lambda(P) \in \Gamma(\wedge^{2k} A^*)$ on the whole of M.

Proposition 4.1. *For any symmetric, invariant, k-multilinear function P, the $2k$-section $\lambda(P)$ is closed:*

$$d_A \lambda(P) = 0. \quad (75)$$

Proof. We compute

$$\begin{aligned} d_A \lambda(P) &= kP(d_A \Omega_j, \ldots, \Omega_j) \\ &= kP(d_A \Omega_j + [\omega_j, \Omega_j], \ldots, \Omega_j) = 0, \end{aligned}$$

where we have used first the linearity and symmetry of P, then the $\mathrm{Ad}\,(G)$-invariance of P, and last the Bianchi identity. ∎

Therefore, to an invariant, symmetric, k-multilinear function $P \in I^k(G)$ we can associate an A-cohomology class $[\lambda(P)] \in H^{2k}(A)$, and in fact we have:

PROPOSITION 4.2. *The cohomology class* $[\lambda(P)]$ *is independent of the* A*-connection used to define it.*

Proof. Suppose we have two A-connections in $P(M,G)$ with connection 1-sections ω^0 and ω^1, and denote by $\lambda^0(P)$ and $\lambda^1(P)$ the $2k$-sections they define through (74). We construct a 1-parameter family of connections with connection 1-section $\omega^t = t\omega^1 + (1-t)\omega^0$, $t \in [0,1]$, and we denote by Ω^t its curvature 2-section.

By the transformation rule (23) for the local connection 1-sections, the difference $\omega_j^{1,0} = \omega_j^1 - \omega_j^0$ is a $\mathfrak{g}$-valued 1-section, and we get a well defined $(2k-1)$-section $\lambda^{1,0}(P)$ by setting

$$\lambda^{1,0}(P)(\alpha_1,\ldots,\alpha_{2k-1}) = \sum_\sigma C_\sigma \int_0^1 P(\omega_j^{1,0}(\alpha_{\sigma(1)}), \Omega_j^t(\alpha_{\sigma(2)},\alpha_{\sigma(3)}),\ldots,\Omega_j^t(\alpha_{\sigma(2k-2)},\alpha_{\sigma(2k-1)}))dt. \tag{76}$$

where $C_\sigma = k(-1)^\sigma$, and the sum is over all permutations in S_{2k-1}. We claim that

$$d_A\lambda^{1,0}(P) = \lambda^1(P) - \lambda^0(P), \tag{77}$$

so $[\lambda^1(P)] = [\lambda^0(P)]$.

To prove (77), we note that if we differentiate the structure equation (21) we obtain

$$\frac{d}{dt}\Omega_j^t = d_A\omega_j^{1,0} + [\omega_j^t, \omega_j^{1,0}]. \tag{78}$$

Hence, using Bianchi's identity, we have

$$
\begin{aligned}
k d_A \int_0^1 & P(\omega_j^{1,0}, \Omega_j^t, \dots, \Omega_j^t) dt = \\
&= k \int_0^1 P(d_A \omega_j^{1,0}, \Omega_j^t, \dots, \Omega_j^t) \\
&\qquad\qquad + P(\omega_j^{1,0}, d_A \Omega_j^t, \dots, \Omega_j^t) + P(\omega_j^{1,0}, \Omega_j^t, \dots, d_A \Omega_j^t) dt \\
&= k \int_0^1 P(\frac{d}{dt} \Omega_j^t - [\omega_j^t, \omega_j^{1,0}], \Omega_j^t, \dots, \Omega_j^t) \\
&\qquad\qquad - P(\omega_j^{1,0}, [\omega_j^t, \Omega_j^t], \dots, \Omega_j^t) - P(\omega_j^{1,0}, \Omega_j^t, \dots, [\omega_j^t, \Omega_j^t]) dt \\
&= k \int_0^1 P(\frac{d}{dt} \Omega_j^t, \Omega_j^t, \dots, \Omega_j^t) dt \\
&= \int_0^1 \frac{d}{dt} P(\Omega_j^t, \Omega_j^t, \dots, \Omega_j^t) dt = P(\Omega_j^1, \dots, \Omega_j^1) - P(\Omega_j^0, \dots, \Omega_j^0),
\end{aligned}
$$

so the claim follows. ▮

If we set

$$
I^\bullet(G) = \bigoplus_{k \geq 0} I^k(G),
$$

the assignment $P \mapsto [\lambda(P)]$ gives a map $I^\bullet(G) \to H^\bullet(A)$, which is in fact a ring homomorphism, and which we call the *A-Chern-Weil homomorphism* of the Lie algebroid. The fact that this map is a ring homomorphism follows, for example, from the following proposition:

Proposition 4.3. *The following diagram commutes*

$$
\begin{array}{ccc}
I^\bullet(G) & \longrightarrow & H^\bullet_{de\ Rham}(M) \\
& \searrow & \downarrow \#^* \\
& & H^\bullet(A)
\end{array}
$$

where on the top row we have the usual Chern-Weil homomorphism.

Proof. Choose an A-connection in P which is induced by some covariant connection. Given $P \in I^k(G)$, this covariant connection gives a closed $(2k)$-form $\tilde{\lambda}(P)$ defined by a formula analogous to (74), and which induces the usual Chern-Weil homomorphism $I^\bullet(G) \to H^\bullet_{\text{de Rham}}(M)$. We check easily that

$$\#^*\tilde{\lambda}(P) = \lambda(P),$$

so the proposition follows. ▮

Recall that the ring of invariant polynomials $I^\bullet(GL_q(\mathbb{R}))$ is generated by elements $P_k \in I^k(GL_q(\mathbb{R}))$ such that $P_k(X, \dots, X) = \sigma_k(X)$, where $\{\sigma_1, \dots, \sigma_q\}$ are the *elementary symmetric functions* defined by:

$$\det(\mu I - \frac{1}{2\pi}X) = \mu^q + \sigma_1(X)\mu^{q-1} + \cdots + \sigma_q(X).$$

Now consider a real vector bundle $p_E : E \to M$, with $\operatorname{rank} E = q$, and let $p : P \to M$ be the associated principal bundle with structure group $GL_q(\mathbb{R})$. Choosing a Lie algebroid connection on P one defines the *kth A-Pontrjagin class* of E as

$$p_k(E, A) = [\lambda(P_{2k})] \in H^{4k}(A).$$

As usual, one does not need to consider the classes for odd k since we have

$$[\lambda(P_{2k-1})] = 0,$$

as can be seen by choosing a connection compatible with a riemannian metric. It is clear from Proposition 4.3 that

$$p_k(E, A) = \#^* p_k(E).$$

where $p_k(E)$ are the usual Pontrjagin classes of E.

To compute these invariants one uses the A-derivative operator ∇ on E, associated with the A-connection, and proceeds as follows. For 1-sections $\alpha, \beta \in \Gamma(A)$, the curvature tensor R defines a linear map $R_{\alpha,\beta} = R(\alpha, \beta) : E_x \to E_x$ which satisfies $R_{\alpha,\beta} = -R_{\beta,\alpha}$, and so $(\alpha, \beta) \to R_{\alpha,\beta}$ can be considered as a $\mathfrak{gl}(E)$-valued 2-section. By fixing a basis of local sections for E, we have $E_x \simeq \mathbb{R}^q$ so we have $R_{\alpha,\beta} \in \mathfrak{gl}_q(\mathbb{R})$. (this matrix representation of $R_{\alpha,\beta}$ is defined only up to a change of basis in $\mathbb{R}^q$). Hence, if

$$P : \mathfrak{gl}_q(\mathbb{R}) \times \cdots \times \mathfrak{gl}_q(\mathbb{R}) \to \mathbb{R}$$

is a symmetric, k-multilinear function, $\operatorname{Ad}(GL_q(\mathbb{R}))$-invariant, we a have a $2k$-section $\lambda(R)(P)$ defined by

$$\lambda(R)(P)(\alpha_1, \dots, \alpha_{2k}) = \sum_{\sigma \in S_{2k}} (-1)^\sigma P(R_{\alpha_{\sigma(1),\sigma(2)}}, \dots, R_{\alpha_{\sigma(2k-1),\sigma(2k)}}). \tag{79}$$

It is easy to see that $\lambda(P) = \lambda(R)(P)$, so this gives a procedure to compute the A-Chern-Weil homomorphism and the A-Pontrjagin classes.

Similar considerations apply to other characteristic classes. One can define, e.g., the A-Chern classes $c_k(E, A)$ of a complex vector bundle E and they are just the images by $\#^*$ of the usual Chern classes of E.

The fact that all these classes arise as image by $\#^*$ of well known classes is perhaps a bit disappointing. However, we shall see below that one can define secondary characteristic classes which are true invariants of the Lie algebroid, in the sense that they do not arise as images by $\#^*$ of some de Rham cohomology classes. On the other hand, the tangential Chern-Weil theory is usefull on its own, and have interesting applications already in the case of regular foliations (see e.g. [26]).

4.2. Secondary Characteristic Classes

Whenever a form representing a (primary) characteristic class vanishes one can introduce new (secondary) characteristic classes, a remark that goes back to the original paper of Chern and Simons ([3]). This remark was the starting point for the theory of "exotic" characteristic classes for foliations (see [2]).

On a Lie algebroid a similar construction of "exotic" characteristic classes can be done. This construction generalizes the construction given in [10] for the case of a Poisson manifold, where it is was shown that Poisson secondary characteristic classes give information on the topology, as well as, the geometry of the symplectic foliation.

In the theory of (regular) foliations, the secondary characteristic classes appear when we compare two connections on the normal bundle each from a distinguished class. In the case of a Lie algebroid the "formal difference" $E = A \ominus T^*M$ plays the role of the (co)normal bundle, and again we compare two connections, each from a distinguished class. So on a Lie algebroid $\pi : A \to M$, with $\operatorname{rank} A = r$ and $\dim M = m$, we consider the following data:

(i) A basic connection with associated A-derivative operator ∇, so on E we have an induced connection (see section 3.4) $\nabla^1 = \nabla \oplus \check{\nabla}$;

(ii) A metric connection ∇^0 on E, i.e., we have a covariant connection $\bar{\nabla}^0$ on E, which preserves some metric g on E, and we take $\nabla^0_\alpha = \bar{\nabla}^0_{\#\alpha}$;

Given an invariant, symmetric, k-multilinear function $P \in I^k(GL(r + m, \mathbb{R}))$ we consider the $(2k-1)$-section $\lambda^{1,0}(P) \in \Gamma(\wedge^{2k-1} A^*)$ given by (76).

Proposition 4.4. *If k is odd, $\lambda^{1,0}(P)$ is a d_A-closed $(2k-1)$-section.*

Proof. According to (77) we have

$$d_A\lambda^{1,0}(P) = \lambda^1(P) - \lambda^0(P).$$

and we claim that $\lambda^1(P) = \lambda^0(P) = 0$ if k is odd (these are the vanishing primary classes that we mentioned to above).

The proof that $\lambda^0(P) = 0$ is standard: we can choose an orthonormal basis of sections for E so that the curvature 2-sections take there values in $\mathfrak{so}(r+m,\mathbb{R})$. But if $X \in \mathfrak{so}(r+m,\mathbb{R})$, we have $P_k(X) = 0$ for any elementary symmetric function, since k is odd. Hence we obtain $\lambda^0(P) = 0$.

Consider now the connection ∇^1. Given $x \in M$ we choose local coordinates (x^j, y^j) around x and a basis of sections $\left\{\alpha^1,\dots,\alpha^r\right\}$ as in the Local Splitting Theorem. Then $\left\{dx^i, dy^j, \alpha^s\right\}$ form a basis for E, and for the canonical skew-symmetric bilinear form $(\ ,\ )$ given by (69) the only non-vanishing pairs are:

$$(\alpha^i, dx^i) = 1 = -(dx^i, \alpha^i), \qquad (\alpha^s, dy^j) = b^{sj} = -(dy^j, \alpha^s).$$

where $b^{sj} = b^{sj}(y)$. Since ∇^1 is induced by a basic connection, it is compatible with the Lie algebroid structure so from proposition 2.10 we conclude:

$$\begin{aligned}
(\nabla^1_\gamma\alpha^i, \alpha^l) &= -(\alpha^i, \nabla^1_\gamma\alpha^l),\\
(\nabla^1_\gamma\alpha^i, dx^l) &= -(\alpha^i, \nabla^1_\gamma dx^l), \quad (1 \le i, l \le q)\\
(\nabla^1_\gamma dx^i, dx^l) &= -(dx^i, \nabla^1_\gamma dx^l).
\end{aligned}$$

On the other hand, from Proposition 3.4 we find

$$R^1(\alpha,\beta)dy^j|_x = R^1(\alpha,\beta)\alpha^s|_x = 0, \quad (j, s > q).$$

It follows that $R^1(\alpha,\beta)_x$ is represented in the basis $(dx^i, \alpha^i, dy^j, \alpha^s)$ by a matrix of the form:

$$\begin{pmatrix} B & 0 \\ C & 0 \end{pmatrix}, \tag{80}$$

with B a $(2q \times 2q)$ symplectic matrix. Now, if A is any matrix of this form, it is clear that $\det(\mu I - A) = \det(\mu I - \tilde{A})$, where $\tilde{A}$ is the same as A with $C = 0$, i.e., $\tilde{A}$ is symplectic. But if $\tilde{A}$ is symplectic, we have $P_k(A) = 0$ for any elementary symmetric function, since k is odd. Hence $\lambda^1(P) = 0$. ∎

Next we want to check that the cohomology class of $\lambda^{1,0}(P)$ is independent of the connections used to define it.

Given 3 connections with local connection 1-sections $\omega_j^0, \omega_j^1, \omega_j^2$ we consider a family of connections with local connection 1-sections $\omega_j^{s,t} = (1 - s - t)\omega_j^0 + s\omega_j^1 + t\omega_j^2$, where (s,t) vary in the standard 2-simplex Δ_2. We introduce a $(2k-2)$-vector field $\lambda^{2,1,0}(P)$ by a formula analogous to (79) and (76):

$$\lambda^{2,1,0}(P) = k \sum_{\sigma\in S_{2k-2}} (-1)^\sigma \int_{\Delta_2} P(\omega_j^{1,0}, \omega_j^{2,0}, \Omega_j^{s,t}, \dots, \Omega_j^{s,t}) dtds, \tag{81}$$

and just like in the proof of Proposition 4.2, one shows that

$$\delta\lambda^{2,1,0}(P) = \lambda^{1,0}(P) - \lambda^{2,0}(P) + \lambda^{2,1}(P). \tag{82}$$

Proposition 4.5. *The cohomology class $[\lambda^{1,0}(P)]$ is independent of the connections used to define it.*

Proof. Let ∇^1 and $\tilde\nabla^1$ (resp. ∇^0 and $\tilde\nabla^0$) be basic connections (resp. riemannian connections). It follows from (82) that

$$\begin{aligned}\lambda(\nabla^1,\nabla^0)(P) - \lambda(\tilde\nabla^1,\tilde\nabla^0)(P) = \delta\lambda(\tilde\nabla^1,\nabla^0,\tilde\nabla^0)(P) + \delta\lambda(\tilde\nabla^1,\nabla^1,\nabla^0)(P)\\ - \lambda(\tilde\nabla^1,\nabla^1)(P) - \lambda(\nabla^0,\tilde\nabla^0)(P).\end{aligned}$$

Hence, it is enough to show that the cohomology classes of $\lambda(\tilde\nabla^1,\nabla^1)(P)$ and $\lambda(\nabla^0,\tilde\nabla^0)(P)$ are trivial.

Consider first the basic connections $\tilde\nabla^1$ and ∇^1. The linear combination $\nabla^{1,t} = (1-t)\nabla^1 + t\tilde\nabla^1$ is also a basic connection. If $x \in M$, we fix splitting coordinates (x^i, y^j) around x and sections $\{\alpha^1, \dots, \alpha^r\}$ as in the proof of proposition 4.4. Then we see that, with respect to the basis $\{dx^i, \alpha^i, dy^j, \alpha^s\}$, the matrix representations of $\tilde\nabla^1_\alpha - \nabla^1_\alpha$ and $R^t(\alpha,\beta)$ are of the form (80). Hence, we conclude that if $P \in I^k(GL(m,\mathbb{R}))$, with k odd,

$$P(\tilde\nabla^1_{\alpha_1} - \nabla^1_{\alpha_1}, R^t(\alpha_2,\alpha_3), \dots, R^t(\alpha_{2k-2},\alpha_{2k-1})) = 0.$$

Therefore, $\lambda(\tilde\nabla^1,\nabla^1)(P) = 0$, whenever $\tilde\nabla^1$ and ∇^1 are basic connections.

Now consider the riemannian connections ∇^0 and $\tilde\nabla^0$. The linear combination $\nabla^{0,t} = (1-t)\tilde\nabla^0 + t\nabla^0$ is also a riemannian connection. All these connections are induced from covariant riemannian connections $\bar\nabla^0$, $\tilde{\bar\nabla}^0$ and $\bar\nabla^{0,t}$, and we can define a differential form $\lambda(\bar\nabla^0, \tilde{\bar\nabla}^0)(P)$ of degree $(2k-1)$ by a formula analogous to (76). Moreover, this form is closed (because k is odd), and $\#^*\lambda(\bar{\nabla}^0, \tilde{\bar{\nabla}}^0)(P) = \lambda(\nabla^0, \tilde{\nabla}^0)(P)$. It follows from the homotopy invariance of $H^\bullet_{\text{de Rham}}(M)$, using a suspension argument, as in the usual theory of secondary characteristic classes of foliations (see [2], page 29), that

$$[\lambda(\bar{\nabla}^0, \tilde{\bar{\nabla}}^0)(P)] = [\lambda(\bar{\nabla}^0, \bar{\nabla}^0)(P)].$$

Hence, the cohomology class $[\lambda(\bar{\nabla}^0, \tilde{\bar{\nabla}}^0)(P)]$ vanishes and so does the class $[\lambda(\nabla^0, \tilde{\nabla}^0)(P)]$. ▮

Remark. We have used a riemannian connection of the special form $\nabla^0_\alpha = \bar{\nabla}^0_{\#\alpha}$. On the other hand, in general, a Lie algebroid does not admit a compatible A-connection of the form $\tilde{\nabla}_{\#\alpha}$. Hence, the basic connections are "genuine" A-connections, i.e., not induced by any covariant connection.

Definition 4.1. The *secondary characteristic classes* $\{m_k(A)\}$ of a Lie algebroid are the Lie algebroid cohomology classes

$$m_k(A) = [\lambda^{1,0}(P_k)] \in H^{2k-1}(A), \qquad (k = 1, 3, \dots). \tag{83}$$

where P_k are the elementary symmetric polynomials.

Note that, by the remark above, these secondary characteristic classes are "genuine" Lie algebroid cohomology classes, i.e., they do not lie in the image of $\#^* : H^\bullet_{\text{de Rham}}(M) \to H^\bullet(A)$ (see also the examples below where one can have trivial de Rham cohomology and non-zero secondary characteristic classes).

Remark. In general, one can only define the characteristic classes m_k for k odd. Assume, however, that A admits flat riemannian connections and flat basic connections. Then the proofs of Propositions 4.4 and 4.5 can be carried through, in the class of flat connections, for *any* k. Hence, in this case, one can define characteristic classes m_k for *any* k.

4.3. The Modular Class

The modular class of a Lie algebroid was introduced in [31], and further discussed in [9]. Extensions to more general algebraic settings were given in [15, 16, 34].

Let us start by recalling the construction given in [9]. Consider the line bundle $Q_A = \wedge^r A \otimes \wedge^m T^*M$. It is easy to check that on this line bundle we have a flat A-connection ∇ defined by:

$$\nabla_\alpha(\alpha^1 \wedge \cdots \wedge \alpha^r \otimes \mu) = \sum_{j=1}^{r} \alpha^1 \wedge \cdots \wedge [\alpha, \alpha^j] \wedge \cdots \wedge \alpha^r \otimes \mu + \alpha^1 \wedge \cdots \wedge \alpha^r \otimes \mathcal{L}_{\#\alpha}\mu, \quad (84)$$

whenever $\alpha, \alpha^1, \dots, \alpha^r \in \Gamma(A)$ and $\mu \in \Gamma(\wedge^m T^*M)$.

Now assume first that Q_A is trivial. Then we have a global section $s \in \Gamma(Q_A)$ so that

$$\nabla_\alpha s = \theta_s(\alpha)s, \qquad \forall \alpha \in \Gamma(A).$$

Since ∇ is flat, we see that θ_s defines a section of $\Gamma(A^*)$ which is closed: $d_A\theta_s = 0$. If s' is another global section in $\Gamma(Q_A)$, we have $s' = as$ for some nonvanishing smooth function a on M, and we find

$$\theta_{s'} = \theta_s + d_A \log |a|.$$

Therefore, we have a well defined cohomology class

$$\operatorname{mod}(A) \equiv [\theta_s] \in H^1(A)$$

which is independent of the section s. If the line bundle Q_A is not trivial one considers the square $L = Q_A \otimes Q_A$, which is trivial, and defines

$$\operatorname{mod}(A) = \frac{1}{2}[\theta_s],$$

for some global section $s \in \Gamma(L)$.

Definition 4.2. The class $\operatorname{mod}(A)$ is called the modular class of the Lie algebroid A.

As was argued in [9] one can think of global sections of Q_A (or $Q_A \otimes Q_A$) as “transverse measures” to A. The modular class is trivial iff there exists a transverse measure which is invariant under the flows of $X_\alpha \equiv (X_{f_\alpha}, \#\alpha)$, for every section $\alpha \in \Gamma(A)$. Hence, the modular class is an obstruction lying in the first Lie algebroid cohomology group $H^1(A)$ to the existence of a transverse invariant measure to A.

THEOREM 4.1. *For any Lie algebroid A*

$$m_1(A) = \frac{1}{2\pi}\mathrm{mod}\,(A). \tag{85}$$

Proof. Choose a basic connection ∇^1 and a riemannian connection ∇^0 relative to some metric on $E = A \ominus T^*M$. We consider the transverse measure s to A associated with this metric. We claim that

$$\lambda^{1,0}(\mathrm{tr}\,) = \theta_s, \tag{86}$$

so (85) follows.

Observe that it is enough to show that (86) holds on the regular points of M, since the set of regular points is an open dense set and both sides are smooth sections in $\Gamma(A^*)$. So assume that $x \in M$ is a regular point where rank $= q$, and pick coordinates (x^i) around x and a basis of sections $\{\alpha^1, \ldots, \alpha^r\}$ as in the Local Splitting Theorem. Then s is given locally by:

$$s = (\det g)^{\frac{1}{2}}\alpha^1 \wedge \cdots \wedge \alpha^r \otimes dx^1 \wedge \cdots \wedge dx^m,$$

where $g = (g^{ij}(x))$ is the matrix of inner products formed by elements in $\{\alpha^s, dx^i\}$.

As in the proofs of the previous section, one computes the trace of the operator ∇^1_α relative to the basis $\{\alpha^s, dx^i\}$ to be

$$\mathrm{tr}\,\nabla^1_{\alpha^s} = \sum_{u>q} c^{su}_u + \sum_{j>q} \frac{\partial b^{sj}}{\partial x^j}.$$

Also, since ∇^0 is a metric connection, we find:

$$\begin{aligned} 0 =& \bar{\nabla}^0_{\#\alpha^s} s \\ =& \#\alpha^s((\det g)^{\frac{1}{2}})\alpha^1 \wedge \cdots \wedge \alpha^r \otimes dx^1 \wedge \cdots \wedge dx^m + \\ &+ (\det g)^{\frac{1}{2}}(\bar{\nabla}^0_{\#\alpha^s}\alpha^1 \wedge \cdots \wedge \alpha^r \otimes dx^1 \wedge \cdots \wedge dx^m + \\ &\qquad + \cdots + \alpha^1 \wedge \cdots \wedge \alpha^r \otimes dx^1 \wedge \cdots \wedge \bar{\nabla}^0_{\#\alpha^s} dx^m) \\ =& \left(\#\alpha^s((\det g)^{\frac{1}{2}}) + (\det g)^{\frac{1}{2}}\mathrm{tr}\,\bar{\nabla}^0_{\#\alpha^s}\right)\alpha^1 \wedge \cdots \wedge \alpha^r \otimes dx^1 \wedge \cdots \wedge dx^m. \end{aligned}$$

So we conclude that:

$$\operatorname{tr}(\nabla^1_{\alpha^s} - \nabla^0_{\alpha^s})s = \#\alpha^s((\det g)^{\frac{1}{2}})\alpha^1 \wedge \cdots \wedge \alpha^r \otimes dx^1 \wedge \cdots \wedge dx^m + (\sum_{u>q} c^{su}_u + \sum_{j>q} \frac{\partial b^{sj}}{\partial x^j})s. \quad (87)$$

On the other hand, a straight forward computation using (84) and the various relations in the Local Splitting Theorem at a regular point, shows that

$$\begin{aligned}\nabla_{\alpha^s} s &= \#\alpha^s((\det g)^{\frac{1}{2}})\alpha^1 \wedge \cdots \wedge \alpha^r \otimes dx^1 \wedge \cdots \wedge dx^m \\ &\quad + (\det g)^{\frac{1}{2}}[\alpha^s, \alpha^1] \wedge \cdots \wedge \alpha^r \otimes dx^1 \wedge \cdots \wedge dx^m \\ &\quad + \cdots + (\det g)^{\frac{1}{2}}\alpha^1 \wedge \cdots \wedge \alpha^r \otimes dx^1 \wedge \cdots \wedge \mathcal{L}_{\alpha^s} dx^m \\ &= \#\alpha^s((\det g)^{\frac{1}{2}})\alpha^1 \wedge \cdots \wedge \alpha^r \otimes dx^1 \wedge \cdots \wedge dx^m + \\ &\quad + (\sum_{u>q} c^{su}_u + \sum_{j>q} \frac{\partial b^{sj}}{\partial x^j})s. \qquad (88)\end{aligned}$$

Comparing (87) and (88) gives

$$\nabla_\alpha s = \operatorname{tr}(\nabla^1_\alpha - \nabla^0_\alpha)s,$$

so relation (86) holds and the theorem follows. ▮

Remark. In a recent preprint [5], M. Crainic proposes a different approach to secondary characteristic classes for vector bundles admitting flat A-connections. Let E be any vector bundle over M which admits a flat$A$-connection and is trivial as a vector bundle. Then for each symmetric, G-invariant, $2k-1$-multilinear function P one can define a $2k-1$-section $\nu(P) \in \Gamma(\wedge^{2k-1}A^*)$ by setting

$$\nu(P)(\alpha_1, \ldots, \alpha_{2k}) = \sum_{\sigma \in S_{2k}} (-1)^\sigma P(\omega_j(\alpha_{\sigma(1)}), \ldots, \omega_j(\alpha_{\sigma(2k-1)})), \quad (89)$$

where $\alpha_1, \ldots, \alpha_{2k}$ are global sections of A. Here ω_j denotes a connection local 1-section which, since we are assuming that E is trivial, is actually globally defined. One can check that $\nu(P)$ is closed so actually defines a cohomology class in $H^{2k-1}(A)$.

If E is not trivial as vector bundle, one needs a kind of Cěch cohomology argument given in [5] to define these classes. These classes generalize the classes θ_s constructed above for a line bundle admitting a flat A-connection.

The "normal bundle" $E = A \ominus T^*M$ in general admits no flat A-connection (this is clear already in the Poisson case, where $A = T^*M$) and hence it is not obvious if one can use this approach to define the secondary classes $m_k(A)$. Perhaps, as stated in [5], it is possible to extend this approach to so called *representations up to homotopy* (in our notation, *flat connections up to homotopy*) so one can use the "adjoint representation" of A (see [9] for details).

4.4. Examples

We now consider some of the classes of Lie algebroids that we have mentioned in section 1 and compute their secondary characteristic classes.

Regular Foliations. Let $\mathcal{F}$ be a regular foliation, and denote by $A = T\mathcal{F} \subset TM$ the associated integrable subbundle. Observe that a section $\alpha \in \Gamma(A)$ is just a vector field on M tangent to $\mathcal{F}$, so A is a Lie algebroid with anchor the inclusion $A \subset TM$ and $[\ ,\ ]$ the usual Lie bracket of vector fields.

First choose some riemannian connection in M determining a splitting

$$T^*(M) = A^* \oplus \nu^*(\mathcal{F}),$$

where $\nu^*(\mathcal{F}))$ is the conormal bundle to the foliation. We have an A-riemannian connection $\check{\nabla}^0$ such that:

$$\check{\nabla}^0_\alpha(\beta + \gamma) = \nabla^{0,\|}_{\#\alpha}\beta + \nabla^{0,\perp}_{\#\alpha}\gamma,$$

where β and γ, are sections of $A^* = T^*(\mathcal{F})$ and $\nu^*(\mathcal{F})$, and $\nabla^{0,\|}$ and $\nabla^{0,\perp}$, are covariant riemannian connections in these bundles. We choose on $E = A \ominus T^*M$ the A-riemannian connection $\nabla^0_\alpha = \nabla^{0,\|}_{\#\alpha} \oplus \check{\nabla}^0_\alpha$. Note that we are using the same notation for a connection on a vector bundle and on its dual, but in fact, taking ∇^0 on E is essentially equivalent to take $\nabla^{0,\|}$ on $\nu^*(\mathcal{F})$.

Now we take as a basic connection (in the sense of definition 3.3) a connection ∇^1 on $E \simeq A \oplus A^* \oplus \nu^*(\mathcal{F})$ of the form $\nabla^1 = \nabla^{1,\|} \oplus \nabla^{1,\|} \oplus \nabla^{1,\perp}$ where $\nabla^{1,\|}$ is a connection in A and $\nabla^{1,\perp}$ is just a basic connection in $\nu^*(\mathcal{F})$ in the usual sense of foliation theory (see [2], p. 33). A straightforward computation shows that

$$\lambda(\nabla^1, \nabla^0)(P)(\alpha_1, \dots, \alpha_{2k-1}) = \lambda(\nabla^{1,\perp}, \nabla^{0,\perp})(P)((\#\alpha_1, \dots, \#\alpha_{2k-1}).$$

Recall that in foliation theory (see [2], p. 66) the forms

$$c_k = \lambda(\nabla^{1,\perp})(\tilde{P}_k), \qquad (1 \le k \le q)$$
$$h_{2k-1} = \lambda(\nabla^{1,\perp}, \nabla^{0,\perp})(\tilde{P}_{2k-1}), \qquad (1 \le 2k-1 \le q),$$

satisfy

$$dc_k = 0, \qquad (1 \le k \le \operatorname{corank}(\mathcal{F})) \tag{90}$$
$$dh_{2k-1} = c_{2k-1}, \qquad (1 \le 2k-1 \le \operatorname{corank}(\mathcal{F})). \tag{91}$$

and so they can be used to define a homomorphism of graded algebras

$$H^*(WO_q) \to H^*(M),$$

where $H^*(WO_q)$ is the relative Gelfand-Fuks cohomology of formal vector fields in $\mathbb{R}^q$. This homomorphism is independent of the connections and its image are the exotic or secondary characteristic classes of foliation theory.

Observe that the $(2k-1)$-forms $h_{2k-1} = \lambda(\nabla^{1,\perp}, \nabla^{0,\perp})(P_k)$ are not closed in general, but are closed along the leaves, so its image under # is a closed $(2k-1)$-section of $T^*\mathcal{F}^*$. Hence, $\#h_{2k-1}$ defines a tangential cohomology class, and one has

$$m_{2k-1}(T\mathcal{F}) = [\#h_{2k-1}] \tag{92}$$

but, in general, m_{2k-1} is not in the image of $\#^* : H^\bullet_{\text{de Rham}}(M) \to H^\bullet(A)$. A simple consequence of this relationship is that, for a regular foliation, the characteristic classes $m_k(T\mathcal{F})$ vanish for $2k-1 > \operatorname{corank}(\mathcal{F})$.

We point out that these classes were known to people working in foliation theory (see e.g. [13]).

Poisson Manifolds. Let (M, Π) be a Poisson manifold, so $A = T^*M$. A basic connection on A is a basic contravariant connection ∇ in the sense of [10]. and we can take for ∇^1 on $E = T^*M \ominus T^*M$ the connection $\nabla^1 = \nabla \oplus \nabla$. Also, we let $\bar{\nabla}$ be some riemannian connection on T^*M and set $\nabla^0 = \bar{\nabla} \oplus \bar{\nabla}$. It is clear that

$$\lambda(\nabla^1, \nabla^0)(P) = 2\lambda(\nabla, \bar{\nabla})(P),$$

and it follows that the characteristic classes we have defined for $A = T^*M$ are equal to twice the characteristic classes we have defined in [10] for the case of a Poisson manifold:

$$m_k(T^*M) = 2m_k(M).$$

A special case where computations can be made explicitly is when $M = \mathfrak{g}^*$ with the Lie-Poisson structure. We have shown in [10] that these classes are represented by Lie algebra cocycles given by the general formula:

$$m_k(\mathfrak{g}^*)(v_1, \ldots, v_{2k-1}) = \frac{1}{(2\pi)^k} \sum_{\sigma \in S_{2k-1}} K_k(v_{\sigma(1)}, [v_{\sigma(2)}, v_{\sigma(3)}], \ldots, [v_{\sigma(2k-2)}, v_{\sigma(2k-1)}]) \quad (93)$$

where:

$$K_j(v_1, \ldots, v_j) \equiv \operatorname{tr}(\operatorname{ad} v_1 \cdots \operatorname{ad} v_j).$$

Transformation Lie algebroids. Consider an infinitesimal action $\rho : \mathfrak{g} \to \mathcal{X}^1(M)$ of a Lie algebra $\mathfrak{g}$ on a manifold M, so $A = M \times \mathfrak{g}$. Sections of A can be identified with $\mathfrak{g}$-valued functions on M, so if $v \in \mathfrak{g}$ we identify it with a constant section.

There is a canonical choice of basic A-connection on A, namely the unique A-connection which for constant sections satisfies:

$$\nabla_v w = [v, w].$$

This connection is compatible with the Lie algebroid since $\#\nabla_v w = \check{\nabla}_v \# w$ where $\check{\nabla}$ is the A-connection on TM which for any constant section $v \in \mathfrak{g}$ and vector field $X \in \mathcal{X}^1(M)$ satisfies

$$\nabla_v X = \mathcal{L}_{\rho(v)} X.$$

Hence we take the connection ∇^1 on $E = A \ominus T^*M$ given by

$$\nabla^1_v(w + \omega) = [v, w] - \mathcal{L}_{\rho(v)}\omega.$$

Now pick some riemannian metric on M and consider the flat metric on $\mathfrak{g}$. This gives a riemannian connection ∇^0 on E and we can then compute the secondary characteristic classes. Formula (76) shows that the classes $\lambda^{1,0}(P)$, in general, will depend on the curvature R^t of $t\nabla^1 + (1-t)\nabla^0$ in some intricate way. However, for the first characteristic class where $P = \frac{1}{2\pi}\operatorname{tr}$, there is no dependence on the curvature, and we find explicitly:

$$m_1(A)(x) = \frac{1}{2\pi}\operatorname{tr}\operatorname{ad} x - \operatorname{div}(\rho(x))$$

(again, we view $x \in \mathfrak{g}$ has a constant section of A), where div is the divergence operator on vector fields defined by the metric on M.

Assume further that $M = V$ is a vector space and ρ is a linear action (a Lie algebra representation). In this case we can also choose on V a flat metric and it is possible to compute all characteristic classes. Let $\tilde{\rho}$ denote the direct sum of representations $\operatorname{ad} \oplus \rho^*$ on $\mathfrak{g} \times V^*$, where $\rho^* : \mathfrak{g} \to \mathfrak{gl}(V^*)$ is the dual representation to $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. Then the general formula for the characteristic classes m_k is:

$$m_k(A)(x_1, \ldots, x_{2k-1}) = \\ C_k \sum_{\sigma \in S_{2k-1}} P_k(\tilde{\rho}(v_{\sigma(1)}), \tilde{\rho}([v_{\sigma(2)}, v_{\sigma(3)}]), \ldots, \tilde{\rho}([v_{\sigma(2k-2)}, v_{\sigma(2k-1)}])) \quad (94)$$

where P_k are the elementary symmetric polynomials and C_k is some numerical factor.

For example, if we let

$$\rho = \operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$$

be the coadjoint action of $\mathfrak{g}$ we see that $m_k(A)$ is twice the Lie algebra cohomology classes given by (93), i.e., they coincide with the characteristic classes for $A = T^*M$, where we take $M = \mathfrak{g}^*$ with the Lie-Poisson bracket.

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