Lie’s Third Theorem

Rui Loja Fernandes

Departamento de Matemática
Instituto Superior Técnico

UAB Math Dep Colloquium
Outline

1. Classical Lie Theory
   - Historical Origins
   - Finite dimensional Lie groups and Lie algebras

2. Lie Theory beyond finite dimensions
   - Motivation
   - Examples

3. Lie Groupoid Theory
   - Groupoids
   - Lie Groupoids
   - Lie Algebroids
   - Geometric Lie theory

4. Lie III revisited
   - Obstructions to integrability
   - The proof
Symmetries of Differential Equations

Sophus Lie, influenced by Felix Klein, proposed:

**Definition**

The **group of symmetries** of a differential equation:

$$\Delta(x, y, \ldots, u, v, \ldots, u_x, v_x, u_{xx}, \ldots) = 0,$$

is the set of all transformation of the independent variables \((x, y, \ldots)\) and of the dependent variables \((u, v, \ldots)\) that transform solutions to solutions.
Lie aimed (and achieved) a **Galois theory** for differential equations:

- he proved that if the group of symmetries is *solvable* then the differential equation can be integrated by quadratures.
- he found a method to compute the group of symmetries.

Unlike the permutation groups of symmetries of algebraic equations, Lie’s symmetry groups are *continuous*.
Symmetries of Differential Equations

Lie aimed (and achieved) a **Galois theory** for differential equations:

- he proved that if the group of symmetries is *solvable* then the differential equation can be integrated by quadratures.
- he found a method to compute the group of symmetries.

Unlike the permutation groups of symmetries of algebraic equations, Lie’s symmetry groups are **continuous**.
Symmetries of Differential Equations

Lie aimed (and achieved) a **Galois theory** for differential equations:

- he proved that if the group of symmetries is *solvable* then the differential equation can be integrated by quadratures.
- he found a method to compute the group of symmetries.

Unlike the permutation groups of symmetries of algebraic equations, Lie’s symmetry groups are **continuous**.
Lie aimed (and achieved) a *Galois theory* for differential equations:

- he proved that if the group of symmetries is *solvable* then the differential equation can be integrated by quadratures.
- he found a method to compute the group of symmetries.

Unlike the permutation groups of symmetries of algebraic equations, Lie’s symmetry groups are *continuous*.
Example: The heat equation

The symmetry group of the heat equation:

$$u_t = u_{xx}$$

is generated by the following transformations:

$$(x, t, u) \mapsto (x + \varepsilon, t, u)$$

$$(x, t, u) \mapsto (x, t + \varepsilon, u)$$

$$(x, t, u) \mapsto (x, t, e^{\varepsilon}u)$$

$$(x, t, u) \mapsto \left( \frac{x}{1 - 4\varepsilon t}, \frac{t}{1 - 4\varepsilon t}, u\sqrt{1 - 4\varepsilon t}e^{\frac{-\varepsilon x^2}{1 - 4\varepsilon t}} \right)$$

where $$\varepsilon \in \mathbb{R}$$ and $$\alpha(x, t)$$ is an arbitrary solution of the heat equation.
Example: The heat equation

The symmetry group of the heat equation:

\[ u_t = u_{xx} \]

is generated by the following transformations:

\[
\begin{align*}
(x, t, u) &\mapsto (x + \varepsilon, t, u) \\
(x, t, u) &\mapsto (x, t + \varepsilon, u) \\
(x, t, u) &\mapsto (x, t, e^{\varepsilon} u) \\
(x, t, u) &\mapsto \left( \frac{x}{1-4\varepsilon t}, \frac{t}{1-4\varepsilon t}, u\sqrt{1 - 4\varepsilon t} e^{-\frac{\varepsilon x^2}{4\varepsilon t}} \right)
\end{align*}
\]

where \( \varepsilon \in \mathbb{R} \) and \( \alpha(x, t) \) is an arbitrary solution of the heat equation.
Problem

How can one find the symmetry group $G_\Delta$ of a given differential equation $\Delta = 0$?

Each 1-parameter group of symmetries:

$\mathbb{R} \ni \varepsilon \mapsto T_\varepsilon \in G_\Delta,$

determines an infinitesimal symmetry, i.e., a vector field:

$$X(x, y, \ldots, u, v \ldots) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} T_\varepsilon(x, y, \ldots, u, v, \ldots)$$

Lie found that the infinitesimal symmetries of $\Delta$ are the solutions of a system of first order linear p.d.e.

$\implies$ systematic method to compute symmetries
From global to infinitesimal

Problem

How can one find the symmetry group $G_\Delta$ of a given differential equation $\Delta = 0$?

Each 1-parameter group of symmetries:

$$\mathbb{R} \ni \varepsilon \mapsto T_\varepsilon \in G_\Delta,$$

determines an infinitesimal symmetry, i.e., a vector field:

$$X(x, y, \ldots, u, v \ldots) = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} T_\varepsilon(x, y, \ldots, u, v, \ldots)$$

Lie found that the infinitesimal symmetries of $\Delta$ are the solutions of a system of first order linear p.d.e.

$\implies$ systematic method to compute symmetries
From global to infinitesimal

**Problem**

*How can one find the symmetry group $G_\Delta$ of a given differential equation $\Delta = 0$?*

Each 1-parameter group of symmetries:

$$\mathbb{R} \ni \varepsilon \mapsto T_\varepsilon \in G_\Delta,$$

determines an **infinitesimal symmetry**, i.e., a vector field:

$$X(x, y, \ldots, u, v \ldots) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} T_\varepsilon(x, y, \ldots, u, v, \ldots)$$

Lie found that the infinitesimal symmetries of $\Delta$ are the solutions of a system of first order linear p.d.e.

$\Rightarrow$ systematic method to compute symmetries

Rui Loja Fernandes

Lie's Third Theorem
From global to infinitesimal

Problem

How can one find the symmetry group \( G_\Delta \) of a given differential equation \( \Delta = 0 \)?

Each 1-parameter group of symmetries:

\[
\mathbb{R} \ni \varepsilon \mapsto T_\varepsilon \in G_\Delta,
\]

determines an infinitesimal symmetry, i.e., a vector field:

\[
X(x, y, \ldots, u, v \ldots) = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} T_\varepsilon(x, y, \ldots, u, v, \ldots)
\]

Lie found that the infinitesimal symmetries of \( \Delta \) are the solutions of a system of first order linear p.d.e.

\[\implies \text{systematic method to compute symmetries}\]
From global to infinitesimal and back

Lie also noted that:

- The vector space $g_\Delta$ of all infinitesimal symmetries is closed under the commutator of vector fields:

$$X_1, X_2 \in g_\Delta \implies [X_1, X_2] \in g_\Delta.$$

Lie claimed that:

**Theorem**

Any space $g$ of vector fields closed under the commutator is the set of infinitesimal symmetries of a group of symmetries $G$.

- Is this really true?
Lie also noted that:

- The vector space $g_\Delta$ of all infinitesimal symmetries is closed under the commutator of vector fields:

$$X_1, X_2 \in g_\Delta \implies [X_1, X_2] \in g_\Delta.$$

Lie claimed that:

**Theorem**

*K Any space $g$ of vector fields closed under the commutator is the set of infinitesimal symmetries of a group of symmetries $G.*

Is this really true?
From global to infinitesimal and back

Lie also noted that:

- The vector space $\mathfrak{g}_\Delta$ of all infinitesimal symmetries is closed under the commutator of vector fields:

$$X_1, X_2 \in \mathfrak{g}_\Delta \implies [X_1, X_2] \in \mathfrak{g}_\Delta.$$

Lie claimed that:

**Theorem**

Any space $\mathfrak{g}$ of vector fields closed under the commutator is the set of infinitesimal symmetries of a group of symmetries $G$.

- Is this really true?
Lie groups and Lie algebras

Definition

A **Lie group** is a manifold $G$ together with a group structure on $G$ such that the product and inversion are smooth:

$$G \times G \to G, \quad (g, h) \mapsto gh, \quad G \to G, \quad g \mapsto g^{-1}.$$ 

Definition

A **Lie algebra** is a vector space $g$ together with a bilinear, skew-symmetric, bracket $[\cdot, \cdot] : g \times g \to g$, which satisfies the Jacobi identity:

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0.$$ 

Rui Loja Fernandes

Lie's Third Theorem
Lie groups and Lie algebras

Definition

A **Lie group** is a manifold $G$ together with a group structure on $G$ such that the product and inversion are smooth:

$$G \times G \rightarrow G, \ (g, h) \mapsto gh, \quad G \rightarrow G, \ g \mapsto g^{-1}.$$ 

Definition

A **Lie algebra** is a vector space $g$ together with a bilinear, skew-symmetric, bracket $[\cdot, \cdot] : g \times g \rightarrow g$, which satisfies the Jacobi identity:

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0.$$
Let $G$ be a *finite dimensional* Lie group. Its Lie algebra $\mathfrak{g} = \mathcal{L}(G)$ is constructed as follows:

- As a vector space, $\mathfrak{g} := T_e G$;
- Bracket: given $u \in \mathfrak{g}$ let $\tilde{u}$ be the right invariant vector field with $\tilde{u}|_e = u$. The bracket of $u, v \in \mathfrak{g}$ is given by:

$$[u, v] := [\tilde{u}, \tilde{v}]|_e$$
From Lie groups to Lie algebras

Let $G$ be a **finite dimensional** Lie group. Its Lie algebra $\mathfrak{g} = \mathcal{L}(G)$ is constructed as follows:

- As a vector space, $\mathfrak{g} := T_e G$;
- Bracket: given $u \in \mathfrak{g}$ let $\tilde{u}$ be the right invariant vector field with $\tilde{u}|_e = u$. The bracket of $u, v \in \mathfrak{g}$ is given by:

$$[u, v] := [\tilde{u}, \tilde{v}]|_e$$
From Lie groups to Lie algebras

Let $G$ be a finite dimensional Lie group. Its Lie algebra $\mathfrak{g} = \mathcal{L}(G)$ is constructed as follows:

- As a vector space, $\mathfrak{g} := T_e G$;
- Bracket: given $u \in \mathfrak{g}$ let $\tilde{u}$ be the right invariant vector field with $\tilde{u}|_e = u$. The bracket of $u, v \in \mathfrak{g}$ is given by:

$$[u, v] := [\tilde{u}, \tilde{v}]|_e$$
From Lie groups to Lie algebras

Let $G$ be a *finite dimensional* Lie group. Its Lie algebra $\mathfrak{g} = \mathcal{L}(G)$ is constructed as follows:

- As a vector space, $\mathfrak{g} := T_eG$;
- Bracket: given $u \in \mathfrak{g}$ let $\tilde{u}$ be the right invariant vector field with $\tilde{u}|_e = u$. The bracket of $u, v \in \mathfrak{g}$ is given by:

$$[u, v] := [\tilde{u}, \tilde{v}]|_e$$
Let $G$ be a *finite dimensional* Lie group. Its Lie algebra $\mathfrak{g} = \mathcal{L}(G)$ is constructed as follows:

- As a vector space, $\mathfrak{g} := T_eG$;
- Bracket: given $u \in \mathfrak{g}$ let $\tilde{u}$ be the right invariant vector field with $\tilde{u}|_e = u$. The bracket of $u, v \in \mathfrak{g}$ is given by:

$$[u, v] := [\tilde{u}, \tilde{v}]|_e$$
From Lie groups to Lie algebras

Let $G$ be a finite dimensional Lie group. Its Lie algebra $\mathfrak{g} = \mathcal{L}(G)$ is constructed as follows:

- As a vector space, $\mathfrak{g} := T_e G$;
- Bracket: given $u \in \mathfrak{g}$ let $\tilde{u}$ be the right invariant vector field with $\tilde{u}|_e = u$. The bracket of $u, v \in \mathfrak{g}$ is given by:

$$[u, v] := [\tilde{u}, \tilde{v}]|_e$$
Let $G$ be a *finite dimensional* Lie group. Its Lie algebra $\mathfrak{g} = \mathcal{L}(G)$ is constructed as follows:

- As a vector space, $\mathfrak{g} := T_eG$;
- Bracket: given $u \in \mathfrak{g}$ let $\tilde{u}$ be the right invariant vector field with $\tilde{u}|_e = u$. The bracket of $u, v \in \mathfrak{g}$ is given by:

$$[u, v] := [\tilde{u}, \tilde{v}]|_e$$
### Examples

<table>
<thead>
<tr>
<th>Lie Group</th>
<th>Lie Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>General Linear Group: $GL(n) = { A \in M_n(\mathbb{R}) : \det A \neq 0 }$</td>
<td>$\mathfrak{gl}(n) = { A \in M_n(\mathbb{R}) }$</td>
</tr>
<tr>
<td>Special Linear Group: $SL(n) = { A \in GL_n(\mathbb{R}) : \det A = 1 }$</td>
<td>$\mathfrak{sl}(n) = { A \in \mathfrak{gl}(n) : \text{tr} A = 0 }$</td>
</tr>
<tr>
<td>Special Orthogonal Group: $SO(n) = { A \in SL(n, \mathbb{R}) : AA^T = I }$</td>
<td>$\mathfrak{so}(n) = { A \in \mathfrak{sl}(\mathbb{R}) : A + A^T = 0 }$</td>
</tr>
<tr>
<td>Special Unitary Group: $SU(n) = { A \in SL(n, \mathbb{C}) : A\bar{A}^T = I }$</td>
<td>$\mathfrak{su}(n) = { A \in \mathfrak{sl}(\mathbb{C}) : A + \bar{A}^T = 0 }$</td>
</tr>
<tr>
<td>Symplectic Group: $Sp(n) = { A \in GL(2n, \mathbb{R}) : AJ + JA^T = 0 }$</td>
<td>$\mathfrak{sp}(n) = { A \in \mathfrak{gl}(2n, \mathbb{R}) : AJ + JA^T = 0 }$</td>
</tr>
<tr>
<td>Group of isometries of $(M, g)$: $G = { \phi : M \to M \mid \phi \text{ preserves } g }$</td>
<td>$g = { X \in \mathfrak{X}(M) \mid \mathcal{L}_X g = 0 }$</td>
</tr>
<tr>
<td>Group of symplectomorphisms of $(M, \omega)$: $G = { \phi : M \to M \mid \phi^* \omega = \omega }$</td>
<td>$g = { X \in \mathfrak{X}(M) \mid \mathcal{L}_X \omega = 0 }$</td>
</tr>
</tbody>
</table>
### Examples

<table>
<thead>
<tr>
<th><strong>Lie Group</strong></th>
<th><strong>Lie Algebra</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>General Linear Group: $GL(n) = {A \in M_n(\mathbb{R}) : \det A \neq 0}$</td>
<td>$gl(n) = {A \in M_n(\mathbb{R})}$</td>
</tr>
<tr>
<td>Special Linear Group: $SL(n) = {A \in GL_n(\mathbb{R}) : \det A = 1}$</td>
<td>$sl(n) = {A \in gl(n) : \text{tr} A = 0}$</td>
</tr>
<tr>
<td>Special Orthogonal Group: $SO(n) = {A \in SL(n, \mathbb{R}) : AA^T = I}$</td>
<td>$so(n) = {A \in sl(\mathbb{R}) : A + A^T = 0}$</td>
</tr>
<tr>
<td>Special Unitary Group: $SU(n) = {A \in SL(n, \mathbb{C}) : A\bar{A}^T = I}$</td>
<td>$su(n) = {A \in sl(n, \mathbb{C}) : A + \bar{A}^T = 0}$</td>
</tr>
<tr>
<td>Symplectic Group: $Sp(n) = {A \in GL(2n, \mathbb{R}) : AJA^T = J}$</td>
<td>$sp(n) = {A \in gl(2n, \mathbb{R}) : AJ + JA^T = 0}$</td>
</tr>
<tr>
<td>Group of isometries of $(M, g)$: $G = {\phi : M \to M</td>
<td>\phi \text{ preserves } g}$</td>
</tr>
<tr>
<td>Group of symplectomorphisms of $(M, \omega)$: $G = {\phi : M \to M</td>
<td>\phi^* \omega = \omega}$</td>
</tr>
</tbody>
</table>
## Examples

<table>
<thead>
<tr>
<th><strong>Lie Group</strong></th>
<th><strong>Lie Algebra</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>General Linear Group: ( GL(n) = { A \in M_n(\mathbb{R}) : \det A \neq 0 } )</td>
<td>( \mathfrak{gl}(n) = { A \in M_n(\mathbb{R}) } )</td>
</tr>
<tr>
<td>Special Linear Group: ( SL(n) = { A \in GL_n(\mathbb{R}) : \det A = 1 } )</td>
<td>( \mathfrak{sl}(n) = { A \in \mathfrak{gl}(n) : \text{tr} A = 0 } )</td>
</tr>
<tr>
<td>Special Orthogonal Group: ( SO(n) = { A \in SL(n, \mathbb{R}) : AA^T = I } )</td>
<td>( \mathfrak{so}(n) = { A \in \mathfrak{sl}(\mathbb{R}) : A + A^T = 0 } )</td>
</tr>
<tr>
<td>Special Unitary Group: ( SU(n) = { A \in SL(n, \mathbb{C}) : A\overline{A}^T = I } )</td>
<td>( \mathfrak{su}(n) = { A \in \mathfrak{sl}(n, \mathbb{C}) : A + \overline{A}^T = 0 } )</td>
</tr>
<tr>
<td>Symplectic Group: ( Sp(n) = { A \in GL(2n, \mathbb{R}) : AJA^T = J } )</td>
<td>( \mathfrak{sp}(n) = { A \in \mathfrak{gl}(2n, \mathbb{R}) : AJ + JA^T = 0 } )</td>
</tr>
<tr>
<td>Group of isometries of ((M, g)): ( G = { \phi : M \to M \mid \phi \text{ preserves } g } )</td>
<td>( g = { X \in \mathfrak{X}(M) \mid \mathcal{L}_X g = 0 } )</td>
</tr>
<tr>
<td>Group of symplectomorphisms of ((M, \omega)): ( G = { \phi : M \to M \mid \phi^*\omega = \omega } )</td>
<td>( g = { X \in \mathfrak{X}(M) \mid \mathcal{L}_X \omega = 0 } )</td>
</tr>
</tbody>
</table>
## Examples

<table>
<thead>
<tr>
<th><strong>Lie Group</strong></th>
<th><strong>Lie Algebra</strong></th>
</tr>
</thead>
</table>
| **General Linear Group:**  
$GL(n) = \{ A \in M_n(\mathbb{R}) : \det A \neq 0 \} $ | $gl(n) = \{ A \in M_n(\mathbb{R}) \} $ |
| **Special Linear Group:**  
$SL(n) = \{ A \in GL_n(\mathbb{R}) : \det A = 1 \} $ | $sl(n) = \{ A \in gl(n) : \text{tr } A = 0 \} $ |
| **Special Orthogonal Group:**  
$SO(n) = \{ A \in SL(n, \mathbb{R}) : AA^T = I \} $ | $so(n) = \{ A \in sl(n) : A + A^T = 0 \} $ |
| **Special Unitary Group:**  
$SU(n) = \{ A \in SL(n, \mathbb{C}) : A \overline{A}^T = I \} $ | $su(n) = \{ A \in sl(n, \mathbb{C}) : A + \overline{A}^T = 0 \} $ |
| **Symplectic Group:**  
$Sp(n) = \{ A \in GL(2n, \mathbb{R}) : AJA^T = J \} $ | $sp(n) = \{ A \in gl(2n, \mathbb{R}) : AJ + JA^T = 0 \} $ |
| **Group of isometries of** $(M, g)$:  
$G = \{ \phi : M \rightarrow M \mid \phi \text{ preserves } g \} $ | $g = \{ X \in \mathfrak{X}(M) \mid L_Xg = 0 \} $ |
| **Group of symplectomorphisms of** $(M, \omega)$:  
$G = \{ \phi : M \rightarrow M \mid \phi^*\omega = \omega \} $ | $g = \{ X \in \mathfrak{X}(M) \mid L_X\omega = 0 \} $ |
## Examples

<table>
<thead>
<tr>
<th>Lie Group</th>
<th>Lie Algebra</th>
</tr>
</thead>
</table>
| General Linear Group:  
$GL(n) = \{ A \in M_n(\mathbb{R}) : \det A \neq 0 \}$ | $gl(n) = \{ A \in M_n(\mathbb{R}) \}$ |
| Special Linear Group:  
$SL(n) = \{ A \in GL_n(\mathbb{R}) : \det A = 1 \}$ | $sl(n) = \{ A \in gl(n) : \text{tr} A = 0 \}$ |
| Special Orthogonal Group:  
$SO(n) = \{ A \in SL(n, \mathbb{R}) : AA^T = I \}$ | $so(n) = \{ A \in sl(\mathbb{R}) : A + A^T = 0 \}$ |
| Special Unitary Group:  
$SU(n) = \{ A \in SL(n, \mathbb{C}) : A\overline{A}^T = I \}$ | $su(n) = \{ A \in sl(n, \mathbb{C}) : A + \overline{A}^T = 0 \}$ |
| Symplectic Group:  
$Sp(n) = \{ A \in GL(2n, \mathbb{R}) : AJA^T = J \}$ | $sp(n) = \{ A \in gl(2n, \mathbb{R}) : AJ + JA^T = 0 \}$ |
| Group of isometries of $(M, g)$:  
$G = \{ \phi : M \to M | \phi \text{ preserves } g \}$ | $g = \{ X \in \mathfrak{X}(M) | \mathcal{L}_X g = 0 \}$ |
| Group of symplectomorphisms of $(M, \omega)$:  
$G = \{ \phi : M \to M | \phi^*\omega = \omega \}$ | $g = \{ X \in \mathfrak{X}(M) | \mathcal{L}_X \omega = 0 \}$ |
### Examples

<table>
<thead>
<tr>
<th><strong>Lie Group</strong></th>
<th><strong>Lie Algebra</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>General Linear Group:</td>
<td></td>
</tr>
<tr>
<td>$GL(n) = {A \in M_n(\mathbb{R}) : \det A \neq 0}$</td>
<td>$\mathfrak{gl}(n) = {A \in M_n(\mathbb{R})}$</td>
</tr>
<tr>
<td>Special Linear Group:</td>
<td></td>
</tr>
<tr>
<td>$SL(n) = {A \in GL_n(\mathbb{R}) : \det A = 1}$</td>
<td>$\mathfrak{sl}(n) = {A \in \mathfrak{gl}(n) : \text{tr} A = 0}$</td>
</tr>
<tr>
<td>Special Orthogonal Group:</td>
<td></td>
</tr>
<tr>
<td>$SO(n) = {A \in SL(n, \mathbb{R}) : AA^T = I}$</td>
<td>$\mathfrak{so}(n) = {A \in \mathfrak{sl}(\mathbb{R}) : A + A^T = 0}$</td>
</tr>
<tr>
<td>Special Unitary Group:</td>
<td></td>
</tr>
<tr>
<td>$SU(n) = {A \in SL(n, \mathbb{C}) : AA^\dagger = I}$</td>
<td>$\mathfrak{su}(n) = {A \in \mathfrak{sl}(\mathbb{C}) : A + A^\dagger = 0}$</td>
</tr>
<tr>
<td>Symplectic Group:</td>
<td></td>
</tr>
<tr>
<td>$Sp(n) = {A \in GL(2n, \mathbb{R}) : AJA^T = J}$</td>
<td>$\mathfrak{sp}(n) = {A \in \mathfrak{gl}(2n, \mathbb{R}) : AJ + JA^T = 0}$</td>
</tr>
<tr>
<td>Group of isometries of $(M, g)$</td>
<td></td>
</tr>
<tr>
<td>$G = {\phi : M \to M \mid \phi \text{ preserves } g}$</td>
<td>$g = {X \in \mathfrak{X}(M) \mid \mathcal{L}_X g = 0}$</td>
</tr>
<tr>
<td>Group of symplectomorphisms of $(M, \omega)$</td>
<td></td>
</tr>
<tr>
<td>$G = {\phi : M \to M \mid \phi^*\omega = \omega}$</td>
<td>$g = {X \in \mathfrak{X}(M) \mid \mathcal{L}_X \omega = 0}$</td>
</tr>
</tbody>
</table>
### Examples

<table>
<thead>
<tr>
<th>Lie Group</th>
<th>Lie Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>General Linear Group: $GL(n) = {A \in M_n(\mathbb{R}) : \det A \neq 0}$</td>
<td>$\mathfrak{gl}(n) = {A \in M_n(\mathbb{R})}$</td>
</tr>
<tr>
<td>Special Linear Group: $SL(n) = {A \in GL_n(\mathbb{R}) : \det A = 1}$</td>
<td>$\mathfrak{sl}(n) = {A \in \mathfrak{gl}(n) : \text{tr } A = 0}$</td>
</tr>
<tr>
<td>Special Orthogonal Group: $SO(n) = {A \in SL(n, \mathbb{R}) : AA^T = I}$</td>
<td>$\mathfrak{so}(n) = {A \in \mathfrak{sl}(\mathbb{R}) : A + A^T = 0}$</td>
</tr>
<tr>
<td>Special Unitary Group: $SU(n) = {A \in SL(n, \mathbb{C}) : A\overline{A}^T = I}$</td>
<td>$\mathfrak{su}(n) = {A \in \mathfrak{sl}(\mathbb{C}) : A + \overline{A}^T = 0}$</td>
</tr>
<tr>
<td>Symplectic Group: $Sp(n) = {A \in GL(2n, \mathbb{R}) : AJA^T = J}$</td>
<td>$\mathfrak{sp}(n) = {A \in \mathfrak{gl}(2n, \mathbb{R}) : AJ + JA^T = 0}$</td>
</tr>
<tr>
<td>Group of isometries of $(M, g)$: $G = {\phi : M \to M \mid \phi \text{ preserves } g}$</td>
<td>$\mathfrak{g} = {X \in \mathfrak{X}(M) \mid \mathcal{L}_X g = 0}$</td>
</tr>
<tr>
<td>Group of symplectomorphisms of $(M, \omega)$: $G = {\phi : M \to M \mid \phi^* \omega = \omega}$</td>
<td>$\mathfrak{g} = {X \in \mathfrak{X}(M) \mid \mathcal{L}_X \omega = 0}$</td>
</tr>
</tbody>
</table>
### Examples

<table>
<thead>
<tr>
<th><strong>Lie Group</strong></th>
<th><strong>Lie Algebra</strong></th>
</tr>
</thead>
</table>
| General Linear Group:  
$GL(n) = \{A \in M_n(\mathbb{R}) : \det A \neq 0\}$ | $gl(n) = \{A \in M_n(\mathbb{R})\}$ |
| Special Linear Group:  
$SL(n) = \{A \in GL_n(\mathbb{R}) : \det A = 1\}$ | $sl(n) = \{A \in gl(n) : \text{tr } A = 0\}$ |
| Special Orthogonal Group:  
$SO(n) = \{A \in SL(n, \mathbb{R}) : AA^T = I\}$ | $so(n) = \{A \in sl(\mathbb{R}) : A + A^T = 0\}$ |
| Special Unitary Group:  
$SU(n) = \{A \in SL(n, \mathbb{C}) : AA^\dagger = I\}$ | $su(n) = \{A \in sl(n, \mathbb{C}) : A + A^\dagger = 0\}$ |
| Symplectic Group:  
$Sp(n) = \{A \in GL(2n, \mathbb{R}) : AJA^T = J\}$ | $sp(n) = \{A \in gl(2n, \mathbb{R}) : AJ + JA^T = 0\}$ |
| Group of isometries of $(M, g)$:  
$G = \{\phi : M \to M \mid \phi \text{ preserves } g\}$ | $g = \{X \in \mathcal{X}(M) \mid \mathcal{L}_X g = 0\}$ |
| Group of symplectomorphisms of $(M, \omega)$:  
$G = \{\phi : M \to M \mid \phi^* \omega = \omega\}$ | $g = \{X \in \mathcal{X}(M) \mid \mathcal{L}_X \omega = 0\}$ |
From Lie algebras to Lie groups

Theorem (Lie I)

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. There exists a unique (up to isomorphism) 1-connected Lie group $\tilde{G}$ with Lie algebra $\mathfrak{g}$.

Theorem (Lie II)

Let $G$ and $H$ be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, where $G$ is 1-connected. Given a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$, there exists a unique Lie group homomorphism $\Phi : G \to H$ with $(\Phi)_* = \phi$.

Theorem (Lie III)

For every Lie algebra $\mathfrak{g}$ there exists a Lie group $G$ with Lie algebra $\mathfrak{g}$.
From Lie algebras to Lie groups

**Theorem (Lie I)**

*Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. There exists a unique (up to isomorphism) 1-connected Lie group $\tilde{G}$ with Lie algebra $\mathfrak{g}$.***

**Theorem (Lie II)**

*Let $G$ and $H$ be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, where $G$ is 1-connected. Given a Lie algebra homomorphism $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$, there exists a unique Lie group homomorphism $\Phi : G \rightarrow H$ with $(\Phi)_* = \phi$.***

**Theorem (Lie III)**

*For every Lie algebra $\mathfrak{g}$ there exists a Lie group $G$ with Lie algebra $\mathfrak{g}$.***
**From Lie algebras to Lie groups**

<table>
<thead>
<tr>
<th>Theorem (Lie I)</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>Let G be a Lie group with Lie algebra</em> $\mathfrak{g}$. <em>There exists a unique (up to isomorphism) 1-connected Lie group</em> $\tilde{G}$ <em>with Lie algebra</em> $\mathfrak{g}$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem (Lie II)</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>Let G and H be Lie groups with Lie algebras</em> $\mathfrak{g}$ <em>and</em> $\mathfrak{h}$, <em>where G is 1-connected. Given a Lie algebra homomorphism</em> $\phi : \mathfrak{g} \to \mathfrak{h}$, <em>there exists a unique Lie group homomorphism</em> $\Phi : G \to H$ <em>with</em> $(\Phi)_* = \phi$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem (Lie III)</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>For every Lie algebra</em> $\mathfrak{g}$ <em>there exists a Lie group G with Lie algebra</em> $\mathfrak{g}$.</td>
</tr>
</tbody>
</table>
From Lie algebras to Lie groups

- Sophus Lie results were only local and in written in terms of groups of transformations.
- The global theory was worked out much later by Élie Cartan and Herman Weyl.
- Lie I and II are not hard to prove. Correct proofs of Lie III were given only by Cartan in 1936 (algebraic), using results of Ado, and Van Est in 1953 (geometric).
From Lie algebras to Lie groups

- Sophus Lie results were only local and in written in terms of groups of transformations.
- The global theory was worked out much later by Élie Cartan and Herman Weyl.
- Lie I and II are not hard to prove. Correct proofs of Lie III were given only by Cartan in 1936 (algebraic), using results of Ado, and Van Est in 1953 (geometric).
From Lie algebras to Lie groups

- Sophus Lie results were only local and in written in terms of groups of transformations.
- The global theory was worked out much later by Élie Cartan and Herman Weyl.
- Lie I and II are not hard to prove. Correct proofs of Lie III were given only by Cartan in 1936 (algebraic), using results of Ado, and Van Est in 1953 (geometric).
Infinite dimensional Lie groups

Symmetry groups of differential equations can be infinite dimensional (e.g., the heat equation).

More general, infinite dimensional Lie groups appear naturally in other settings (e.g., groups of diffeomorphisms in differential geometry/topology, field theories, fluid mechanics, etc.)

- Are Lie’s theorems true for infinite dimensional Lie groups?
Infinite dimensional Lie groups

Symmetry groups of differential equations can be infinite dimensional (e.g., the heat equation).

More general, infinite dimensional Lie groups appear naturally in other settings (e.g., groups of diffeomorphisms in differential geometry/topology, field theories, fluid mechanics, etc.)

Are Lie’s theorems true for infinite dimensional Lie groups?
Infinite dimensional Lie groups

Symmetry groups of differential equations can be infinite dimensional (e.g., the heat equation).

More general, infinite dimensional Lie groups appear naturally in other settings (e.g., groups of diffeomorphisms in differential geometry/topology, field theories, fluid mechanics, etc.)

- Are Lie’s theorems true for infinite dimensional Lie groups?
Example I [Van Est & Korthagen, 1964]

\[ g_0 := \{ X : [0, 1] \to \mathfrak{su}(2) | \int_0^1 X(t)dt = 0 \} \text{ with pointwise bracket;} \]

Take the skew-symmetric bilinear form \( \tau : g_0 \times g_0 : \to \mathbb{R} : \)

\[ \tau(X, Y) := \int_0^1 \text{tr} \left( \int_0^t X(s)ds \circ Y(t) \right) dt. \]

and form the central extension \( g = \mathbb{R} \times g_0 : \)

\[ 0 \to \mathbb{R} \to g \to g_0 \to 0 \]

relative to \( \tau \) so that: \( [(a, X), (b, Y)]_g := (\tau(X, Y), [X, Y]_{g_0}). \)

Theorem

The extension \( g \) is a Banach Lie algebra but there is no Banach Lie group with Lie algebra \( g. \)
Example I [Van Est & Korthagen, 1964]

\( g_0 := \{ X : [0, 1] \to su(2) | \int_0^1 X(t)dt = 0 \} \) with pointwise bracket;

Take the skew-symmetric bilinear form \( \tau : g_0 \times g_0 \to \mathbb{R} : \)

\[ \tau(X, Y) := \int_0^1 \text{tr} \left( \int_0^t X(s)ds \circ Y(t) \right) dt. \]

and form the central extension \( g = \mathbb{R} \times g_0 : \)

\[
\begin{array}{cccc}
0 & \rightarrow & \mathbb{R} & \rightarrow & g & \rightarrow & g_0 & \rightarrow & 0 \\
\end{array}
\]

relative to \( \tau \) so that: \( [(a, X), (b, Y)]_g := (\tau(X, Y), [X, Y]_{g_0}) . \)

**Theorem**

The extension \( g \) is a Banach Lie algebra but there is no Banach Lie group with Lie algebra \( g \).
Example I [Van Est & Korthagen, 1964]

\[ g_0 := \{ X : [0, 1] \to \mathfrak{su}(2) | \int_0^1 X(t)dt = 0 \} \]

with pointwise bracket;

Take the skew-symmetric bilinear form \( \tau : g_0 \times g_0 \to \mathbb{R} : \)

\[ \tau(X, Y) := \int_0^1 \text{tr} \left( \int_0^t X(s)ds \circ Y(t) \right) dt. \]

and form the central extension \( g = \mathbb{R} \times g_0 : \)

\[
egin{array}{cccc}
0 & \longrightarrow & \mathbb{R} & \longrightarrow & g & \longrightarrow & g_0 & \longrightarrow & 0 \\
\end{array}
\]

relative to \( \tau \) so that: \( [(a, X), (b, Y)]_g := (\tau(X, Y), [X, Y]_{g_0}) \).

**Theorem**

*The extension \( g \) is a Banach Lie algebra but there is no Banach Lie group with Lie algebra \( g \).*
Example II [Hamilton, 1982; Milnor, 1983]

\(M\) - a compact manifold

- The group \(\text{Diff}(M)\) is a Fréchet Lie group;
- \(\text{Diff}(M)\) has Lie algebra \(T_{\text{id}}M = \mathfrak{X}(M)\), with usual Lie bracket of vector fields;

However, Lie II fails:

**Theorem**

*If a diffeomorphism of the circle without fixed points is the time-1 flow of vector field then it must be conjugate to a rotation.*

There are diffeomorphisms as close to the identity as we wish which are not the time-1 flow of a vector field!
Example II [Hamilton, 1982; Milnor, 1983]

\(M\) - a compact manifold

- The group \(\text{Diff}(M)\) is a Fréchet Lie group;
- \(\text{Diff}(M)\) has Lie algebra \(T_{\text{id}}M = \mathfrak{X}(M)\), with usual Lie bracket of vector fields;

However, Lie II fails:

**Theorem**

*If a diffeomorphism of the circle without fixed points is the time-1 flow of vector field then it must be conjugate to a rotation.*

There are diffeomorphisms as close to the identity as we wish which are not the time-1 flow of a vector field!
Example II [Hamilton, 1982; Milnor, 1983]

$M$ - a compact manifold

- The group $\text{Diff}(M)$ is a Fréchet Lie group;
- $\text{Diff}(M)$ has Lie algebra $T_{\text{id}}M = \mathfrak{x}(M)$, with usual Lie bracket of vector fields;

However, Lie II fails:

**Theorem**

*If a diffeomorphism of the circle without fixed points is the time-1 flow of vector field then it must be conjugate to a rotation.*

There are diffeomorphisms as close to the identity as we wish which are not the time-1 flow of a vector field!
Example II [Hamilton, 1982; Milnor, 1983]

$M$ - a compact manifold

- The group $\text{Diff}(M)$ is a Fréchet Lie group;
- $\text{Diff}(M)$ has Lie algebra $T_{\text{id}}M = \mathfrak{x}(M)$, with usual Lie bracket of vector fields;

However, Lie II fails:

**Theorem**

*If a diffeomorphism of the circle without fixed points is the time-1 flow of vector field then it must be conjugate to a rotation.*

There are diffeomorphisms as close to the identity as we wish which are not the time-1 flow of a vector field!
Example II [Hamilton, 1982; Milnor, 1983]

\( M \) - a compact manifold

- The group \( \text{Diff}(M) \) is a Fréchet Lie group;
- \( \text{Diff}(M) \) has Lie algebra \( T_{\text{id}}M = \mathfrak{X}(M) \), with usual Lie bracket of vector fields;

However, Lie II fails:

**Theorem**

*If a diffeomorphism of the circle without fixed points is the time-1 flow of vector field then it must be conjugate to a rotation.*

There are diffeomorphisms as close to the identity as we wish which are not the time-1 flow of a vector field!
Dificulties with infinite dimensional Lie groups are enormous...

...but there is a way out, using Lie groupoids.
Dificulties with infinite dimensional Lie groups are enormous...

...but there is a way out, using Lie groupoids.
A **groupoid** is a small category where every morphism is an isomorphism.

\[ G \equiv \text{set of morphisms} \quad M \equiv \text{set of objects}. \]
A groupoid is a small category where every morphism is an isomorphism.

\[ \mathcal{G} \equiv \text{set of morphisms} \quad M \equiv \text{set of objects}. \]
A groupoid is a small category where every morphism is an isomorphism.

\[ G \equiv \text{set of morphisms} \quad M \equiv \text{set of objects}. \]

- **source** and **target** maps:

\[ t(g) \quad s(g) \]

\[ G \xrightarrow{t} M \]

- **product**:

\[ G^{(2)} = \{(h, g) \in G \times G : s(h) = t(g)\} \]

\[ m : G^{(2)} \rightarrow G \]

\[ R_g : s^{-1}(t(g)) \rightarrow s^{-1}(s(g)) \]
A **groupoid** is a small category where every morphism is an isomorphism.

\[ \mathcal{G} \equiv \text{set of morphisms} \quad M \equiv \text{set of objects.} \]

- **source** and **target** maps:

  \[ \begin{array}{ccc}
  \downarrow \quad g & \quad \downarrow \quad t \\
  \bullet & \quad \bullet \\
  t(g) & \quad s(g)
  \end{array} \]

- **product**:

  \[ \begin{array}{ccc}
  \downarrow \quad hg & \quad \downarrow \quad m \\
  \bullet & \quad \bullet \\
  t(h) & \quad s(h) = t(g) & \quad s(g)
  \end{array} \]

\[ g^{(2)} = \{(h, g) \in \mathcal{G} \times \mathcal{G} : s(h) = t(g)\} \]

\[ m : g^{(2)} \rightarrow \mathcal{G} \]

\[ R_g : s^{-1}(t(g)) \rightarrow s^{-1}(s(g)) \]
Groupoids

A **groupoid** is a small category where every morphism is an isomorphism.

\[ G \equiv \text{set of morphisms} \quad M \equiv \text{set of objects}. \]

- **identity:**
  \[ \epsilon : M \leftrightarrow G \]

- **inverse:**
  \[ \iota : G \longrightarrow G \]

Rui Loja Fernandes

Lie's Third Theorem
A **groupoid** is a small category where every morphism is an isomorphism.

\[ G \equiv \text{set of morphisms} \quad M \equiv \text{set of objects}. \]

- **identity:**

  \[ \epsilon : M \hookrightarrow G \]

- **inverse:**

  \[ \iota : G \longrightarrow G \]

\[ t(g) \bullet \quad \bullet s(g) \]

\[ g \quad g^{-1} \]
Example: Fundamental groupoid of a space

Let $X$ be any topological space. Look at continuous curves $\gamma : [0, 1] \to X$.
Example: Fundamental groupoid of a space

Let $X$ be any topological space. Look at continuous curves $\gamma : [0, 1] \to X$. The fundamental groupoid of $X$ is:

$$\Pi(X) = \{ [\gamma] | \gamma : [0, 1] \to X \}.$$
Example: Fundamental groupoid of a space

\( X \) any topological space
Look at continuous curves \( \gamma : [0, 1] \rightarrow X \)
Example: Fundamental groupoid of a space

$X$ any topological space
Look at continuous curves $\gamma : [0, 1] \rightarrow X$
Example: Fundamental groupoid of a space

$X$ any topological space
Look at continuous curves $\gamma : [0, 1] \to X$
Example: Fundamental groupoid of a space

$X$ any *topological* space
Look at *continuous* curves $\gamma : [0, 1] \rightarrow X$
Example: Fundamental groupoid of a space

$X$ any *topological* space

Look at *continuous* curves $\gamma : [0, 1] \to X$

$[\gamma] \equiv$ homotopy class of $\gamma$
Example: Fundamental groupoid of a space

$X$ any topological space

Look at continuous curves $\gamma : [0, 1] \to X$

$[\gamma] \equiv \text{homotopy class of } \gamma \quad (\text{e.g. } [\gamma_0] = [\gamma_1] \text{ but } [\gamma_0] \neq [\eta]).$
Example: Fundamental groupoid of a space

$X$ any *topological* space

Look at *continuous* curves $\gamma : [0, 1] \to X$

$[\gamma] \equiv$ homotopy class of $\gamma$ (e.g. $[\gamma_0] = [\gamma_1]$ but $[\gamma_0] \neq [\eta]$).

The *fundamental groupoid* of $X$ is:

$$\Pi(X) = \{ [\gamma] \mid \gamma : [0, 1] \to X \}. $$
Example: Fundamental groupoid of a space

For the fundamental groupoid

$$\Pi(X) = \{[\gamma] \mid \gamma : [0, 1] \to X\}$$

the structure maps are:

- *source/target* give initial/final points: $s([\gamma]) = \gamma(0)$, $t([\gamma]) = \gamma(1)$;
- *product* is concatenation of curves: $[\gamma] \cdot [\eta] = [\gamma \cdot \eta]$;
- *units* are the constant curves: $1_x = [\gamma]$, where $\gamma(t) = x$;
- *inverse* is the opposite curve: $[\gamma]^{-1} = [\bar{\gamma}]$, where $\bar{\gamma}(t) = \gamma(1 - t)$. 
Example: Fundamental groupoid of a space

For the fundamental groupoid

\[ \Pi(X) = \{[\gamma] \mid \gamma : [0, 1] \rightarrow X\} \]

the structure maps are:

- **source/target** give initial/final points: \( s([\gamma]) = \gamma(0), t([\gamma]) = \gamma(1); \)
- **product** is concatenation of curves: \( [\gamma] \cdot [\eta] = [\gamma \cdot \eta]; \)
- **units** are the constant curves: \( 1_x = [\gamma], \text{ where } \gamma(t) = x; \)
- **inverse** is the opposite curve: \( [\gamma]^{-1} = [\bar{\gamma}], \text{ where } \bar{\gamma}(t) = \gamma(1 - t). \)
Example: Fundamental groupoid of a space

For the fundamental groupoid

\[ \Pi(X) = \{ [\gamma] | \gamma : [0, 1] \to X \} \]

the structure maps are:

- **source/target** give initial/final points: \( s([\gamma]) = \gamma(0), \ t([\gamma]) = \gamma(1) \);
- **product** is concatenation of curves: \([\gamma] \cdot [\eta] = [\gamma \cdot \eta]\);
- **units** are the constant curves: \( 1_x = [\gamma] \), where \( \gamma(t) = x \);
- **inverse** is the opposite curve: \( [\gamma]^{-1} = [\overline{\gamma}] \), where \( \overline{\gamma}(t) = \gamma(1 - t) \).
Example: Fundamental groupoid of a space

For the fundamental groupoid

\[ \Pi(X) = \{[\gamma] \mid \gamma : [0, 1] \to X\} \]

the structure maps are:

- **source/target** give initial/final points: \( s([\gamma]) = \gamma(0), t([\gamma]) = \gamma(1) \);
- **product** is concatenation of curves: \( [\gamma] \cdot [\eta] = [\gamma \cdot \eta] \);
- **units** are the constant curves: \( 1_x = [\gamma] \), where \( \gamma(t) = x \);
- **inverse** is the opposite curve: \( [\gamma]^{-1} = [\bar{\gamma}] \), where \( \bar{\gamma}(t) = \gamma(1 - t) \).
Example: Fundamental groupoid of a space

For the fundamental groupoid

\[ \Pi(X) = \{ [\gamma] \mid \gamma : [0, 1] \to X \} \]

the structure maps are:

- **source/target** give initial/final points: \( s([\gamma]) = \gamma(0), t([\gamma]) = \gamma(1); \)
- **product** is concatenation of curves: \( [\gamma] \cdot [\eta] = [\gamma \cdot \eta]; \)
- **units** are the constant curves: \( 1_x = [\gamma], \text{ where } \gamma(t) = x; \)
- **inverse** is the opposite curve: \( [\gamma]^{-1} = [\gamma], \text{ where } \gamma(t) = \gamma(1 - t). \)
Lie groupoids

**Definition (Charles Ehresmann, 1950’s)**

A **Lie groupoid** is a groupoid where $\mathcal{G}$ and $M$ are manifolds and all structure maps are smooth.

**Examples**

- A Lie group $G$ is a Lie groupoid: $\mathcal{G} := G \rightrightarrows \{\ast\}$;
- For a manifold $M$, $\Pi(M) \rightrightarrows M$ and $M \times M \rightrightarrows M$ are Lie groupoids;
- For a foliation $\mathcal{F}$, $\Pi(\mathcal{F}) \rightrightarrows M$ is a Lie groupoid;
- Given a an action of a Lie group $G$ on a manifold $M$ can form the action groupoid: $\mathcal{G} := G \times M \rightrightarrows M$: $(g, x) \cdot (h, y) = (gh, y)$, if $x = h \cdot y$. 

Rui Loja Fernandes

Lie's Third Theorem
Lie groupoids

Definition (Charles Ehresmann, 1950’s)

A Lie groupoid is a groupoid where $\mathcal{G}$ and $M$ are manifolds and all structure maps are smooth.

Examples

- A Lie group $G$ is a Lie groupoid: $\mathcal{G} := G \rightrightarrows \{\ast\}$;
- For a manifold $M$, $\Pi(M) \rightrightarrows M$ and $M \times M \rightrightarrows M$ are Lie groupoids;
- For a foliation $\mathcal{F}$, $\Pi(\mathcal{F}) \rightrightarrows M$ is a Lie groupoid;
- Given a an action of a Lie group $G$ on a manifold $M$ can form the action groupoid: $\mathcal{G} := G \times M \rightrightarrows M$: $(g, x) \cdot (h, y) = (gh, y)$, if $x = h \cdot y$. 

Rui Loja Fernandes

Lie’s Third Theorem
Lie groupoids

Definition (Charles Ehresmann, 1950’s)

A Lie groupoid is a groupoid where $\mathcal{G}$ and $M$ are manifolds and all structure maps are smooth.

Examples

- A Lie group $G$ is a Lie groupoid: $\mathcal{G} := G \rightrightarrows \{\ast\}$;
- For a manifold $M$, $\Pi(M) \rightrightarrows M$ and $M \times M \rightrightarrows M$ are Lie groupoids;
- For a foliation $\mathcal{F}$, $\Pi(\mathcal{F}) \rightrightarrows M$ is a Lie groupoid;
- Given a an action of a Lie group $G$ on a manifold $M$ can form the action groupoid: $\mathcal{G} := G \times M \rightrightarrows M$: $(g, x) \cdot (h, y) = (gh, y)$, if $x = h \cdot y$. 
Lie groupoids

Definition (Charles Ehresmann, 1950’s)

A **Lie groupoid** is a groupoid where $\mathcal{G}$ and $M$ are manifolds and all structure maps are smooth.

Examples

- A Lie group $G$ is a Lie groupoid: $\mathcal{G} := G \rightrightarrows \{\ast\}$;
- For a manifold $M$, $\Pi(M) \rightrightarrows M$ and $M \times M \rightrightarrows M$ are Lie groupoids;
- For a foliation $\mathcal{F}$, $\Pi(\mathcal{F}) \rightrightarrows M$ is a Lie groupoid;
- Given a an action of a Lie group $G$ on a manifold $M$ can form the action groupoid: $\mathcal{G} := G \times M \rightrightarrows M$: $(g, x) \cdot (h, y) = (gh, y)$, if $x = h \cdot y$. 
Lie groupoids

Definition (Charles Ehresmann, 1950’s)

A **Lie groupoid** is a groupoid where $\mathcal{G}$ and $M$ are manifolds and all structure maps are smooth.

Examples

- A Lie group $G$ is a Lie groupoid: $\mathcal{G} := G \xrightarrow{\sim} \{\ast\}$;
- For a manifold $M$, $\Pi(M) \xrightarrow{\sim} M$ and $M \times M \xrightarrow{\sim} M$ are Lie groupoids;
- For a foliation $\mathcal{F}$, $\Pi(\mathcal{F}) \xrightarrow{\sim} M$ is a Lie groupoid;
- Given an action of a Lie group $G$ on a manifold $M$ can form the action groupoid: $\mathcal{G} := G \times M \xrightarrow{\sim} M$:
  $$(g, x) \cdot (h, y) = (gh, y), \text{ if } x = h \cdot y.$$
Lie groupoids vs (infinite dimensional) Lie groups

**Definition**

A **bisection** of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a smooth map $b : M \to \mathcal{G}$ such that $s \circ b : M \to M$ and $t \circ b : M \to M$ are diffeomorphisms.
Lie groupoids vs (infinite dimensional) Lie groups

Definition

A **bisection** of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a smooth map $b : M \rightarrow \mathcal{G}$ such that $s \circ b : M \rightarrow M$ and $t \circ b : M \rightarrow M$ are diffeomorphisms.
Lie groupoids vs (infinite dimensional) Lie groups

**Definition**

A *bisection* of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a smooth map $b : M \to \mathcal{G}$ such that $s \circ b : M \to M$ and $t \circ b : M \to M$ are diffeomorphisms.
**Definition**

A **bisection** of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a smooth map $b : M \to \mathcal{G}$ such that $s \circ b : M \to M$ and $t \circ b : M \to M$ are diffeomorphisms.
Lie groupoids vs (infinite dimensional) Lie groups

**Definition**

A **bisection** of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a smooth map $b : M \to \mathcal{G}$ such that $s \circ b : M \to M$ and $t \circ b : M \to M$ are diffeomorphisms.

![Diagram of Lie groupoids and bisections](image.png)
A **bisection** of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a smooth map $b : M \to \mathcal{G}$ such that $s \circ b : M \to M$ and $t \circ b : M \to M$ are diffeomorphisms.

The group of bisections $\Gamma(\mathcal{G})$ is a Fréchet Lie group (usually, infinite dimensional):

- If $\mathcal{G} = \mathcal{G} \rightrightarrows \{\ast\}$, then $\Gamma(\mathcal{G}) = \mathcal{G}$;
- If $\mathcal{G} = M \times M \rightrightarrows M$, then $\Gamma(\mathcal{G}) = \text{Diff}(M)$. 

```
Rui Loja Fernandes
```

```
Lie's Third Theorem
```
Lie groupoids vs (infinite dimensional) Lie groups

Definition

A **bisection** of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a smooth map $b : M \to \mathcal{G}$ such that $s \circ b : M \to M$ and $t \circ b : M \to M$ are diffeomorphisms.
Lie groupoids vs (infinite dimensional) Lie groups

Definition

A **bisection** of a Lie groupoid $G \Rightarrow M$ is a smooth map $b : M \to G$ such that $s \circ b : M \to M$ and $t \circ b : M \to M$ are diffeomorphisms.

- The group of bissections $\Gamma(G)$ is a Fréchet Lie group (usually, infinite dimensional):
Definition

A \textbf{bisection} of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a smooth map $b : M \to \mathcal{G}$ such that $s \circ b : M \to M$ and $t \circ b : M \to M$ are diffeomorphisms.

- The group of bisections $\Gamma(\mathcal{G})$ is a Fréchet Lie group (usually, infinite dimensional):
  - If $\mathcal{G} = G \rightrightarrows \{\ast\}$, then $\Gamma(\mathcal{G}) = G$;
**Definition**

A **bisection** of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a smooth map $b : M \to \mathcal{G}$ such that $s \circ b : M \to M$ and $t \circ b : M \to M$ are diffeomorphisms.

The group of bissections $\Gamma(\mathcal{G})$ is a Fréchet Lie group (usually, infinite dimensional):

- If $\mathcal{G} = G \rightrightarrows \{\ast\}$, then $\Gamma(\mathcal{G}) = G$;
- If $\mathcal{G} = M \times M \rightrightarrows M$, then $\Gamma(\mathcal{G}) = \text{Diff}(M)$;
Lie algebroids

Definition

A **Lie algebroid** is a vector bundle $A \to M$ with:

(i) a Lie bracket $[ , ]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$;

(ii) a bundle map $\rho : A \to TM$ (the anchor);

such that:

$$[\alpha, f\beta]_A = f[\alpha\beta]_A + \rho(\alpha)(f)\beta, \quad (f \in C^\infty(M), \alpha, \beta \in \Gamma(A)).$$

- The space of sections $\Gamma(A)$ is a Fréchet Lie algebra (usually infinite dimensional).
- $\text{Im } \rho \subset TM$ is integrable $\Rightarrow$ **characteristic foliation** of $M$;
A Lie algebroid is a vector bundle $A \to M$ with:

(i) a Lie bracket $[ , ]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$;
(ii) a bundle map $\rho : A \to TM$ (the anchor);

such that:

$$[\alpha, f\beta]_A = f[\alpha\beta]_A + \rho(\alpha)(f)\beta, \quad (f \in C^\infty(M), \alpha, \beta \in \Gamma(A)).$$

- The space of sections $\Gamma(A)$ is a Fréchet Lie algebra (usually infinite dimensional).
- $\text{Im} \rho \subset TM$ is integrable $\Rightarrow$ characteristic foliation of $M$;
A **Lie algebroid** is a vector bundle $A \to M$ with:

(i) a Lie bracket $[\ , 
\ ]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$;

(ii) a bundle map $\rho : A \to TM$ (the *anchor*);

such that:

$$[\alpha, f\beta]_A = f[\alpha\beta]_A + \rho(\alpha)(f)\beta, \quad (f \in C^\infty(M), \alpha, \beta \in \Gamma(A)).$$

- The space of sections $\Gamma(A)$ is a Fréchet Lie algebra (usually infinite dimensional).
- $\text{Im } \rho \subset TM$ is integrable $\Rightarrow$ characteristic foliation of $M$;
Lie algebroids

Definition

A Lie algebroid is a vector bundle $A \rightarrow M$ with:

(i) a Lie bracket $\left[\cdot, \cdot\right]_A : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$;

(ii) a bundle map $\rho : A \rightarrow TM$ (the anchor);

such that:

$$\left[\alpha, f\beta\right]_A = f\left[\alpha\beta\right]_A + \rho(\alpha)(f)\beta, \quad (f \in C^\infty(M), \alpha, \beta \in \Gamma(A)).$$

- The space of sections $\Gamma(A)$ is a Fréchet Lie algebra (usually infinite dimensional).
- $\text{Im} \rho \subset TM$ is integrable $\Rightarrow$ characteristic foliation of $M$. 

Rui Loja Fernandes

Lie's Third Theorem
Lie algebroids

Examples

- **Flows.** For $X \in \mathfrak{x}(M)$, the associated Lie algebroid is:

  \[ A = M \times \mathbb{R}, \quad [f, g]_A := fX(g) - gX(f), \quad \rho(f) = fX. \]

  Leaves of $A$ are the orbits of $X$.

- **Actions.** For an infinitesimal $g$-action $\phi : g \to \mathfrak{x}(M)$, the associated Lie algebroid is:

  \[ A = M \times g, \quad \rho(x, \xi) = \phi(\xi)_x, \]

  \[ [f, g]_A(x) = [f(x), g(x)]_g + \mathcal{L}_{\rho(f(x))}g(x) - \mathcal{L}_{\rho(g(x))}f(x). \]

  Leaves of $A$ are the orbits of the action.
Lie algebroids

Examples

- **Flows.** For $X \in \mathfrak{X}(M)$, the associated Lie algebroid is:
  \[
  A = M \times \mathbb{R}, \quad [f, g]_A := fX(g) - gX(f), \quad \rho(f) = fX.
  \]
  Leaves of $A$ are the orbits of $X$.

- **Actions.** For an infinitesimal $\mathfrak{g}$-action $\phi : \mathfrak{g} \to \mathfrak{X}(M)$, the associated Lie algebroid is:
  \[
  A = M \times \mathfrak{g}, \quad \rho(x, \xi) = \phi(\xi)x,
  \]
  \[
  [f, g]_A(x) = [f(x), g(x)]_{\mathfrak{g}} + \mathcal{L}_{\rho(f(x))}g(x) - \mathcal{L}_{\rho(g(x))}f(x).
  \]
  Leaves of $A$ are the orbits of the action.
Lie algebroids

Examples

- **Foliations.** For $\mathcal{F} \in \text{Fol}_K(M)$, the associated Lie algebroid is:

  $$A = T\mathcal{F}, \quad [X, Y]_A = [X, Y], \quad \rho = \text{id}.$$ 

  Leaves of $A$ are the leaves of $\mathcal{F}$.

- **Prequantization.** For $\omega \in \Omega^2(M)$, closed, the associated Lie algebroid is: $A = TM \otimes \mathbb{R}$, $\rho(X, a) = X$,

  $$[(X, f), (Y, g)]_A = ([X, Y], X(g) - Y(f) - \omega(X, Y)).$$

  There is only leaf of $A$, which is $M$ itself.
Lie algebroids

Examples

- **Foliations.** For $\mathcal{F} \in \text{Fol}_k(M)$, the associated Lie algebroid is:

  \[ A = T\mathcal{F}, \quad [X, Y]_A = [X, Y], \quad \rho = \text{id}. \]

  Leaves of $A$ are the leaves of $\mathcal{F}$.

- **Prequantization.** For $\omega \in \Omega^2(M)$, closed, the associated Lie algebroid is: $A = TM \otimes \mathbb{R}$, $\rho(X, a) = X$,

  \[ [(X, f), (Y, g)]_A = ([X, Y], X(g) - Y(f) - \omega(X, Y)). \]

  There is only leaf of $A$, which is $M$ itself.
From Lie groupoids to Lie algebroids

**Theorem**

*Every Lie groupoid* $\mathcal{G} \rightarrowtail M$ *determines a Lie algebroid* $A \rightarrow M$. 
Theorem

Every Lie groupoid $\mathcal{G} \rightrightarrows M$ determines a Lie algebroid $A \to M$. 

From Lie groupoids to Lie algebroids
From Lie groupoids to Lie algebroids

**Theorem**

*Every Lie groupoid $\mathcal{G} \rightarrow M$ determines a Lie algebroid $A \rightarrow M$.***

$$A = \ker d \circ s_{|M}$$
Theorem

Every Lie groupoid \( \mathcal{G} \cong M \) determines a Lie algebroid \( A \to M \).

\[ A = \ker d \ s \big|_M \]
From Lie groupoids to Lie algebroids

**Theorem**

*Every Lie groupoid $\mathcal{G} \rightrightarrows M$ determines a Lie algebroid $A \to M$.***
From Lie groupoids to Lie algebroids

**Theorem**

*Every Lie groupoid* \( \mathcal{G} \rightrightarrows M \) *determines a Lie algebroid* \( A \to M \).*

\[
A = \ker d_s \\
\rho = dt
\]
From Lie groupoids to Lie algebroids

**Theorem**

*Every Lie groupoid \( \mathcal{G} \rightrightarrows M \) determines a Lie algebroid \( A \rightrightarrows M \).*

\[
A = \ker d_s \bigg|_M \quad \rho = dt \bigg|_A
\]
From Lie groupoids to Lie algebroids

**Theorem**

*Every Lie groupoid $\mathcal{G} \rightrightarrows M$ determines a Lie algebroid $A \to M$.***

\[
A = \ker d\rho \bigg|_M \quad \rho = dt \bigg|_A \quad [\alpha, \beta] = [X^\alpha, X^\beta]
\]
From Lie algebroids to Lie groupoids

Theorem (Lie I)

Let $\mathcal{G}$ be a Lie groupoid with Lie algebroid $A$. There exists a unique (up to isomorphism) source 1-connected Lie groupoid $\tilde{\mathcal{G}}$ with Lie algebroid $A$.

Theorem (Lie II)

Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groupoids with Lie algebroids $A$ and $B$, where $\mathcal{G}$ is source 1-connected. Given a Lie algebroid homomorphism $\phi : A \to B$, there exists a unique Lie groupoid homomorphism $\Phi : \mathcal{G} \to \mathcal{H}$ with $(\Phi)_* = \phi$.

... but Lie III does not hold!
From Lie algebroids to Lie groupoids

Theorem (Lie I)

Let $\mathcal{G}$ be a Lie groupoid with Lie algebroid $A$. There exists a unique (up to isomorphism) source 1-connected Lie groupoid $\tilde{\mathcal{G}}$ with Lie algebroid $A$.

Theorem (Lie II)

Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groupoids with Lie algebroids $A$ and $B$, where $\mathcal{G}$ is source 1-connected. Given a Lie algebroid homomorphism $\phi : A \to B$, there exists a unique Lie groupoid homomorphism $\Phi : \mathcal{G} \to \mathcal{H}$ with $(\Phi)_* = \phi$.

... but Lie III does not hold!
From Lie algebroids to Lie groupoids

Theorem (Lie I)

Let $\mathcal{G}$ be a Lie groupoid with Lie algebroid $A$. There exists a unique (up to isomorphism) source 1-connected Lie groupoid $\tilde{\mathcal{G}}$ with Lie algebroid $A$.

Theorem (Lie II)

Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groupoids with Lie algebroids $A$ and $B$, where $\mathcal{G}$ is source 1-connected. Given a Lie algebroid homomorphism $\phi : A \to B$, there exists a unique Lie groupoid homomorphism $\Phi : \mathcal{G} \to \mathcal{H}$ with $(\Phi)_* = \phi$.

... but Lie III does not hold!
A non-integrable Lie algebroid

- Fix \( \omega \in \Omega^2(M) \), closed, and take the associated Lie algebroid \( A = TM \oplus \mathbb{R} \).

**Theorem**

The Lie algebroid \( A \) integrates to a Lie groupoid \( \mathcal{G} \) iff the group of spherical periods of \( \omega \):

\[
N_x := \left\{ \int_\gamma \omega \mid \gamma \in \pi_2(M, x) \right\} \subset \mathbb{R}
\]

is discrete.

**Example**

If \( M = S^2 \times S^2 \) and \( \omega = dA \oplus \lambda dA \), then \( N_x \) is discrete iff \( \lambda \in \mathbb{Q} \).
A non-integrable Lie algebroid

Fix $\omega \in \Omega^2(M)$, closed, and take the associated Lie algebroid $A = TM \oplus \mathbb{R}$.

**Theorem**

*The Lie algebroid $A$ integrates to a Lie groupoid $\mathcal{G}$ iff the group of spherical periods of $\omega$:

$$N_x := \{ \int_\gamma \omega \mid \gamma \in \pi_2(M, x) \} \subset \mathbb{R}$$

is discrete.*

**Example**

If $M = S^2 \times S^2$ and $\omega = dA \oplus \lambda dA$, then $N_x$ is discrete iff $\lambda \in \mathbb{Q}$. 
A non-integrable Lie algebroid

Fix $\omega \in \Omega^2(M)$, closed, and take the associated Lie algebroid $A = TM \oplus \mathbb{R}$.

**Theorem**

*The Lie algebroid $A$ integrates to a Lie groupoid $G$ iff the group of spherical periods of $\omega$:*

$$N_x := \left\{ \int_\gamma \omega \mid \gamma \in \pi_2(M, x) \right\} \subset \mathbb{R}$$

*is discrete.*

**Example**

If $M = S^2 \times S^2$ and $\omega = dA \oplus \lambda dA$, then $N_x$ is discrete iff $\lambda \in \mathbb{Q}$. 

Rui Loja Fernandes  
Lie's Third Theorem
The obstructions to integrability are completely described by:

**Theorem (Crainic & RLF, 2003)**

For a Lie algebroid $A$, there exist monodromy groups $N_x \subset A_x$ such that $A$ is integrable iff the groups $N_x$ are uniformly discrete for $x \in M$.

Each $N_x$ is the image of a monodromy map:

$$\partial : \pi_2(L, x) \to \widehat{G(g_x)}$$

with $L$ the leaf through $x$ and $g_x := \ker \rho_x$ the isotropy Lie algebra.

**Corollary**

A Lie algebroid $A$ is integrable provided either of the following hold:

(i) All leaves have finite $\pi_2$;

(ii) The isotropy Lie algebras have trivial center.
Obstructions to integrability

The obstructions to integrability are completely described by:

Theorem (Crainic & RLF, 2003)

For a Lie algebroid $A$, there exist monodromy groups $N_x \subset A_x$ such that $A$ is integrable iff the groups $N_x$ are uniformly discrete for $x \in M$.

Each $N_x$ is the image of a monodromy map:

$$\partial : \pi_2(L, x) \to \widetilde{G}(g_x)$$

with $L$ the leaf through $x$ and $g_x := \text{Ker} \rho_x$ the isotropy Lie algebra.

Corollary

A Lie algebroid $A$ is integrable provided either of the following hold:

(i) All leaves have finite $\pi_2$;

(ii) The isotropy Lie algebras have trivial center.
Obstructions to integrability

The *obstructions to integrability* are completely described by:

**Theorem (Crainic & RLF, 2003)**

For a Lie algebroid $A$, there exist monodromy groups $N_x \subset A_x$ such that $A$ is integrable iff the groups $N_x$ are uniformly discrete for $x \in M$.

Each $N_x$ is the image of a monodromy map:

$$\partial : \pi_2(L, x) \to \widetilde{G(g_x)}$$

with $L$ the leaf through $x$ and $g_x := \text{Ker } \rho_x$ the isotropy Lie algebra.

**Corollary**

A Lie algebroid $A$ is integrable provided either of the following hold:

(i) All leaves have finite $\pi_2$;

(ii) The isotropy Lie algebras have trivial center.
Obstructions to integrability

The obstructions to integrability are completely described by:

**Theorem (Crainic & RLF, 2003)**

For a Lie algebroid $A$, there exist monodromy groups $N_x \subset A_x$ such that $A$ is integrable iff the groups $N_x$ are uniformly discrete for $x \in M$.

Each $N_x$ is the image of a monodromy map:

$$\partial : \pi_2(L, x) \to \widetilde{G(g_x)}$$

with $L$ the leaf through $x$ and $g_x := \text{Ker} \rho_x$ the isotropy Lie algebra.

**Corollary**

A Lie algebroid $A$ is integrable provided either of the following hold:

(i) All leaves have finite $\pi_2$;

(ii) The isotropy Lie algebras have trivial center.
Obstructions to integrability

The *obstructions to integrability* are completely described by:

**Theorem (Crainic & RLF, 2003)**

*For a Lie algebroid $A$, there exist monodromy groups $N_x \subset A_x$ such that $A$ is integrable iff the groups $N_x$ are uniformly discrete for $x \in M$.*

Each $N_x$ is the image of a monodromy map:

$$\partial : \pi_2(L, x) \to \widetilde{G(g_x)}$$

with $L$ the leaf through $x$ and $g_x := \text{Ker } \rho_x$ the isotropy Lie algebra.

**Corollary**

*A Lie algebroid $A$ is integrable provided either of the following hold:

(i) All leaves have finite $\pi_2$;

(ii) The isotropy Lie algebras have trivial center.*
Obstructions to integrability

The *obstructions to integrability* are completely described by:

**Theorem (Crainic & RLF, 2003)**

*For a Lie algebroid \( A \), there exist monodromy groups \( N_x \subset A_x \) such that \( A \) is integrable iff the groups \( N_x \) are uniformly discrete for \( x \in M \).*

Each \( N_x \) is the image of a monodromy map:

\[
\partial : \pi_2(L, x) \to \widetilde{G(\mathfrak{g}_x)}
\]

with \( L \) the leaf through \( x \) and \( \mathfrak{g}_x := \text{Ker} \rho_x \) the isotropy Lie algebra.

**Corollary**

*A Lie algebroid \( A \) is integrable provided either of the following hold:

(i) All leaves have finite \( \pi_2 \);

(ii) The isotropy Lie algebras have trivial center.*
Proof: The Weinstein groupoid

Notations

- An **A-path** is a Lie algebroid map $TI \rightarrow A$;
- An **A-homotopy** is a Lie algebroid map $T(I \times I) \rightarrow A$;

Definition

For a Lie algebroid $\pi: A \rightarrow M$, the **Weinstein Groupoid** of $A$ is:

\[ G(A) = P(A) / \sim \]

where

- $s: G(A) \rightarrow M, \quad [a] \mapsto \pi(a(0))$
- $t: G(A) \rightarrow M, \quad [a] \mapsto \pi(a(1))$
- $M \hookrightarrow G(A), \quad x \mapsto [0_x]$
Proof: The Weinstein groupoid

Notations

- An \textbf{A-path} is a Lie algebroid map $Tl \rightarrow A$;
- An \textbf{A-homotopy} is a Lie algebroid map $T(l \times l) \rightarrow A$;

Definition

For a Lie algebroid $\pi : A \rightarrow M$, the \textbf{Weinstein Groupoid} of $A$ is:

$$\mathcal{G}(A) = P(A)/\sim \quad \text{where}$$

- $s : \mathcal{G}(A) \rightarrow M$, $[a] \mapsto \pi(a(0))$
- $t : \mathcal{G}(A) \rightarrow M$, $[a] \mapsto \pi(a(1))$
- $M \hookrightarrow \mathcal{G}(A)$, $x \mapsto [0_x]$
Proof: The Weinstein groupoid

Notations

- An **A-path** is a Lie algebroid map $Tl \to A$;
- An **A-homotopy** is a Lie algebroid map $T(l \times l) \to A$;

**Definition**

For a Lie algebroid $\pi : A \to M$, the **Weinstein Groupoid** of $A$ is:

$$\mathcal{G}(A) = P(A)/ \sim \quad \text{where} \quad s : \mathcal{G}(A) \to M, \quad [a] \mapsto \pi(a(0))$$

$$t : \mathcal{G}(A) \to M, \quad [a] \mapsto \pi(a(1))$$

$$M \leftrightarrow \mathcal{G}(A), \quad x \mapsto [0_x]$$
Proof: The Weinstein groupoid

Notations

- An **A-path** is a Lie algebroid map $Tl \rightarrow A$;
- An **A-homotopy** is a Lie algebroid map $T(l \times l) \rightarrow A$;

Definition

For a Lie algebroid $\pi : A \rightarrow M$, the **Weinstein Groupoid** of $A$ is:

$$\mathcal{G}(A) = P(A)/\sim \quad \text{where} \quad \begin{align*}
    s : \mathcal{G}(A) \rightarrow M, & \quad [a] \mapsto \pi(a(0)) \\
    t : \mathcal{G}(A) \rightarrow M, & \quad [a] \mapsto \pi(a(1)) \\
    M \leftrightarrow \mathcal{G}(A), & \quad x \mapsto [0_x]
\end{align*}$$
Proof: The Weinstein groupoid and monodromy

Lemma

- $\mathcal{G}(A)$ is a topological groupoid with source 1-connected fibers;
- $A$ is integrable iff $\mathcal{G}(A)$ is smooth (for the quotient topology);

Fix leaf $L \subset M$ and $x \in L$:

$$
0 \longrightarrow g_L \longrightarrow A_L \not
\# \longrightarrow TL \longrightarrow 0
$$

$$
\downarrow
$$

$$
\cdots \longrightarrow \pi_2(L, x) \xrightarrow{\partial} G(g_L)_x \longrightarrow G(A)_x \longrightarrow \pi_1(L, x) \longrightarrow 1
$$

The monodromy group at $x$ is: $N_x(A) := \text{Im} \partial \subset Z(g_L)$. 

Rui Loja Fernandes

Lie Theory beyond finite dimensions

Lie Groupoid Theory

Lie III revisited

Obstructions to integrability

The proof
Proof: The Weinstein groupoid and monodromy

Lemma

- \( \mathcal{G}(A) \) is a topological groupoid with source 1-connected fibers;
- A is integrable iff \( \mathcal{G}(A) \) is smooth (for the quotient topology);

Fix leaf \( L \subset M \) and \( x \in L \):

\[
\begin{array}{c}
0 \rightarrow g_L \rightarrow A_L \xrightarrow{\#} TL \rightarrow 0 \\
\downarrow \\
\cdots \rightarrow \pi_2(L, x) \xrightarrow{\partial} \mathcal{G}(g_L)_x \rightarrow \mathcal{G}(A)_x \rightarrow \pi_1(L, x) \rightarrow 1
\end{array}
\]

The monodromy group at \( x \) is: \( N_x(A) := \text{Im} \partial \subset Z(g_L) \).
Proof: The Weinstein groupoid and monodromy

Lemma

- \( \mathcal{G}(A) \) is a topological groupoid with source 1-connected fibers;
- \( A \) is integrable iff \( \mathcal{G}(A) \) is smooth (for the quotient topology);

Fix leaf \( L \subset M \) and \( x \in L \):

\[
0 \longrightarrow g_L \longrightarrow A_L \# \longrightarrow TL \longrightarrow 0
\]

\[
\cdots \longrightarrow \pi_2(L, x) \xrightarrow{\partial} \mathcal{G}(g_L)_x \longrightarrow \mathcal{G}(A)_x \longrightarrow \pi_1(L, x) \longrightarrow 1
\]

The monodromy group at \( x \) is: \( N_x(A) := \text{Im} \partial \subset Z(g_L) \).
Proof: The Weinstein groupoid and monodromy

Lemma

- $\mathcal{G}(A)$ is a topological groupoid with source 1-connected fibers;
- $A$ is integrable iff $\mathcal{G}(A)$ is smooth (for the quotient topology);

Fix leaf $L \subset M$ and $x \in L$:

\[
0 \longrightarrow g_L \longrightarrow A_L \overset{\#}{\longrightarrow} TL \longrightarrow 0
\]

\[
\downarrow
\]

\[
\cdots \longrightarrow \pi_2(L, x) \overset{\partial}{\longrightarrow} \mathcal{G}(g_L)_x \longrightarrow \mathcal{G}(A)_x \longrightarrow \pi_1(L, x) \longrightarrow 1
\]

The **monodromy group** at $x$ is: $N_x(A) := \text{Im} \, \partial \subset Z(g_L)$. 

Rui Loja Fernandes

Lie's Third Theorem
Proof: The obstructions

To measure the discreteness of $N_x(A)$ we set:

$$r(x) := d(N_x - \{0\}, \{0\}) \quad \text{(with } d(\emptyset, \{0\}) = +\infty).$$

Theorem (Crainic & RLF, 2003)

A Lie algebroid is integrable iff both the following conditions hold:

(i) Each monodromy group is discrete, i.e., $r(x) > 0$,

(ii) The monodromy groups are uniformly discrete, i.e.,

$$\liminf_{y \to x} r(y) > 0,$$

for all $x \in M$.

...in many examples it is possible to compute the monodromy...
Proof: The obstructions

To measure the discreteness of $N_x(A)$ we set:

$$r(x) := d(N_x - \{0\}, \{0\}) \quad (\text{with } d(\emptyset, \{0\}) = +\infty).$$

Theorem (Crainic & RLF, 2003)

A Lie algebroid is integrable iff both the following conditions hold:

(i) Each monodromy group is discrete, i.e., $r(x) > 0$,

(ii) The monodromy groups are uniformly discrete, i.e.,

$$\liminf_{y \to x} r(y) > 0,$$

for all $x \in M$.

...in many examples it is possible to compute the monodromy...
Proof: The obstructions

To measure the discreteness of \( N_x(A) \) we set:

\[
    r(x) := d(N_x - \{0\}, \{0\}) \quad \text{(with } d(\emptyset, \{0\}) = +\infty \text{)}.
\]

Theorem (Crainic & RLF, 2003)

A Lie algebroid is integrable iff both the following conditions hold:

(i) Each monodromy group is discrete, i.e., \( r(x) > 0 \),

(ii) The monodromy groups are uniformly discrete, i.e.,

\[
    \liminf_{y \to x} r(y) > 0,
\]

for all \( x \in M \).

...in many examples it is possible to compute the monodromy...
Proof: The obstructions

To measure the discreteness of $N_x(A)$ we set:

$$r(x) := d(N_x - \{0\}, \{0\}) \quad \text{(with } d(\emptyset, \{0\}) = +\infty)\).$$

**Theorem (Crainic & RLF, 2003)**

A Lie algebroid is integrable iff both the following conditions hold:

(i) Each monodromy group is discrete, i.e., $r(x) > 0$,

(ii) The monodromy groups are uniformly discrete, i.e.,
$$\lim \inf_{y \to x} r(y) > 0,$$

for all $x \in M$.

...in many examples it is possible to compute the monodromy...
Proof: The obstructions

To measure the discreteness of \( N_x(A) \) we set:

\[
r(x) := d(N_x - \{0\}, \{0\}) \quad \text{(with } d(\emptyset, \{0\}) = +\infty).\]

**Theorem (Crainic & RLF, 2003)**

*A Lie algebroid is integrable iff both the following conditions hold:*

(i) *Each monodromy group is discrete, i.e.,* \( r(x) > 0 \),

(ii) *The monodromy groups are uniformly discrete, i.e.,*

\[
\liminf_{y \to x} r(y) > 0,
\]

*for all* \( x \in M \).

...in many examples it is possible to compute the monodromy...