

# Integrability of Lie brackets

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## Abstract

In this paper we present the solution to a longstanding problem of differential geometry: Lie's third theorem for Lie algebroids. We show that the integrability problem is controlled by two computable obstructions. As applications we derive, explain and improve the known integrability results, we establish integrability by local Lie groupoids, we clarify the smoothness of the Poisson sigma-model for Poisson manifolds, and we describe other geometrical applications.

## Contents

0. Introduction
1.  $A$ -paths and homotopy
  - 1.1.  $A$ -paths
  - 1.2.  $A$ -paths and connections
  - 1.3. Homotopy of  $A$ -paths
  - 1.4. Representations and  $A$ -paths
2. The Weinstein groupoid
  - 2.1. The groupoid  $\mathcal{G}(A)$
  - 2.2. Homomorphisms
  - 2.3. The exponential map
3. Monodromy
  - 3.1. Monodromy groups
  - 3.2. A second-order monodromy map
  - 3.3. Computing the monodromy
  - 3.4. Measuring the monodromy

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\*The first author was supported in part by NWO and a Miller Research Fellowship. The second author was supported in part by FCT through program POCTI and grant POCTI/1999/MAT/33081.

*Key words and phrases.* Lie algebroid, Lie groupoid.

4. Obstructions to integrability
  - 4.1. The main theorem
  - 4.2. The Weinstein groupoid as a leaf space
  - 4.3. Proof of the main theorem
5. Examples and applications
  - 5.1. Local integrability
  - 5.2. Integrability criteria
  - 5.3. Transversally parallelizable foliations

#### Appendix A. Flows

- A.1. Flows and infinitesimal flows
- A.2. The infinitesimal flow of a section

#### References

## 0. Introduction

This paper is concerned with the general problem of integrability of geometric structures. The geometric structures we consider are always associated with local Lie brackets  $[ \ , \ ]$  on sections of some vector bundles, or what one calls *Lie algebroids*. A Lie algebroid can be thought of as a generalization of the tangent bundle, the locus where infinitesimal geometry takes place. Roughly speaking, the general integrability problem asks for the existence of a “space of arrows” and a product which unravels the infinitesimal structure. These global objects are usually known as *Lie groupoids* (or *differentiable groupoids*) and in this paper we shall give the precise obstructions to integrate a Lie algebroid to a Lie groupoid. For an introduction to this problem and a brief historical account we refer the reader to the recent monograph [3]. More background material and further references can be found in [17], [18].

To describe our results, let us start by recalling that a *Lie algebroid* over a manifold  $M$  consists of a vector bundle  $A$  over  $M$ , endowed with a Lie bracket  $[ \ , \ ]$  on the space of sections  $\Gamma(A)$ , together with a bundle map  $\# : A \rightarrow TM$ , called *the anchor*. One requires the induced map  $\# : \Gamma(A) \rightarrow \mathcal{X}^1(M)$  <sup>(1)</sup> to be a Lie algebra map, and also the Leibniz identity

$$[\alpha, f\beta] = f[\alpha, \beta] + \# \alpha(f)\beta,$$

to hold, where the vector field  $\# \alpha$  acts on  $f$ .

For any  $x \in M$ , there is an induced Lie bracket on

$$\mathfrak{g}_x \equiv \text{Ker}(\#_x) \subset A_x$$

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<sup>1</sup>We denote by  $\Omega^r(M)$  and  $\mathcal{X}^r(M)$ , respectively, the spaces of differential  $r$ -forms and  $r$ -multivector fields on a manifold  $M$ . If  $E$  is a bundle over  $M$ ,  $\Gamma(E)$  will denote the space of global sections.

which makes it into a Lie algebra. In general, the dimension of  $\mathfrak{g}_x$  varies with  $x$ . The image of  $\#$  defines a smooth generalized distribution in  $M$ , in the sense of Sussmann ([26]), which is integrable. When we restrict to a leaf  $L$  of the associated foliation, the  $\mathfrak{g}_x$ 's are all isomorphic and fit into a Lie algebra bundle  $\mathfrak{g}_L$  over  $L$  (see [17]). In fact, there is an induced Lie algebroid

$$A_L = A|_L$$

which is transitive (i.e. the anchor is surjective), and  $\mathfrak{g}_L$  is the kernel of its anchor map. A general Lie algebroid  $A$  can be thought of as a singular foliation on  $M$ , together with transitive algebroids  $A_L$  over the leaves  $L$ , glued in some complicated way.

Before giving the definitions of Lie groupoids and integrability of Lie algebroids, we illustrate the problem by looking at the following basic examples:

- For algebroids over a point (i.e. Lie algebras), the integrability problem is solved by Lie's third theorem on the integrability of (finite-dimensional) Lie algebras by Lie groups;
- For algebroids with zero anchor map (i.e. bundles of Lie algebras), it is Douady-Lazard [10] extension of Lie's third theorem which ensures that the Lie groups integrating each Lie algebra fiber fit into a smooth bundle of Lie groups;
- For algebroids with injective anchor map (i.e. involutive distributions  $\mathcal{F} \subset TM$ ), the integrability problem is solved by Frobenius' integrability theorem.

Other fundamental examples come from Élie Cartan's infinite continuous groups (Singer and Sternberg, [25]), the integrability of infinitesimal actions of Lie algebras on manifolds (Palais, [24]), abstract Atiyah sequences (Almeida and Molino, [2]; Mackenzie, [17]), of Poisson manifolds (Weinstein, [27]) and of algebras of vector fields (Nistor, [22]). These, together with various other examples will be discussed in the forthcoming sections.

Let us look closer at the most trivial example. A vector field  $X \in \mathcal{X}^1(M)$  is the same as a Lie algebroid structure on the trivial line bundle  $\mathbb{L} \rightarrow M$ : the anchor is just multiplication by  $X$ , while the Lie bracket on  $\Gamma(\mathbb{L}) \simeq C^\infty(M)$  is given by  $[f, g] = X(f)g - fX(g)$ . The integrability result here states that a vector field is integrable to a local flow. It may be useful to think of the flow  $\Phi_X^t$  as a collection of arrows  $x \rightarrow \Phi_X^t(x)$  between the different points of the manifold, which can be composed by the rule  $\Phi_X^t \Phi_X^s = \Phi_X^{s+t}$ . The points which can be joined by such an arrow with a given point  $x$  form the orbit of  $\Phi_X$  (or the integral curve of  $X$ ) through  $x$ .

The general integrability problem is similar: it asks for the existence of a “space of arrows” and a partially defined multiplication, which unravels the infinitesimal structure  $(A, [ \ , \ ], \#)$ . In a more precise fashion, a *groupoid* is a small category  $\mathcal{G}$  with all arrows invertible. If the set of objects (points) is  $M$ , we say that  $\mathcal{G}$  is a groupoid over  $M$ . We shall denote by the same letter  $\mathcal{G}$  the space of arrows, and write

$$\begin{array}{c} \mathcal{G} \\ \mathbf{s} \downarrow \quad \downarrow \mathbf{t} \\ M \end{array}$$

where  $\mathbf{s}$  and  $\mathbf{t}$  are the source and target maps. If  $g, h \in \mathcal{G}$  the product  $gh$  is defined only for pairs  $(g, h)$  in the set of composable arrows

$$\mathcal{G}^{(2)} = \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid \mathbf{t}(h) = \mathbf{s}(g)\},$$

and we denote by  $g^{-1} \in \mathcal{G}$  the inverse of  $g$ , and by  $1_x = x$  the identity arrow at  $x \in M$ . If  $\mathcal{G}$  and  $M$  are topological spaces, all the maps are continuous, and  $\mathbf{s}$  and  $\mathbf{t}$  are open surjections, we say that  $\mathcal{G}$  is a *topological groupoid*. A *Lie groupoid* is a groupoid where the space of arrows  $\mathcal{G}$  and the space of objects  $M$  are smooth manifolds, the source and target maps  $\mathbf{s}, \mathbf{t}$  are submersions, and all the other structure maps are smooth. We require  $M$  and the  $\mathbf{s}$ -fibers  $\mathcal{G}(x, -) = \mathbf{s}^{-1}(x)$ , where  $x \in M$ , to be Hausdorff manifolds, but it is important to allow the total space  $\mathcal{G}$  of arrows to be non-Hausdorff: simple examples arise even when integrating Lie algebra bundles [10], while in foliation theory it is well known that the monodromy groupoid of a foliation is non-Hausdorff if there are vanishing cycles. For more details see [3].

As in the case of Lie groups, any Lie groupoid  $\mathcal{G}$  has an associated Lie algebroid  $A = A(\mathcal{G})$ . As a vector bundle, it is the restriction to  $M$  of the bundle  $T^{\mathbf{s}}\mathcal{G}$  of  $\mathbf{s}$ -vertical vector fields on  $\mathcal{G}$ . Its fiber at  $x \in M$  is the tangent space at  $1_x$  of the  $\mathbf{s}$ -fibers  $\mathcal{G}(x, -) = \mathbf{s}^{-1}(x)$ , and the anchor map is just the differential of the target map  $\mathbf{t}$ . To define the bracket, one shows that  $\Gamma(A)$  can be identified with  $\mathcal{X}_{\text{inv}}^{\mathbf{s}}(\mathcal{G})$ , the space of  $\mathbf{s}$ -vertical, right-invariant, vector fields on  $\mathcal{G}$ . The standard formula of Lie brackets in terms of flows shows that  $\mathcal{X}_{\text{inv}}^{\mathbf{s}}(\mathcal{G})$  is closed under  $[\cdot, \cdot]$ . This induces a Lie bracket on  $\Gamma(A)$ , which makes  $A$  into a Lie algebroid.

We say that a Lie algebroid  $A$  is integrable if there exists a Lie groupoid  $\mathcal{G}$  inducing  $A$ . The extension of Lie’s theory (Lie’s first and second theorem) to Lie algebroids has a promising start.

**THEOREM (Lie I).** *If  $A$  is an integrable Lie algebroid, then there exists a (unique)  $\mathbf{s}$ -simply connected Lie groupoid integrating  $A$ .*

This has been proved in [20] (see also [17] for the transitive case). A different argument, which is just an extension of the construction of the smooth structure on the universal cover of a manifold (cf. Theorem 1.13.1 in [11]), will be presented below. Here  $\mathfrak{s}$ -simply connected means that the  $\mathfrak{s}$ -fibers  $\mathfrak{s}^{-1}(x)$  are simply connected. The Lie groupoid in the theorem is often called the *monodromy groupoid* of  $A$ , and will be denoted by  $\text{Mon}(A)$ . For the simple examples above,  $\text{Mon}(TM)$  is the homotopy groupoid of  $M$ ,  $\text{Mon}(\mathcal{F})$  is the monodromy groupoid of the foliation  $\mathcal{F}$ , while  $\text{Mon}(\mathfrak{g})$  is the unique simply-connected Lie group integrating  $\mathfrak{g}$ .

The following result is standard (we refer to [19], [20], although the reader may come across it in various other places). See also Section 2 below.

**THEOREM (Lie II).** *Let  $\phi : A \rightarrow B$  be a morphism of integrable Lie algebroids, and let  $\mathcal{G}$  and  $\mathcal{H}$  be integrations of  $A$  and  $B$ . If  $\mathcal{G}$  is  $\mathfrak{s}$ -simply connected, then there exists a (unique) morphism of Lie groupoids  $\Phi : \mathcal{G} \rightarrow \mathcal{H}$  integrating  $\phi$ .*

In contrast with the case of Lie algebras or foliations, there is no Lie's third theorem for general Lie algebroids. Examples of nonintegrable Lie algebroids are known (we will see several of them in the forthcoming sections) and, up to now, no good explanation for this failure was known. For transitive Lie algebroids, there is a cohomological obstruction due to Mackenzie ([17]), which may be regarded as an extension to non-abelian groups of the Chern class of a circle bundle, and which gives a necessary and sufficient criterion for integrability. Other various integrability criteria one finds in the literature are (apparently) nonrelated: some require a nice behavior of the Lie algebras  $\mathfrak{g}_x$ , some require a nice topology of the leaves of the induced foliation, and most of them require regular algebroids. A good understanding of this failure should shed some light on the following questions:

- *Is there a (computable) obstruction to the integrability of Lie algebroids?*
- *Is the integrability problem a local one?*
- *Are Lie algebroids locally integrable?*

In this paper we provide answers to these questions. We show that the obstruction to integrability comes from the relation between the topology of the leaves of the induced foliation and the Lie algebras defined by the kernel of the anchor map.

We will now outline our integrability result. Given an algebroid  $A$  and  $x \in M$ , we will construct certain (monodromy) subgroups  $N_x(A) \subset A_x$ , which lie in the center of the Lie algebra  $\mathfrak{g}_x = \text{Ker}(\#_x)$ : they consist of those elements

$v \in Z(\mathfrak{g}_x)$  which are *homotopic* to zero (see §1). As we shall explain, these groups arise as the image of a second-order monodromy map

$$\partial : \pi_2(L_x) \rightarrow \mathcal{G}(\mathfrak{g}_x),$$

which relates the topology of the leaf  $L_x$  through  $x$  with the simply connected Lie group  $\mathcal{G}(\mathfrak{g}_x)$  integrating the Lie algebra  $\mathfrak{g}_x = \text{Ker}(\#_x)$ . From a conceptual point of view, the monodromy map can be viewed as an analogue of a boundary map of the homotopy long exact sequence of a fibration (namely  $0 \rightarrow \mathfrak{g}_{L_x} \rightarrow A_{L_x} \rightarrow TL_x \rightarrow 0$ ). In order to measure the discreteness of the groups  $N_x(A)$  we let

$$r(x) = d(0, N_x(A) - \{0\}),$$

where the distance is computed with respect to a (arbitrary) norm on the vector bundle  $A$ . Here we adopt the convention  $d(0, \emptyset) = +\infty$ . We will see that  $r$  is *not* a continuous function. Our main result is:

**THEOREM (Obstructions to Lie III).** *For a Lie algebroid  $A$  over  $M$ , the following are equivalent:*

- (i)  $A$  is integrable;
- (ii) For all  $x \in M$ ,  $N_x(A) \subset A_x$  is discrete and  $\liminf_{y \rightarrow x} r(y) > 0$ .

We stress that these obstructions are computable in many examples. First of all, the definition of the monodromy map is explicit. Moreover, given a splitting  $\sigma : TL \rightarrow A_L$  of  $\#$  with  $Z(\mathfrak{g}_L)$ -valued curvature 2-form  $\Omega_\sigma$ , we will see that

$$N_x(A) = \left\{ \int_\gamma \Omega_\sigma : \gamma \in \pi_2(L, x) \right\} \subset Z(\mathfrak{g}_x).$$

With this information at hand the reader can already jump to the examples (see §§3.3, 3.4, 4.1 and 5).

As is often the case, the main theorem is just an instance of a more fruitful approach. In fact, we will show that a Lie algebroid  $A$  always admits an “integrating” *topological* groupoid  $\mathcal{G}(A)$ . Although it is not always smooth (in general it is only a leaf space), it does behave like a Lie groupoid. This immediately implies the integrability of Lie algebroids by “local Lie groupoids”, a result which has been assumed to hold since the original works of Pradines in the 1960’s.

The main idea of our approach is as follows: Suppose  $\pi : A \rightarrow M$  is a Lie algebroid which can be integrated to a Lie groupoid  $\mathcal{G}$ . Denote by  $P(\mathcal{G})$  the space of  $\mathcal{G}$ -paths, with the  $C^2$ -topology:

$$P(\mathcal{G}) = \left\{ g : [0, 1] \rightarrow \mathcal{G} \mid g \in C^2, \mathfrak{s}(g(t)) = x, g(0) = 1_x \right\}$$

(paths lying in  $\mathfrak{s}$ -fibers of  $\mathcal{G}$  starting at the identity). Also, denote by  $\sim$  the equivalence relation defined by  $C^1$ -homotopies in  $P(\mathcal{G})$  with fixed end-points. Then we have a standard description of the monodromy groupoid as

$$\text{Mon}(A) = P(\mathcal{G}) / \sim .$$

The source and target maps are the obvious ones, and for two paths  $g, g' \in P(\mathcal{G})$  which are composable (i.e.  $\mathfrak{t}(g(1)) = \mathfrak{s}(g'(0))$ ) we define

$$g' \cdot g(t) \equiv \begin{cases} g(2t), & 0 \leq t \leq \frac{1}{2} \\ g'(2t - 1)g(1), & \frac{1}{2} < t \leq 1. \end{cases}$$

Note that any element in  $P(\mathcal{G})$  is equivalent to some  $g(t)$  with derivatives vanishing at the end-points, and if  $g$  and  $g'$  have this property, then  $g' \cdot g \in P(\mathcal{G})$ . Therefore, this multiplication is associative up to homotopy, so we get the desired multiplication on the quotient space which makes  $\text{Mon}(A)$  into a (topological) groupoid. The construction of the smooth structure on  $\text{Mon}(A)$  is similar to the construction of the smooth structure on the universal cover of a manifold (see e.g. Theorem 1.13.1 in [11]).

Now, any  $\mathcal{G}$ -path  $g$  defines an  $A$ -path  $a$ , i.e. a curve  $a : I \rightarrow A$  defined on the unit interval  $I = [0, 1]$ , with the property that

$$\#a(t) = \frac{d}{dt}\pi(a(t)).$$

The  $A$ -path  $a$  is obtained from  $g$  by differentiation and right translations. This defines a bijection between  $P(\mathcal{G})$  and the set  $P(A)$  of  $A$ -paths and, using this bijection, we can transport homotopy of  $\mathcal{G}$ -paths to an equivalence relation (*homotopy*) of  $A$ -paths. Moreover, this equivalence can be expressed using the infinitesimal data only (§1, below). It follows that a monodromy type groupoid  $\mathcal{G}(A)$  can be constructed without any integrability assumption. This construction of  $\mathcal{G}(A)$ , suggested by Alan Weinstein, in general only produces a *topological* groupoid (§2). Our main task will then be to understand when does the Weinstein groupoid  $\mathcal{G}(A)$  admit the desired smooth structure, and that is where the obstructions show up. We first describe the second-order monodromy map which encodes these obstructions (§3) and we then show that these are in fact the only obstructions to integrability (§4). In the final section, we derive the known integrability criteria from our general result and we give two applications.

*Acknowledgments.* The construction of the groupoid  $\mathcal{G}(A)$  was suggested to us by Alan Weinstein, and is inspired by a “new” proof of Lie’s third theorem in the recent monograph [11] by Duistermaat and Kolk. We are indebted to him for this suggestion as well as many comments and discussions. The same

type of construction, for the special case of Poisson manifolds, appears in the work of Cattaneo and Felder [4]. Though they do not discuss integrability obstructions, their paper was also a source of inspiration for the present work.

We would also like to express our gratitude for additional comments and discussions to Ana Cannas da Silva, Viktor Ginzburg, Kirill Mackenzie, Ieke Moerdijk, Janez Mrčun and James Stasheff.

## 1. $A$ -paths and homotopy

In this section  $A$  is a Lie algebroid over  $M$ ,  $\# : A \rightarrow TM$  denotes the anchor, and  $\pi : A \rightarrow M$  denotes the projection.

In order to construct our main object of study, the groupoid  $\mathcal{G}(A)$  that plays the role of the monodromy groupoid  $\text{Mon}(A)$  for a general (nonintegrable) algebroid, we need the appropriate notion of paths on  $A$ . These are known as  $A$ -paths (or admissible paths) and we shall discuss them in this section.

1.1.  $A$ -paths. We call a  $C^1$  curve  $a : I \rightarrow A$  an  $A$ -path if

$$\#a(t) = \frac{d}{dt}\gamma(t),$$

where  $\gamma(t) = \pi(a(t))$  is the base path (necessarily of class  $C^2$ ). We let  $P(A)$  denote the space of  $A$ -paths, endowed with the topology of uniform convergence.

We emphasize that this is the right notion of paths in the world of algebroids. From this point of view, one should view  $a$  as a bundle map

$$a \, dt : TI \rightarrow A$$

which covers the base path  $\gamma : I \rightarrow M$  and this gives a algebroid morphism  $TI \rightarrow A$ .

Obviously, the base path of an  $A$ -path sits inside a leaf  $L$  of the induced foliation, and so can be viewed as an  $A_L$ -path. The key remark is:

**PROPOSITION 1.1.** *If  $\mathcal{G}$  integrates the Lie algebroid  $A$ , then there is a homeomorphism  $D^R : P(\mathcal{G}) \rightarrow P(A)$  between the space of  $\mathcal{G}$ -paths, and the space of  $A$ -paths ( $D^R$  is called the differentiation of  $\mathcal{G}$ -paths, and its inverse is called the integration of  $A$ -paths.)*

*Proof.* Any  $\mathcal{G}$ -path  $g : I \rightarrow \mathcal{G}$  defines an  $A$ -path  $D^R(g) : I \rightarrow A$  by the formula

$$(D^R g)(t) = (dR_{g(t)^{-1}})_{g(t)} \dot{g}(t),$$

where, for  $h : x \rightarrow y$  an arrow in  $\mathcal{G}$ ,  $R_h : s^{-1}(y) \rightarrow s^{-1}(x)$  is the right multiplication by  $h$ . Conversely, any  $A$ -path  $a$  arises in this way, by integrating (using LIE II) the Lie algebroid morphism  $TI \rightarrow A$  defined by  $a$ . Finally, notice that any Lie groupoid homomorphism  $\phi : I \times I \rightarrow \mathcal{G}$  from the pair groupoid into  $\mathcal{G}$ , is of the form  $\phi(s, t) = g(s)g^{-1}(t)$  for some  $\mathcal{G}$ -path  $g$ .

A more explicit argument, avoiding LIE II, and which also shows that the inverse of  $D^R$  is continuous, is as follows. Given  $a$ , we choose a time-dependent section  $\alpha$  of  $A$  extending  $a$ , i.e. so that

$$a(t) = \alpha(t, \gamma(t)).$$

If we let  $\varphi_\alpha^{t,0}$  be the flow of the right-invariant vector field that corresponds to  $\alpha$ , then  $g(t) = \varphi_\alpha^{t,0}(\gamma(0))$  is the desired  $\mathcal{G}$ -path. Indeed, right-invariance guarantees that this flow is defined for all  $t \in [0, 1]$  and also implies that

$$(D^R g)(t) = (dR_{g(t)^{-1}})_{g(t)}(\alpha(t, g(t))) = \alpha(t, \gamma(t)) = a(t). \quad \square$$

1.2. *A-paths and connections.* Given an  $A$ -connection on a vector bundle  $E$  over  $M$ , most of the classical constructions (which we recover when  $A = TM$ ) extend to Lie algebroids, provided we use  $A$ -paths. This is explained in detail in [13], [12], and here we recall only the results we need.

An  $A$ -connection on a vector bundle  $E$  over  $M$  can be defined by an  $A$ -derivative operator  $\Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$ ,  $(\alpha, u) \mapsto \nabla_\alpha u$  satisfying  $\nabla_{f\alpha} u = f\nabla_\alpha u$ , and  $\nabla_\alpha(fu) = f\nabla_\alpha u + \# \alpha(f)u$ . The curvature of  $\nabla$  is given by the usual formula

$$R_\nabla(\alpha, \beta) = [\nabla_\alpha, \nabla_\beta] - \nabla_{[\alpha, \beta]},$$

and  $\nabla$  is called flat if  $R_\nabla = 0$ . For an  $A$ -connection  $\nabla$  on the vector bundle  $A$ , the torsion of  $\nabla$  is also defined as usual by:

$$T_\nabla(\alpha, \beta) = \nabla_\alpha \beta - \nabla_\beta \alpha - [\alpha, \beta].$$

Given an  $A$ -path  $a$  with base path  $\gamma : I \rightarrow M$ , and  $u : I \rightarrow E$  a path in  $E$  above  $\gamma$ , then the derivative of  $u$  along  $a$ , denoted  $\nabla_a u$ , is defined as usual: choose a time-dependent section  $\xi$  of  $E$  such that  $\xi(t, \gamma(t)) = u(t)$ , then

$$\nabla_a u(t) = \nabla_a \xi^t(x) + \frac{d\xi^t}{dt}(x), \quad \text{at } x = \gamma(t).$$

One has then the notion of parallel transport along  $a$ , denoted  $T_a^t : E_{\gamma(0)} \rightarrow E_{\gamma(t)}$ , and for the special case  $E = A$ , we can talk about the geodesics of  $\nabla$ . Geodesics are  $A$ -paths  $a$  with the property that  $\nabla_a a(t) = 0$ . Exactly as in the classical case, one has existence and uniqueness of geodesics with given initial base point  $x \in M$  and “initial speed”  $a_0 \in A_{x_0}$ .

*Example 1.2.* If  $L$  is a leaf of the foliation induced by  $A$ , then  $\mathfrak{g}_L = \text{Ker}(\#|_L)$  carries a flat  $A_L$ -connection defined by  $\nabla_\alpha\beta = [\alpha, \beta]$ . In particular, for any  $A$ -path  $a$ , the induced parallel transport defines a linear map, called *the linear holonomy of  $a$* ,

$$\text{Hol}(a) : \mathfrak{g}_x \rightarrow \mathfrak{g}_y ,$$

where  $x, y$  are the initial and the end-point of the base path. For more on linear holonomy we refer to [13].

Most of the connections that we will use are induced by a standard  $TM$ -connection  $\nabla$  on the vector bundle  $A$ . Associated with  $\nabla$  there is an obvious  $A$ -connection on the vector bundle  $A$

$$\nabla_\alpha\beta \equiv \nabla_{\#\alpha}\beta.$$

A bit more subtle are the following two  $A$ -connections on  $A$  and on  $TM$ , respectively (see [6]):

$$\bar{\nabla}_\alpha\beta \equiv \nabla_{\#\beta}\alpha + [\alpha, \beta], \quad \bar{\nabla}_\alpha X \equiv \#\nabla_X\alpha + [\#\alpha, X].$$

Note that  $\bar{\nabla}_\alpha\#\beta = \#\bar{\nabla}_\alpha\beta$ , so in the terminology of [13] this means that  $\bar{\nabla}$  is a basic connection on  $A$ . These connections play a fundamental role in the theory of characteristic classes (see [5], [6], [13]).

1.3. *Homotopy of  $A$ -paths.* As we saw above, if  $A$  is integrable,  $A$ -paths are in a bijective correspondence with  $\mathcal{G}$ -paths. Let us see now how one can transport the notion of homotopy to  $P(A)$ , so that it only uses the infinitesimal data (i.e., Lie algebroid data).

Let us fix

$$a_\epsilon(t) = a(\epsilon, t) : I \times I \rightarrow A$$

a *variation of  $A$ -paths*, that is a family of  $A$ -paths  $a_\epsilon$  which is of class  $C^2$  on  $\epsilon$ , with the property that the base paths  $\gamma_\epsilon(t) = \gamma(\epsilon, t) : I \times I \rightarrow M$  have fixed end-points. If  $A$  came from a Lie groupoid  $\mathcal{G}$ , and  $a_\epsilon$  came from  $\mathcal{G}$ -paths  $g_\epsilon$ , then  $g_\epsilon$  would not necessarily give a homotopy between  $g_0$  and  $g_1$ , because the end-points  $g_\epsilon(1)$  may vary. The following lemma describes two distinct ways of controlling the variation  $\frac{d}{d\epsilon}g_\epsilon(1)$ : one way uses a connection on  $A$ , and the other uses flows of sections of a  $A$  (see Appendix A). They both depend only on infinitesimal data.

PROPOSITION 1.3. *Let  $A$  be an algebroid and  $a = a_\epsilon$  a variation of  $A$ -paths.*

- (i) *If  $\nabla$  is a  $TM$ -connection on  $A$  with torsion  $T_\nabla$ , then the solution  $b = b(\epsilon, t)$  of the differential equation*

(1)  $\partial_t b - \partial_\epsilon a = T_\nabla(a, b), \quad b(\epsilon, 0) = 0,$

does not depend on  $\nabla$ . Moreover,  $\#b = \frac{d}{d\epsilon}\gamma$ .

(ii) If  $\xi_\epsilon$  are time-dependent sections of  $A$  such that  $\xi_\epsilon(t, \gamma_\epsilon(t)) = a_\epsilon(t)$ , then  $b(\epsilon, t)$  is given by

(2) 
$$b(\epsilon, t) = \int_0^t \phi_{\xi_\epsilon}^{t,s} \frac{d\xi_\epsilon}{d\epsilon}(s, \gamma_\epsilon(s)) ds,$$

where  $\phi_{\xi_\epsilon}^{t,s}$  denotes the flow of the time-dependent section  $\xi_\epsilon$ .

(iii) If  $\mathcal{G}$  integrates  $A$  and  $g_\epsilon$  are the  $\mathcal{G}$ -paths satisfying  $D^R(g_\epsilon) = a_\epsilon$ , then  $b = D^R(g^t)$ , where  $g^t$  are the paths in  $\mathcal{G}$ :  $\epsilon \rightarrow g^t(\epsilon) = g(\epsilon, t)$ .

This motivates the following definition:

*Definition 1.4.* We say that two  $A$ -paths  $a_0$  and  $a_1$  are equivalent (or homotopic), and write  $a_0 \sim a_1$ , if there exists a variation  $a_\epsilon$  with the property that  $b$  insured by Proposition 1.3 satisfies  $b(\epsilon, 1) = 0$  for all  $\epsilon \in I$ .

When  $A$  admits an integration  $\mathcal{G}$ , then the isomorphism  $D^R : P(\mathcal{G}) \rightarrow P(A)$  of Proposition 1.1 transforms the usual homotopy into the homotopy of  $A$ -paths. Note also that, as  $A$ -paths should be viewed as algebroid morphisms, the pair  $(a, b)$  defining the equivalence of  $A$ -paths should be viewed as a true homotopy

$$adt + bd\epsilon : TI \times TI \rightarrow A$$

in the world of algebroids. In fact, equation (1) is just an explicit way of saying that this is a morphism of Lie algebroids (see [15]).

*Proof of Proposition 1.3.* Obviously, (i) follows from (ii). To prove (ii), let  $\xi_\epsilon$  be as in the statement, and let  $\eta$  be given by

$$\eta(\epsilon, t, x) = \int_0^t \phi_{\xi_\epsilon}^{t,s} \frac{d\xi_\epsilon}{d\epsilon}(s, \Phi_{\#\xi_\epsilon}^{s,t}(x)) ds \in A_x.$$

We may assume that  $\xi_\epsilon$  as compact support. We note that  $\eta$  coincides with the solution of the equation

(3) 
$$\frac{d\eta}{dt} - \frac{d\xi}{d\epsilon} = [\eta, \xi],$$

with  $\eta(\epsilon, 0) = 0$ . Indeed, since

$$\eta(\epsilon, t, -) = \int_0^t (\phi_{\xi_\epsilon}^{s,t})^* \left( \frac{d\xi_\epsilon^s}{d\epsilon} \right) ds \in \Gamma(A),$$

equation (3) immediately follows from the basic formula (A.2) for flows. Also,  $X = \#\xi$  and  $Y = \#\eta$  satisfy a similar equation on  $M$ , and since we have

$X(\epsilon, t, \gamma_\epsilon(t)) = \frac{d\gamma}{dt}$ , it follows that  $Y(\epsilon, t, \gamma_\epsilon(t)) = \frac{d\gamma}{d\epsilon}$ . In other words,  $b(\epsilon, t) = \eta(\epsilon, t, \gamma(\epsilon, t))$  satisfies  $\#b = \frac{d\gamma}{d\epsilon}$ . We now have

$$\partial_t b = \nabla_{\frac{d\gamma}{dt}} \eta + \frac{d\eta}{dt} = \nabla_{\# \xi} \eta + \frac{d\eta}{dt}$$

at  $x = \gamma_\epsilon(t)$ . Subtracting from this the similar formula for  $\partial_\epsilon a$  and using (3) we get

$$\partial_t b - \partial_\epsilon a = \nabla_{\# \xi} \eta - \nabla_{\# \eta} \xi + [\eta, \xi] = T_{\nabla}(\xi, \eta).$$

We are now left proving (iii). Assume that  $\mathcal{G}$  integrates  $A$  and  $g_\epsilon$  are the  $\mathcal{G}$ -paths satisfying  $D^R(g_\epsilon) = a_\epsilon$ . The formula of variation of parameters applied to the right-invariant vector field  $\xi_\epsilon$  shows that

$$\begin{aligned} \frac{\partial g(\epsilon, t)}{\partial \epsilon} &= \int_0^t (d\varphi_{\xi_\epsilon}^{t,s})_{g(\epsilon,s)} \frac{d\xi_\epsilon^s}{d\epsilon} (g(\epsilon, s)) ds \\ &= (dR_{g(\epsilon,t)})_{\gamma_\epsilon(t)} \int_0^t \phi_{\xi_\epsilon}^{t,s} \frac{d\xi_\epsilon^s}{d\epsilon} (\gamma_\epsilon(s)) ds. \end{aligned}$$

But then:

$$D^R(g^t) = \int_0^t \phi_{\xi_\epsilon}^{t,s} \frac{d\xi_\epsilon^s}{d\epsilon} (\gamma_\epsilon(s)) ds = b(\epsilon, t). \quad \square$$

The next lemma gives elementary properties of homotopies of  $A$ -paths:

LEMMA 1.5. *Let  $A$  be a Lie algebroid.*

- (i) *If  $\tau : I \rightarrow I$ , with  $\tau(0) = 0$ ,  $\tau(1) = 1$  is a smooth change of parameter, then any  $A$ -path  $a$  is equivalent to its reparametrization  $a^\tau(t) \equiv \tau'(t)a(\tau(t))$ .*
- (ii) *Any  $A$ -path  $a_0$  is equivalent to a smooth (i.e. of class  $C^\infty$ )  $A$ -path.*
- (iii) *If two smooth  $A$ -paths  $a_0, a_1$  are equivalent, then there exists a smooth homotopy between them.*

*Proof.* To prove (i), we consider the variation

$$a_\epsilon(t) = ((1 - \epsilon) + \epsilon\tau'(t))a((1 - \epsilon)t + \epsilon\tau(t))$$

and we check that the associated  $b$  satisfies  $b(\epsilon, 1) = 0$ . In fact, one can compute by any of the methods of Proposition 1.3:

$$b(\epsilon, t) = (\tau(t) - t)a((1 - \epsilon)t + \epsilon\tau(t)).$$

For example, if we let  $\alpha$  be a time-dependent section which extends the path  $a$ , and define a 1-parameter family of time-dependent sections  $\xi_\epsilon$  by:

$$\xi_\epsilon(t, x) = ((1 - \epsilon) + \epsilon\tau'(t))\alpha((1 - \epsilon)t + \epsilon\tau(t), x),$$

then  $\xi_\epsilon$  extends  $a_\epsilon$  and the family

$$\eta(\epsilon, t, x) = (\tau(t) - t)\alpha((1 - \epsilon)t + \epsilon\tau(t), x)$$

satisfies (3). Hence, we must have  $b(\epsilon, t) = \eta(\epsilon, t, \gamma(\epsilon, t))$  as claimed.

For (ii), note that from the similar claim for ordinary paths on manifolds (see e.g. Theorem 1.13.1 in [11]), we can find a  $C^r$ -homotopy  $\gamma_\epsilon$  between the base path  $\gamma_0$  of  $a_0$  and a smooth path  $\gamma_1$ . Also, we can do it so that  $\gamma_\epsilon$  stays in the same leaf  $L$  as  $\gamma_0$ , and so that  $\gamma_\epsilon(t)$  is smooth in the domain  $t \in [0, 1]$ ,  $\epsilon \in [c, 1]$  for some constant  $0 < c < 1$ . We now choose a smooth splitting  $\sigma : TL \rightarrow A|_L$  of the anchor map, and put  $b(\epsilon, t) = \sigma(\frac{d}{d\epsilon}\gamma_\epsilon(t))$ . Let  $a$  be the solution of the differential equation (1), with the initial conditions  $a(0, t) = a_0(t)$ . Clearly  $a$  is smooth on the domain on which  $b$  is; hence it defines a homotopy between  $a_0$  and the smooth  $A$ -path  $a_1$ . Part (iii) is just a degree-one higher version of part (ii), and can be proved similarly, replacing the path  $a_0$  by the given homotopy between  $a_0$  and  $a_1$  (a similar argument will be presented in detail in the proof of Proposition 3.5).  $\square$

1.4. *Representations and A-paths.* A flat  $A$ -connection on a vector bundle  $E$  defines a *representation* of  $A$  on  $E$ . The terminology is inspired by the case of Lie algebras. There is also an obvious notion of representation of a Lie groupoid  $\mathcal{G}$ : this is a vector bundle  $E$  over the space  $M$  of objects, together with smooth linear actions  $g : E_x \rightarrow E_y$  defined for arrows  $g$  from  $x$  to  $y$  in  $\mathcal{G}$ , satisfying the usual identities. By differentiation, any such representation becomes a representation of the Lie algebroid  $A$  of  $\mathcal{G}$  (see e.g. [5], [15]). Moreover, when  $\mathcal{G} = \text{Mon}(A)$  is the unique  $\mathfrak{s}$ -simply connected Lie groupoid integrating  $A$ , this construction induces a bijection

$$\text{Rep}(\text{Mon}(A)) \cong \text{Rep}(A)$$

between the (semi-rings of equivalence classes of) representations. This is explained in [5], [14], using the integrability of actions of [20], but it follows also from our construction of  $\mathcal{G}(A)$  (see next section) since we have:

PROPOSITION 1.6. *If  $a_0$  and  $a_1$  are equivalent  $A$ -paths from  $x$  to  $y$ . Then for any representation  $E$  of  $A$ , parallel transports  $E_x \rightarrow E_y$  along  $a_0$  and  $a_1$  coincide.*

*Proof.* We first claim that for any  $A$ -connection  $\nabla$  on  $E$ , and homotopy  $adt + bde$  between  $a_0$  and  $a_1$ , we have:

$$\nabla_{a_\epsilon} \nabla_{b_t} u - \nabla_{b_t} \nabla_{a_\epsilon} u = R_\nabla(a, b)u$$

for all paths  $u : I \times I \rightarrow E$  above  $\gamma(\epsilon, t)$ . To see this, let us assume that  $\xi, \eta$  are as in the proof of Proposition 1.3, and let  $s$  be a family of time-dependent

sections of  $E$  so that  $u(\epsilon, t) = s(\epsilon, t, \gamma(\epsilon, t))$ . Then

$$\nabla_{b_t} u = \nabla_{\eta} s + \frac{ds}{d\epsilon}$$

at  $x = \gamma(\epsilon, t)$ . Hence

$$\nabla_{a_\epsilon} \nabla_{b_t} u = \nabla_{\xi} \nabla_{\eta} s + \nabla_{\xi} \left( \frac{ds}{d\epsilon} \right) + \nabla_{\eta} \left( \frac{ds}{dt} \right) + \frac{d^2 s}{d\epsilon dt} + \nabla_{\frac{d\eta}{dt}} s.$$

Subtracting the analogous formula for  $\nabla_{b_t} \nabla_{a_\epsilon} u$  and using (3), we have proved the claim.

When  $\nabla$  is flat, this formula applied to  $u(\epsilon, t) = T_{a_\epsilon}^t(u_0)$ , where  $T_{a_\epsilon}^t$  denotes parallel transport, gives  $\nabla_{a_\epsilon} \nabla_{b_t} u = 0$ . But  $\nabla_{b_t} u = 0$  at  $t = 0$ , hence  $\nabla_{b_t} u = 0$  for all  $t$ 's. Since  $u(0, t) = T_{a_0}^t(u_0)$  it follows that  $u(\epsilon, t) = T_{b_t}^\epsilon T_{a_0}^t(u_0)$ . Therefore  $T_{a_\epsilon}^t = T_{b_t}^\epsilon T_{a_0}^t$ , for all  $\epsilon, t$  and, in particular, for  $\epsilon = t = 1$  we get  $T_{a_1}^1 = T_{a_0}^1$ .  $\square$

Recalling the notion of linear holonomy (cf. Example 1.2) we have:

**COROLLARY 1.7.** *If  $a_0$  and  $a_1$  are equivalent  $A$ -paths from  $x$  to  $y$ , they induce the same linear holonomy maps*

$$\text{Hol}(a_0) = \text{Hol}(a_1) : \mathfrak{g}_x \rightarrow \mathfrak{g}_y.$$

## 2. The Weinstein groupoid

We are now ready to define the Weinstein groupoid  $\mathcal{G}(A)$  of a general Lie algebroid, which in the integrable case will be the unique  $\mathfrak{s}$ -simply connected groupoid integrating  $A$ .

2.1. *The groupoid  $\mathcal{G}(A)$ .* Let  $a_0, a_1$  be two composable  $A$ -paths, i.e. so that  $\pi(a_0(1)) = \pi(a_1(0))$ . We define their concatenation

$$a_1 \odot a_0(t) \equiv \begin{cases} 2a_0(2t), & 0 \leq t \leq \frac{1}{2} \\ 2a_1(2t - 1), & \frac{1}{2} < t \leq 1. \end{cases}$$

This is essentially the multiplication that we need. However,  $a_1 \odot a_0$  is only *piecewise* smooth. One way around this difficulty is allowing for  $A$ -paths which are *piecewise* smooth. Instead, let us fix a cutoff function  $\tau \in C^\infty(\mathbb{R})$  with the following properties:

- (a)  $\tau(t) = 1$  for  $t \geq 1$  and  $\tau(t) = 0$  for  $t \leq 0$ ;
- (b)  $\tau'(t) > 0$  for  $t \in ]0, 1[$ .

For an  $A$ -path  $a$  we denote, as above, by  $a^\tau$  its reparametrization  $a^\tau(t) = \tau'(t)a(\tau(t))$ . We now define the multiplication by

$$a_1a_0 \equiv a_1^\tau \odot a_0^\tau \in P(A).$$

According to Lemma 1.5 (i),  $a_0a_1$  is equivalent to  $a_0 \odot a_1$  whenever  $a_0(1) = a_1(0)$ . We also consider the natural structure maps: source and target  $\mathbf{s}, \mathbf{t} : P(A) \rightarrow M$  which map  $a$  to  $\pi(a(0))$  and  $\pi(a(1))$ , respectively, the identity section  $\varepsilon : M \rightarrow P(A)$  mapping  $x$  to the constant path above  $x$ , and the inverse  $\iota : P(A) \rightarrow P(A)$  mapping  $a$  to  $\bar{a}$  given by  $\bar{a}(t) = -a(1 - t)$ .

**THEOREM 2.1.** *Let  $A$  be a Lie algebroid over  $M$ . Then the quotient*

$$\mathcal{G}(A) \equiv P(A) / \sim$$

*is a  $\mathbf{s}$ -simply connected topological groupoid independent of the choice of cutoff function. Moreover, whenever  $A$  is integrable,  $\mathcal{G}(A)$  admits a smooth structure which makes it into the unique  $\mathbf{s}$ -simply connected Lie groupoid integrating  $A$ .*

*Proof.* If we take the maps on the quotient induced from the structure maps defined above, then  $\mathcal{G}(A)$  is clearly a groupoid. Note that the multiplication on  $P(A)$  was defined so that, whenever  $\mathcal{G}$  integrates  $A$ , the map  $D^R$  of Proposition 1.1 preserves multiplications. Hence the only thing we still have to prove is that  $\mathbf{s}, \mathbf{t} : \mathcal{G}(A) \rightarrow M$  are open maps.

Given  $D \subset \mathcal{G}(A)$  open, we will show that its saturation  $\tilde{D}$  with respect to the equivalence relation  $\sim$  is still open. This follows from the fact, to be shown later in Theorem 4.7, that the equivalence relation can be defined by a foliation on  $P(A)$ .

A more direct argument is to show that for any two equivalent  $A$ -paths  $a_0$  and  $a_1$ , there exists a homeomorphism of  $T : P(A) \rightarrow P(A)$  such that  $T(a) \sim a$  for all  $a$ 's, and  $T(a_0) = a_1$ . To construct such a  $T$  we let  $\eta = \eta(\epsilon, t)$  be a family of time-dependent sections of  $A$  which determines the equivalence  $a_0 \sim a_1$  (see Proposition 1.3), so that  $\eta(\epsilon, 0) = \eta(\epsilon, 1) = 0$  (we may assume  $\eta$  has compact support, so that all the flows involved are everywhere defined). Given an  $A$ -path  $b_0$ , we consider a time-dependent section  $\xi_0$  so that  $\xi_0(t, \gamma_0(t)) = b(t)$  and denote by  $\xi$  the solution of equation (3) with initial condition  $\xi_0$ . If we set  $\gamma_\epsilon(t) = \Phi_{\#\eta t}^{\epsilon, 0} \gamma_0(t)$  and  $b_\epsilon(t) = \xi_\epsilon(t, \gamma_\epsilon(t))$ , then  $T_\eta(b_0) \equiv b_1$  is homotopic to  $b_0$  via  $b_\epsilon$ , and maps  $a_0$  into  $a_1$ .  $\square$

**2.2. Homomorphisms.** Note that, although  $\mathcal{G}(A)$  is not always smooth, in many aspects it behaves like in the smooth (i.e. integrable) case. For instance, we can call a representation of  $\mathcal{G}(A)$  *smooth* if the action becomes smooth when pull backed to  $P(A)$ . Similarly one can talk about smooth functions on  $\mathcal{G}(A)$ , about its tangent space, etc. This subsection and the next are variations on this theme.

PROPOSITION 2.2. *Let  $A$  and  $B$  be Lie algebroids. Then:*

- (i) *Every algebroid homomorphism  $\phi : A \rightarrow B$  determines a smooth groupoid homomorphism  $\Phi : \mathcal{G}(A) \rightarrow \mathcal{G}(B)$  of the associated Weinstein groupoids. If  $A$  and  $B$  are integrable, then  $\Phi_* = \phi$ ;*
- (ii) *Every representation  $E \in \text{Rep}(A)$  determines a smooth representation of  $\mathcal{G}(A)$ , which in the integrable case is the induced representation.*

*Proof.* For (i) we define  $\Phi$  in the only possible way: If  $a \in P(A)$  is an  $A$ -path then  $\phi \circ a$  is an  $A$ -path in  $P(B)$ . Moreover, it is easy to see that if  $a_1 \sim a_2$  are equivalent  $A$ -paths then  $\phi \circ a_1 \sim \phi \circ a_2$ , so we get a well-defined smooth map  $\Phi : \mathcal{G}(A) \rightarrow \mathcal{G}(B)$  by setting

$$\Phi([a]) \equiv [\phi \circ a].$$

This map is clearly a groupoid homomorphism.

Part (ii) follows easily from Proposition 1.6.  $\square$

In particular we see that, as in the smooth case, there is a bijection between the representations of  $A$  and the (smooth) representations of  $\mathcal{G}(A)$ :

$$\text{Rep}(\mathcal{G}(A)) \cong \text{Rep}(A).$$

2.3. *The exponential map.* Assume first that  $\mathcal{G}$  is a Lie groupoid integrating  $A$ , and  $\nabla$  is a  $TM$ -connection on  $A$ . Then the pull-back of  $\nabla$  along the target map  $\mathbf{t}$  defines a family of (right-invariant) connections  $\nabla_x$  on the manifolds  $\mathbf{s}^{-1}(x)$ . The associated exponential maps  $\text{Exp}_{\nabla_x} : A_x = T_x^{\mathbf{s}}\mathcal{G} \rightarrow \mathbf{s}^{-1}(x)$  fit together into a global exponential map [23]

$$\overline{\text{Exp}}_{\nabla} : A \rightarrow \mathcal{G}$$

(defined only on an open neighborhood of the zero section). By standard arguments,  $\overline{\text{Exp}}_{\nabla}$  is a diffeomorphism on a small enough neighborhood of  $M$ .

Now if  $A$  is not integrable, we still have the exponential map associated to a connection  $\nabla$  on  $A$ . It is defined as usual, so  $\text{Exp}_{\nabla}(a)$  is the value at time  $t = 1$  of the geodesic ( $A$ -path) with the initial condition  $a$ . By a slight abuse of notation we view it as a map

$$\text{Exp}_{\nabla} : A \rightarrow P(A).$$

Of course,  $\text{Exp}_{\nabla}$  is only defined on an open neighborhood of  $M$  inside  $A$  consisting of elements whose geodesics are defined for all  $t \in [0, 1]$ . Passing to the quotient, we have an induced exponential map

$$\overline{\text{Exp}}_{\nabla} : A \rightarrow \mathcal{G}(A).$$

For integrable  $A$ , this coincides with the exponential map above.

Note that the exponential map we have discussed so far depends on the choice of the connection. To get an exponential, independent of the connection, recall ([17]) that an *admissible section* of a Lie groupoid  $\mathcal{G}$  is a differentiable map  $\sigma : M \rightarrow \mathcal{G}$ , such that  $\mathbf{s} \circ \sigma(x) = x$  and  $\mathbf{t} \circ \sigma : M \rightarrow M$  is a diffeomorphism. Also, each admissible section  $\sigma \in \Gamma(\mathcal{G})$  determines diffeomorphisms

$$\begin{aligned} \mathcal{G} \ni g \mapsto \sigma g &\equiv \sigma(x)g, & \text{where } x = \mathbf{t}(g), \\ \mathcal{G} \ni g \mapsto g\sigma &\equiv g\sigma(y), & \text{where } \mathbf{t} \circ \sigma(y) = \mathbf{s}(g). \end{aligned}$$

Now, each section  $\alpha \in \Gamma(A)$  can be identified with a right-invariant vector field on  $\mathcal{G}$ , and we denote its flow by  $\varphi_\alpha^t$ . We define an admissible section  $\exp(\alpha)$  of  $\mathcal{G}$  by setting:

$$\exp(\alpha)(x) \equiv \varphi_\alpha^1(x).$$

This gives an exponential map  $\exp : \Gamma(A) \rightarrow \Gamma(\mathcal{G})$  which, in general, is defined only for sections  $\alpha$  sufficiently close to the zero section (e.g., sections with compact support). For more details see also [17], [22].

In the nonintegrable case, we can also define an exponential map  $\exp : \Gamma(A) \rightarrow \Gamma(\mathcal{G}(A))$  to the admissible smooth sections of the Weinstein groupoid as follows. First of all notice that

$$a_\alpha(x)(t) = \alpha(t, \phi_{\# \alpha}^{t,0}(x))$$

defines an  $A$  path  $a_\alpha(x)$  for any  $x \in M$  and for any time-dependent section  $\alpha$  of  $A$  with flow defined up to  $t = 1$  (e.g., if  $\alpha$  is sufficiently close to zero, or if it is compactly supported). This defines a smooth map  $a_\alpha : M \rightarrow P(A)$ . For  $\alpha \in \Gamma(A)$  close enough to the zero section we set

$$\exp(\alpha)(x) = [a_\alpha(x)].$$

Notice that  $a = a_\alpha(x)$  is the unique  $A$ -path with  $a(0) = \alpha(x)$  and  $a(t) = \alpha(\pi(a(t)))$ , for all  $t \in I$ .

In the integrable case these two constructions coincide. Moreover, for a general Lie algebroid, we have the following

**PROPOSITION 2.3.** *Let  $A$  be a Lie algebroid and  $\alpha, \beta \in \Gamma(A)$ . Then, as admissible sections,*

$$\exp(t\alpha) \exp(\beta) \exp(-t\alpha) = \exp(\phi_\alpha^t \beta),$$

where  $\phi_\alpha^t$  denotes the infinitesimal flow of  $\alpha$  (see Appendix A).

*Proof.* First we make the following remark concerning functoriality of  $\exp$ : Let  $\phi : A_1 \rightarrow A_2$  be an isomorphism of Lie algebroids and let  $\Phi : \mathcal{G}(A_1) \rightarrow \mathcal{G}(A_2)$  be the corresponding isomorphism of groupoids (Proposition 2.2 (i)).

If one denotes by  $\tilde{\phi}$  (resp.  $\tilde{\Phi}$ ) the corresponding homomorphism of sections (resp. admissible sections), then we obtain the following commutative diagram:

$$\begin{array}{ccc} \Gamma(\mathcal{G}(A_1)) & \xrightarrow{\tilde{\Phi}} & \Gamma(\mathcal{G}(A_2)) \\ \exp \uparrow & & \uparrow \exp \\ \Gamma(A_1) & \xrightarrow{\tilde{\phi}} & \Gamma(A_2). \end{array}$$

To prove the proposition, it is therefore enough to proof that for the homomorphism  $\Phi_\alpha^t : \mathcal{G}(A) \rightarrow \mathcal{G}(A)$  associated to  $\phi_\alpha^t : A \rightarrow A$  we have:

$$\Phi_\alpha^t(g) = \exp(t\alpha)g \exp(-t\alpha).$$

Equivalently,

$$[\phi_\alpha^t \circ a] = \exp(t\alpha)[a] \exp(-t\alpha)$$

for any  $A$ -path  $a \in \mathcal{G}(A)$ . Now, to prove this, one considers the variation of  $A$ -paths  $a_\varepsilon = \exp(-\varepsilon t\alpha) \cdot (\phi_\alpha^{\varepsilon t} \circ a) \cdot \exp(-\varepsilon t\alpha)$ , and checks that this realizes an equivalence of  $A$ -paths using Proposition 1.3.  $\square$

*Remark 2.4.* Hence  $\mathcal{G}(A)$  behaves in many respects like a smooth manifold, even if  $A$  is not integrable. This might be important in various aspects of noncommutative geometry and its applications to singular foliations and analysis: one might expect that the algebras of pseudodifferential operators and the  $C^*$ -algebra of  $\mathcal{G}(A)$  (see [23]) can be constructed even in the nonintegrable case. A related question is when  $\mathcal{G}(A)$  is a measurable groupoid.

Although the exponential map does exist even in the nonintegrable case, its injectivity on a neighborhood of  $M$  only holds if  $A$  is integrable. One could say that this is *the* difference between the integrable and the nonintegrable cases, as we will see in the next sections. However, our main job is to relate the kernel of the exponential and the geometry of  $A$ , and this is the origin of our obstructions: the monodromy groups described in the next section consist of the simplest elements which belong to this kernel. It turns out that these elements are enough to control the entire kernel.

### 3. Monodromy

Let  $A$  be a Lie algebroid over  $M$ ,  $x \in M$ . In this section we give several descriptions of the (second-order) monodromy groups of  $A$  at  $x$ , which control the integrability of  $A$ .

**3.1. Monodromy groups.** There are several possible ways to introduce the monodromy groups. Our first description is as follows:

*Definition 3.1.* We define  $N_x(A) \subset A_x$  as the subset of the center of  $\mathfrak{g}_x$  formed by those elements  $v \in Z(\mathfrak{g}_x)$  with the property that the constant  $A$ -path  $v$  is equivalent to the trivial  $A$ -path.

Let us denote by  $\mathcal{G}(\mathfrak{g}_x)$  the simply-connected Lie group integrating  $\mathfrak{g}_x$  (equivalently, the Weinstein groupoid associated to  $\mathfrak{g}_x$ ). Also, let  $\mathcal{G}(A)_x$  be the isotropy groups of the Weinstein groupoid  $\mathcal{G}(A)$ :

$$\mathcal{G}(A)_x \equiv \mathfrak{s}^{-1}(x) \cap \mathfrak{t}^{-1}(x) \subset \mathcal{G}(A) .$$

Closely related to the groups  $N_x(A)$  are the following:

*Definition 3.2.* We define  $\tilde{N}_x(A)$  as the subgroup of  $\mathcal{G}(\mathfrak{g}_x)$  which consists of the equivalence classes  $[a] \in \mathcal{G}(\mathfrak{g}_x)$  of  $\mathfrak{g}_x$ -paths with the property that, as an  $A$ -path,  $a$  is equivalent to the trivial  $A$ -path.

The precise relation is as follows:

LEMMA 3.3. *For any Lie algebroid  $A$ , and any  $x \in M$ ,  $\tilde{N}_x(A)$  is a subgroup of  $\mathcal{G}(\mathfrak{g}_x)$  contained in the center  $Z(\mathcal{G}(\mathfrak{g}_x))$ , and its intersection with the connected component  $Z(\mathcal{G}(\mathfrak{g}_x))^0$  of the center is isomorphic to  $N_x(A)$ .*

*Proof.* Given  $g \in \tilde{N}_x(A) \subset \mathcal{G}(\mathfrak{g}_x)$  represented by a  $\mathfrak{g}_x$ -path  $a$ , Proposition 1.6 implies that parallel transport  $T_a : \mathfrak{g}_x \rightarrow \mathfrak{g}_x$  along  $a$  is the identity. On the other hand, since  $a$  sits inside  $\mathfrak{g}_x$ , it is easy to see that  $T_a = \text{ad}_g$ , the adjoint action by the element  $g \in \mathcal{G}(\mathfrak{g}_x)$  represented by  $a$ . This shows that  $g \in Z(\mathcal{G}(\mathfrak{g}_x))$ . The last part follows from the fact that the exponential map induces an isomorphism  $\exp : Z(\mathfrak{g}_x) \rightarrow Z(\mathcal{G}(\mathfrak{g}_x))^0$  (cf., e.g., 1.14.3 in [11]), and  $N_x(A) = \exp^{-1}(\tilde{N}_x(A))$ .  $\square$

Since the group  $\tilde{N}_x(A)$  is always countable (see next section), we obtain:

COROLLARY 3.4. *For any Lie algebroid  $A$ , and any  $x \in M$ , the following are equivalent:*

- (i)  $\tilde{N}_x(A)$  is closed;
- (ii)  $\tilde{N}_x(A)$  is discrete;
- (iii)  $N_x(A)$  is closed;
- (iv)  $N_x(A)$  is discrete.

We remark that a special case of our main theorem shows that the previous assertions are in fact equivalent to the integrability of  $A|_{L_x}$ , the restriction of  $A$  to the leaf through  $x$ .

3.2. *A second-order monodromy map.* Let  $L \subset M$  denote the leaf through  $x$ . We define a homomorphism  $\partial : \pi_2(L, x) \rightarrow \mathcal{G}(\mathfrak{g}_x)$  with image precisely the group  $\tilde{N}_x(A)$ . This second-order monodromy map relates the topology of the leaf through  $x$  with the Lie algebra  $\mathfrak{g}_x$ .

Let  $[\gamma] \in \pi_2(L, x)$  be represented by a smooth path  $\gamma : I \times I \rightarrow L$  which maps the boundary into  $x$ . We choose a morphism of algebroids

$$adt + bde : TI \times TI \rightarrow A_L$$

(i.e.  $(a, b)$  satisfies equation (1)) which lifts  $d\gamma : TI \times TI \rightarrow TL$  via the anchor, and such that  $a(0, t)$ ,  $b(\epsilon, 0)$ , and  $b(\epsilon, 1)$  vanish. This is always possible: for example, we can put  $b(\epsilon, t) = \sigma(\frac{d}{d\epsilon}\gamma(\epsilon, t))$  where  $\sigma : TL \rightarrow A_L$  is a splitting of the anchor map, and take  $a$  to be the unique solution of the differential equation (1) with the initial conditions  $a(0, t) = 0$ . Since  $\gamma$  is constant on the boundary,  $a_1 = a(1, -)$  stays inside the Lie algebra  $\mathfrak{g}_x$ , i.e. defines a  $\mathfrak{g}_x$ -path

$$a_1 : I \rightarrow \mathfrak{g}_x .$$

Its integration (cf. [11], or our Proposition 1.1 applied to the Lie algebra  $\mathfrak{g}_x$ ) defines a path in  $\mathcal{G}(\mathfrak{g}_x)$ , and its end-point is denoted by  $\partial(\gamma)$ .

PROPOSITION 3.5. *The element  $\partial(\gamma) \in \mathcal{G}(\mathfrak{g}_x)$  does not depend on the auxiliary choices we made, and only depends on the homotopy class of  $\gamma$ . Moreover, the resulting map*

$$(4) \quad \partial : \pi_2(L, x) \rightarrow \mathcal{G}(\mathfrak{g}_x)$$

*is a morphism of groups and its image is precisely  $\tilde{N}_x(A)$ .*

Notice the similarity between the construction of  $\partial$  and the construction of the boundary map of the homotopy long exact sequence of a fibration: if we view  $0 \rightarrow \mathfrak{g}_L \rightarrow A_L \rightarrow TL \rightarrow 0$  as analogous to a fibration, the first few terms of the associated long exact sequence will be

$$\dots \rightarrow \pi_2(L, x) \xrightarrow{\partial} \mathcal{G}(\mathfrak{g}_x) \rightarrow \mathcal{G}(A)_x \rightarrow \pi_1(L, x).$$

The exactness at  $\mathcal{G}(\mathfrak{g}_x)$  is precisely the last statement of the proposition. We leave to the reader the (easy) check of exactness at  $\mathcal{G}(A)_x$ .

*Proof of Proposition 3.5.* From the definitions it is clear that  $\text{Im } \partial = \tilde{N}_x(A)$  so all we have to check is that  $\partial$  is well defined. For that we assume that

$$\gamma^i = \gamma^i(\epsilon, t) : I \times I \rightarrow L, \quad i \in \{0, 1\}$$

are homotopic relative to the boundary, and that

$$a^i dt + b^i d\epsilon : TI \times TI \rightarrow A_L. \quad i \in \{0, 1\}$$

are lifts of  $d\gamma^i$  as above. We prove that the paths  $a^i(1, t)$  ( $i \in \{0, 1\}$ ) are homotopic as  $\mathfrak{g}_x$ -paths.

By hypothesis, there is a homotopy  $\gamma^u = \gamma^u(\epsilon, t)$  ( $u \in I$ ) between  $\gamma^0$  and  $\gamma^1$ . We choose a family  $b^u(\epsilon, t)$  joining  $b^0$  and  $b^1$ , such that  $\#(b^u(\epsilon, t)) = \frac{d\gamma^u}{d\epsilon}$  and  $b^u(\epsilon, 0) = b^u(\epsilon, 1) = 0$ . We also choose a family of sections  $\eta$  depending on  $u, \epsilon, t$  such that

$$\eta^u(\epsilon, t, \gamma^u(\epsilon, t)) = b^u(\epsilon, t), \text{ with } \eta = 0 \text{ when } t = 0, 1.$$

As in the proof of Proposition 1.3, let  $\xi$  and  $\theta$  be the solutions of

$$\begin{cases} \frac{d\xi}{d\epsilon} - \frac{d\eta}{dt} = [\xi, \eta], \text{ with } \xi = 0 \text{ when } \epsilon = 0, 1, \\ \frac{d\theta}{d\epsilon} - \frac{d\eta}{du} = [\theta, \eta], \text{ with } \theta = 0 \text{ when } \epsilon = 0, 1. \end{cases}$$

Setting  $u = 0, 1$  we get

$$a^i(\epsilon, t) = \xi^i(\epsilon, t, \gamma^i(\epsilon, t)), \quad i = 0, 1.$$

On the other hand, setting  $t = 0, 1$  we get

$$\theta = 0 \text{ when } t = 0, 1.$$

A brief computation shows that  $\phi \equiv \frac{d\xi}{du} - \frac{d\theta}{dt} - [\xi, \theta]$  satisfies

$$\frac{d\phi}{d\epsilon} = [\phi, \eta],$$

and since  $\phi = 0$  when  $\epsilon = 0$ , it follows that

$$\frac{d\xi}{du} - \frac{d\theta}{dt} = [\xi, \theta].$$

If in this relation we choose  $\epsilon = 1$ , and use  $\theta^u(1, t) = 0$  when  $t = 0, 1$ , we conclude that  $a^i(1, t) = \xi^i(1, t, \gamma^i(1, t))$ ,  $i = 0, 1$ , are equivalent as  $\mathfrak{g}_x$ -paths. □

3.3. *Computing the monodromy.* Let us indicate briefly how the monodromy groups (Definition 3.1 or, alternatively, Definition 3.2), can be explicitly computed in many examples. We consider the short exact sequence

$$0 \rightarrow \mathfrak{g}_L \rightarrow A_L \xrightarrow{\#} TL \rightarrow 0$$

and a linear splitting  $\sigma : TL \rightarrow A_L$  of  $\#$ . The *curvature* of  $\sigma$  is the element  $\Omega_\sigma \in \Omega^2(L; \mathfrak{g}_L)$  defined by:

$$\Omega_\sigma(X, Y) \equiv \sigma([X, Y]) - [\sigma(X), \sigma(Y)] .$$

In favorable cases, the computation of monodromy can be reduced to the following:

LEMMA 3.6. *If there is a splitting  $\sigma$  with the property that its curvature  $\Omega_\sigma$  is  $Z(\mathfrak{g}_L)$ -valued, then*

$$N_x(A) = \left\{ \int_\gamma \Omega_\sigma : [\gamma] \in \pi_2(L, x) \right\} \subset Z(\mathfrak{g}_x)$$

for all  $x \in L$ .

Before we give a proof some explanations are in order.

First of all,  $Z(\mathfrak{g}_L)$  is canonically a flat vector bundle over  $L$ . The corresponding flat connection can be expressed with the help of the splitting  $\sigma$  as

$$\nabla_X \alpha = [\sigma(X), \alpha],$$

and it is easy to see that the definition does not depend on  $\sigma$ . In this way  $\Omega_\sigma$  appears as a 2-cohomology class with coefficients in the local system defined by  $Z(\mathfrak{g}_L)$  over  $L$ , and then the integration is just the usual pairing between cohomology and homotopy. In practice one can always avoid working with local coefficients: if  $Z(\mathfrak{g}_L)$  is not already trivial as a vector bundle, one can achieve this by pulling back to the universal cover of  $L$  (where parallel transport with respect to the flat connection gives the desired trivialization).

Second, we should specify what we mean by integrating forms with coefficients in a local system. Assume  $\omega \in \Omega^2(M; E)$  is a 2-form with coefficients in some flat vector bundle  $E$ . Integrating  $\omega$  over a 2-cycle  $\gamma : \mathbb{S}^2 \rightarrow M$  means (i) taking the pull-back  $\gamma^* \omega \in \Omega^2(\mathbb{S}^2; \gamma^* E)$ , and (ii) integrate  $\gamma^* \omega$  over  $\mathbb{S}^2$ . Here  $\gamma^* E$  should be viewed as a flat vector bundle of  $\mathbb{S}^2$  for the pull-back connection. Notice that the connection enters the integration part, and this matters for the integration to be invariant under homotopy.

*Proof of Lemma 3.6.* We may assume that  $L = M$ , i.e.  $A$  is transitive. In agreement with the comments above, we also assume for simplicity that  $Z(\mathfrak{g})$  is trivial as a vector bundle ( $\mathfrak{g} \equiv \mathfrak{g}_L$ ). The formula above defines a connection  $\nabla^\sigma$  on the entire  $\mathfrak{g}$ . We use  $\sigma$  to identify  $A$  with  $TM \oplus \mathfrak{g}$  so the bracket becomes

$$[(X, v), (Y, w)] = ([X, Y], [v, w] + \nabla_X^\sigma(w) - \nabla_Y^\sigma(v) - \Omega_\sigma(X, Y)).$$

We choose a connection  $\nabla^M$  on  $M$ , and we consider the connection  $\nabla = (\nabla^M, \nabla^\sigma)$  on  $A$ . Note that

$$T_\nabla((X, v), (Y, w)) = (T_{\nabla^M}(X, Y), \Omega_\sigma(X, Y) - [v, w])$$

for all  $X, Y \in TM$ ,  $v, w \in \mathfrak{g}$ . This shows that two  $A$ -paths  $a$  and  $b$  as in Proposition 1.3 will be of the form  $a = (\frac{d\gamma}{dt}, \phi)$ ,  $b = (\frac{d\gamma}{d\epsilon}, \psi)$  where  $\phi, \psi$  are paths in  $\mathfrak{g}$  satisfying

$$\partial_t \psi - \partial_\epsilon \phi = \Omega_\sigma\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}\right) - [\phi, \psi].$$

Now we only have to apply the definition of  $\partial$ : Given  $[\gamma] \in \pi_2(M, x)$ , we choose the lift  $adt + bde$  of  $d\gamma$  with  $\psi = 0$  and

$$\phi = - \int_0^\epsilon \Omega_\sigma \left( \frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon} \right).$$

Then  $\phi$  takes values in  $Z(\mathfrak{g}_x)$ , and we obtain  $\partial[\gamma] = [\int_\gamma \Omega_\sigma]$ . □

*Example 3.7.* Recall (e.g. [17]) that any closed two-form  $\omega \in \Omega^2(M)$  induces an algebroid  $A_\omega = TM \oplus \mathbb{L}$ , where  $\mathbb{L}$  is the trivial line bundle, with anchor  $(X, \lambda) \mapsto X$  and Lie bracket

$$[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f) + \omega(X, Y)).$$

Using the obvious splitting of  $A$ , Lemma 3.3 tells us that

$$N_x(A_\omega) = \left\{ \int_\gamma \omega : [\gamma] \in \pi_2(M, x) \right\} \subset \mathbb{R}$$

is the group of periods of  $\omega$ . Other examples will be discussed in the next sections.

*3.4. Measuring the monodromy.* In order to measure the size of the monodromy groups  $N_x(A)$ , we fix some norm on the Lie algebroid  $A$  and for  $x \in M$  we set

$$r(x) \equiv d(0, N_x(A) - \{0\}),$$

where we adopt the convention that  $d(0, \emptyset) = +\infty$ .

When  $x$  varies on a leaf  $L$  this function varies continuously, since the norm on  $A$  is assumed to vary continuously and the groups  $N_x(A)$  are all isomorphic for  $x \in L$ . On the other hand, when  $x$  varies in a transverse direction the behavior of  $r(x)$  is far from being continuous as illustrated by the following examples:

*Example 3.8.* We take for  $A$  the trivial 3-dimensional vector bundle over  $M = \mathbb{R}^3$ , with basis  $\{e_1, e_2, e_3\}$ . The Lie bracket on  $A$  is defined by

$$\begin{aligned} [e_2, e_3] &= ae_1 + bx^1\bar{n}, \\ [e_3, e_1] &= ae_2 + bx^2\bar{n}, \\ [e_1, e_2] &= ae_3 + bx^3\bar{n} \end{aligned}$$

where  $\bar{n} = \sum_i x^i e_i$  is a central element, and depends on two (arbitrary) smooth functions  $a$  and  $b$  of the radius  $R$ , with  $a(R) > 0$  whenever  $R > 0$ . The anchor is given by

$$\#(e_i) = av_i, \quad i = 1, 2, 3$$

where  $v^i$  is the infinitesimal generator of a rotation about the  $i$ -axis:

$$v_1 = x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3}, \quad v_2 = x^1 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^1}, \quad v_3 = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}.$$

The leaves of the foliation induced on  $\mathbb{R}^3$  are the spheres  $S_R^2$  centered at the origin, and the origin is the only singular point.

We now compute the function  $r$  using the obvious metric on  $A$ . We restrict to a leaf  $S_R^2$  with  $R > 0$ , and as splitting of  $\#$  we choose the map defined by

$$\sigma(v_i) = \frac{1}{a} \left( e_i - \frac{x^i}{R^2} \bar{n} \right).$$

Then we obtain the center-valued 2-form (cf. §3.3)

$$\Omega_\sigma = \frac{bR^2 - a}{a^2 R^4} \omega \bar{n}$$

where  $\omega = x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2$ . Since  $\int_{S_R^2} \omega = 4\pi R^3$  it follows that

$$N(A) \simeq 4\pi \frac{bR^2 - a}{a^2 R} \mathbb{Z} \bar{n} \subset \mathbb{R} \bar{n}.$$

This shows that

$$r(x, y, z) = \begin{cases} +\infty & \text{if } R = 0 \text{ or } a = bR^2, \\ 4\pi \frac{bR^2 - a}{a^2} & \text{otherwise.} \end{cases}$$

So the monodromy might vary in a nontrivial fashion, even nearby regular leaves.

In the previous example the function  $r$  is *not* upper semi-continuous. In the next example we show that  $r$ , in general, is *not* lower semi-continuous. This example also shows that, even if the anchor is injective in a “large set”, one has no control on the way the monodromy groups vary.

*Example 3.9.* We consider a variation of Example 3.8, so we use the same notation. We let  $M = \mathbb{S}^2 \times \mathbb{H}$ , where  $\mathbb{H}$  denotes the quaternions. The Lie algebroid  $\pi : A \rightarrow M$  is trivial as a vector bundle, has rank 3, and relative to a basis of sections  $\{e_1, e_2, e_3\}$  the Lie bracket is defined by  $[e_1, e_2] = e_3$  and cyclic permutations. To define the anchor, we let  $v_1, v_2, v_3$  be the vector fields on  $\mathbb{S}^2$  obtain by restricting the infinitesimal generators of rotations, and we let  $w_1, w_2, w_3$  be the vector fields on  $\mathbb{H}$  corresponding to multiplication by  $\vec{i}, \vec{j}, \vec{k}$ . The anchor of the algebroid is then defined by setting  $\#e_i \equiv (v_i, w_i)$ ,  $i = 1, 2, 3$ . For this Lie algebroid one has:

- the anchor is injective on a dense open set;
- there is exactly one singular leaf, namely the sphere  $\mathbb{S}^2 \times \{0\}$ .

Now observe that the monodromy above the singular leaf is nontrivial, since the restriction of  $A$  to this singular leaf is the central extension algebroid  $T\mathbb{S}^2 \oplus \mathbb{L}$

defined by the area form on  $\mathbb{S}^2$ . For the function  $r$  we have again:

$$r(x) = \begin{cases} r_0 & \text{if } x \in \mathbb{S}^2 \times \{0\} \\ +\infty & \text{otherwise.} \end{cases}$$

Note that in this case  $r_0 > 0$ . This is no accident, as we will see later in Section 5.2.5.

Finally, we give an example where a splitting as in Lemma 3.6 does not exist, and which illustrates how the groups  $N_x$  and  $\tilde{N}_x$  can differ.

*Example 3.10.* Let  $\mathfrak{g}$  be any Lie algebra and  $G$  a simply connected Lie group with Lie algebra  $\mathfrak{g}$ . If we consider the Poisson manifold  $M = \mathfrak{g}^*$  with the Kirillov-Kostant Poisson bracket, and let  $A = T^*\mathfrak{g}^*$  (for the cotangent Lie algebroid of a Poisson manifold; see [4]). We always obtain an integrable Lie algebroid: a source-simply connected Lie groupoid integrating  $A$  is  $\mathcal{G} = T^*G \simeq G \times \mathfrak{g}^*$  with source and target maps

$$\mathbf{s}(g, \xi) = \xi, \quad \mathbf{t}(g, \xi) = \text{Ad}^*g \cdot \xi,$$

and with multiplication  $(g_1, \xi_1) \cdot (g_2, \xi_2) = (g_1g_2, \xi_2)$ , wherever defined.

For a specific example take  $M = \mathfrak{su}^*(3)$ . The symplectic leaves (i.e., the co-adjoint orbits) are isospectral sets, and so we can understand them by looking at their point of intersection with the diagonal matrices with imaginary eigenvalues. There are orbits of dimension 6 (distinct eigenvalues), dimension 4 (two equal eigenvalues) and the origin (all eigenvalues equal). Let us take for example the (singular) orbit  $L$  through

$$x = \begin{pmatrix} i\lambda & 0 & 0 \\ 0 & i\lambda & 0 \\ 0 & 0 & -2i\lambda \end{pmatrix}.$$

Then we find its isotropy subalgebra to be

$$\mathfrak{g}_x = \left\{ \left( \begin{array}{c|c} X & 0 \\ \hline 0 & -\text{tr } X \end{array} \right) : X \in \mathfrak{u}(2) \right\}$$

and so we see that the simply connected Lie group integrating the Lie algebra  $\mathfrak{g}_x$  is  $\mathcal{G}(\mathfrak{g}_x) = \mathbb{R} \times \text{SU}(2)$ . On the other hand, the isotropy group  $\mathcal{G}_x$  is given by

$$\begin{aligned} \mathcal{G}_x &= \{g \in \text{SU}(3) : \text{Ad}^*g \cdot x = x\} \\ &= \left\{ \left( \begin{array}{c|c} g & 0 \\ \hline 0 & \det g^{-1} \end{array} \right) : g \in \text{U}(2) \right\}. \end{aligned}$$

We conclude that the orbit  $L$  is diffeomorphic to  $SU(3)/U(2) = CP(2)$ . In fact, one can show that it is symplectomorphic to  $CP(2)$  with its standard symplectic structure (see [12, Example 3.4.5]). Also, we see that the long exact sequence

$$\dots \rightarrow \pi_2(CP(2), x) \xrightarrow{\partial} \mathcal{G}(\mathfrak{g}_x) \rightarrow \mathcal{G}(T^*M)_x \rightarrow \pi_1(CP(2), x),$$

reduces to:

$$\dots \rightarrow \mathbb{Z} \xrightarrow{\partial} \mathbb{R} \times SU(2) \xrightarrow{\rho} U(2) \rightarrow \{1\},$$

where  $\rho(\theta, A) = e^{i\theta} A$ . We conclude that  $\partial n = (\pi n, (-1)^n I)$ , so that  $\partial$  takes values in the center  $Z(\mathbb{R} \times SU(2)) = \mathbb{R} \times \{\pm I\}$ , and

$$\tilde{N}_x = \text{Im } \partial = 2\mathbb{Z} \times \{\pm I\}, \quad N_x = 2\mathbb{Z}.$$

Since these two groups are distinct, there can be no splitting as in Lemma 3.6. Another argument is that such a splitting would define a flat connection on the conormal bundle  $\nu^*(L) = \text{Ker } \#|_L$ , and since  $L = CP(2)$  is 1-connected, it would follow that the conormal bundle would be a trivial bundle. This is not possible. In fact, the total Stiefel-Whitney class of  $CP(2)$  is nontrivial, and hence so is the Stiefel-Whitney class of the normal bundle, when we embed  $CP(2)$  in any euclidean space. So the conormal bundle cannot be trivial.

#### 4. Obstructions to integrability

In this section we first state our main result which gives the obstructions to integrability, and give a few examples. We then give another description of the Weinstein groupoid which is more suitable for proving the theorem.

4.1. *The main theorem.* Let  $A$  be a Lie algebroid over  $M$ . Using the notations introduced above, our main result is the following:

**THEOREM 4.1.** *A Lie algebroid  $A$  over  $M$  is integrable if and only if:*

- (i) *Longitudinal obstruction:*  $N_x(A) \subset A_x$  is discrete (i.e.,  $r(x) \neq 0$ ),
- (ii) *Transverse obstruction:*  $\liminf_{y \rightarrow x} r(y) > 0$ ,

for all  $x \in M$ .

The next examples illustrate this result and show that these two obstructions are independent.

*Example 4.2.* In this example, nonintegrability is forced by the first obstruction. We simply take the central extension Lie algebroid  $A_\omega = TM \oplus \mathbb{L}$  associated with a closed 2-form on  $M$  with a noncyclic group of periods (cf. Example 3.7). Then  $r(x) = 0$  so the first obstruction ensures us that  $A_\omega$  is

nonintegrable. We point out that this is a well-known counter-example to integrability (cf. e.g. [3, p. 118]) which is usually approached through the theory of transversally parallelizable foliations (see also §5.3 below).

*Example 4.3.* Let us give an example of a regular Lie algebroid with trivial first obstruction, while the second one is not. Take  $\mathcal{F}$  to be the trivial foliation of  $M = N \times T$  with leaves  $N \times \{t\}$ ,  $t \in T$ . Also we choose a closed 2-form  $\omega$  on  $N$  with cyclic group of periods and we set  $\omega_t = \phi(t)\omega$ , where  $\phi$  is some smooth function on  $T$ . Since the pull-back of  $\omega_t$  to any leave is closed, we obtain the central extension Lie algebroid  $A_{\omega_t} = \mathcal{F} \oplus \mathbb{L}$ , as in Example 3.7, the leaves of which are the leaves of  $\mathcal{F}$ .

The “first obstruction” is satisfied for all leaves, but clearly the “second obstruction” is not satisfied at the points  $t_0 \in T$  with the property that  $\phi(t_0) = 0$  and  $\phi$  is not locally constant at  $t_0$ .

*Example 4.4.* Consider the Lie algebroid  $A$  over  $\mathbb{R}^3$  discussed in Example 3.8. Then  $A$  satisfies the first obstruction, but it does not satisfy the second obstruction at points where  $aR^2 - b$  vanishes (without vanishing identically in some neighborhood of the point) and also at the origin if  $\liminf_{R \rightarrow 0} \frac{bR^2 - a}{a^2} = 0$ .

For example, if we choose  $a = R^2$ ,  $b = R^3 + 1$ , the resulting Lie algebroid  $A$  over  $\mathbb{R}^3$  has the following two properties:

- (a) Its restriction to  $\mathbb{R}^3 - 0$  is integrable;
- (b) Its restriction to any disk around the origin is not integrable (because of the second obstruction at  $x = 0$ ).

*Example 4.5.* Let us explain Weinstein’s example of a nonintegrable regular Poisson manifold given in [27] (see also [4, §6]). He takes  $M = \mathbb{R}^3 - \{0\} \cong \mathfrak{su}(2)^* - \{0\}$  with the Kirillov-Poisson structure scaled by a function  $f(R)$  depending on the radius. The associated cotangent Lie algebroid is in fact  $T^*M = A|_{\mathbb{R}^3 - \{0\}}$ , where  $A$  is the Lie algebroid of Examples 3.8 and 4.4 with  $a = f$ ,  $b = \frac{1}{R}f'$ . Its integrability is then controlled by

$$r(R) = 4\pi \frac{Rf' - f}{f^2} = -A'(R),$$

where  $A(R) = \frac{4\pi R}{f}$  is the symplectic area.

We refer to Section 5 for various integrability criteria that can be deduced from the theorem, including all criteria that have appeared before in the literature.

4.2. *The Weinstein groupoid as a leaf space.* Before we can proceed with the proof of our main result, we need a better control on the equivalence relation defining the Weinstein groupoid  $\mathcal{G}(A)$ . In this section we will show that  $\mathcal{G}(A)$  is the leaf space of a foliation  $\mathcal{F}(A)$  on  $P(A)$ , of finite codimension, the leaves of which are precisely the equivalence classes of the homotopy relation  $\sim$  of  $A$ -paths.

As before,  $A$  is a fixed Lie algebroid over  $M$ . We will use the following notations when working in local coordinates: we let  $x = (x^1, \dots, x^n)$  denote local coordinates on  $M$ , and we denote by  $\{e_1, \dots, e_k\}$  a (local) basis of  $A$  over this chart. The anchor and the bracket of  $A$  decompose as

$$\#e_p = \sum_i b_p^i \frac{\partial}{\partial x^i}, \quad [e_p, e_q] = \sum_r c_{pq}^r e_r,$$

and an  $A$ -path  $a$  can be written as  $a(t) = \sum_p a^p(t)e_p$ .

Let us first describe the smooth structure on  $P(A)$ . We consider the larger space  $\tilde{P}(A)$  of all  $C^1$ -curves  $a : I \rightarrow A$  with base path  $\gamma = \pi \circ a$  of class  $C^2$ . It has an obvious structure of Banach manifold, and its tangent space  $T_a(\tilde{P}(A))$  consists of curves  $U : I \rightarrow TA$  such that  $U(t) \in T_{a(t)}A$ . Using a  $TM$ -connection  $\nabla$  on  $A$ , such curves can be viewed as pairs  $(u, \phi)$  formed by a curve  $u : I \rightarrow A$  over  $\gamma$  and a curve  $\phi : I \rightarrow TM$  over  $\gamma$  (namely, the vertical and horizontal component of  $U$ ).

LEMMA 4.6.  *$P(A)$  is a (Banach) submanifold of  $\tilde{P}(A)$ . Moreover, given a connection  $\nabla$  on  $A$ , the tangent space  $T_a P(A)$  consists of the paths  $U = (u, \phi)$  such that*

$$\#u = \bar{\nabla}_a \phi.$$

*Proof.* We consider the smooth map  $F : \tilde{P}(A) \rightarrow \tilde{P}(TM)$  given by

$$F(a) = \#a - \frac{d}{dt} \pi \circ a.$$

Clearly  $P(A) = F^{-1}(Q)$ , where  $Q$  is the subspace of  $\tilde{P}(TM)$  consisting of zero paths. Fix  $a \in P(A)$ , with base path  $\gamma = \pi \circ a$ , and let us compute the image of  $U = (u, \phi) \in T_a \tilde{P}(A)$  under the differential

$$(dF)_a : T_a \tilde{P}(A) \rightarrow T_{0_\gamma} \tilde{P}(TM).$$

The result will be some path  $t \mapsto (dF)_a \cdot U(t) \in T_{0_{\gamma(t)}} TM$ ; hence, using the canonical splitting  $T_{0_x} TM \cong T_x M \oplus T_x M$ , it will have a horizontal and vertical component. We claim that for any connection  $\nabla$ , if  $(u, \phi)$  are the components of  $U$ , then

$$((dF)_a \cdot U)^{\text{hor}} = \phi, \quad ((dF)_a \cdot U)^{\text{ver}} = \#u - \bar{\nabla}_a \phi.$$

Note that this immediately implies that  $F$  is transverse to  $Q$ , so the assertion of the proposition follows. Since this decomposition is independent of the connection  $\nabla$  and it is local (we can look at restrictions of  $a$  to smaller intervals), we may assume that we are in local coordinates, and that  $\nabla$  is the standard flat connection. We now use the notation above, and we denote by  $\frac{\partial}{\partial x^i}$  the horizontal basis of  $T_{0_x}TM$ , and by  $\frac{\delta}{\delta x^i}$  the vertical basis. A simple computation shows that the horizontal component of  $(dF)_a(u, \phi)$  is  $\sum \phi^i \frac{\partial}{\partial x^i}$ , while its vertical component is

$$\sum_j \left( -\dot{\phi}^j(t) + \sum_p u^p(t) b_p^j(\gamma(t)) + \sum_{p,i} a^p(t) \phi^i(t) \frac{\partial b_p^j}{\partial x^i}(\gamma(t)) \right) \frac{\delta}{\delta x^j}.$$

That this is precisely  $\#u - \bar{\nabla}_a \phi$  immediately follows by computing

$$\begin{aligned} \bar{\nabla}_{e_p} \frac{\partial}{\partial x^i} &= \# \nabla_{\frac{\partial}{\partial x^i}} e_p - \left[ \frac{\partial}{\partial x^i}, \# e_p \right] \\ &= - \sum_j \frac{\partial b_p^j}{\partial x^i} \frac{\partial}{\partial x^j}. \end{aligned} \quad \square$$

We now construct an involutive sub-bundle  $\mathcal{F}(A)$  of  $TP(A)$ , i.e. a foliation on  $P(A)$ . Let us fix a connection  $\nabla$  on  $A$ , and let  $a$  be an  $A$ -path with base path  $\gamma$ . We denote by  $\mathcal{P}_{0,\gamma}(A)$  the space of all  $C^2$ -paths  $b : I \rightarrow A$  such that  $b(t) \in A_{\gamma(t)}$  and  $b(0) = b(1) = 0$ . For any such  $b$ , we have a tangent vector  $X_{b,a} \in T_a P(A)$  with components  $(u, \phi)$  relative to the connection  $\nabla$  given by

$$u = \bar{\nabla}_a b, \quad \phi = \#b.$$

Lemma 4.6 shows that these are indeed tangent to  $P(A)$ , and we set:

$$\mathcal{F}_a(A) \equiv \{X_{b,a} \in T_a P(A) : b \in \mathcal{P}_{0,\gamma}(A)\}.$$

Some geometric insight to this sub-bundle can be obtain by considering the Lie algebra of time-depending sections of  $A$  vanishing at the end-points:

$$P_0\Gamma(A) = \left\{ I \ni t \mapsto \eta_t \in \Gamma(A) : \eta_0 = \eta_1 = 0, \eta \text{ is of class } C^2 \text{ in } t \right\}.$$

For any such section  $\eta$  we consider the induced path  $b(t) = \eta(t, \gamma(t))$  and put  $X_{\eta,a} \equiv X_{b,a}$ . The resulting map

$$P_0\Gamma(A) \rightarrow \mathcal{X}(P(A)), \quad \eta \mapsto X_\eta$$

is an action of the Lie algebra  $P_0\Gamma(A)$  on  $P(A)$ .

We now show that the foliation  $\mathcal{F}(A)$  is in fact the same as the partition of  $P(A)$  into equivalent classes of  $A$ -paths:

PROPOSITION 4.7. *For a Lie algebroid  $A$ :*

- (i) *The spaces  $\mathcal{F}_a(A)$  do not depend on the choice of the connection  $\nabla$ . More precisely, for any  $\eta \in P_0\Gamma(A)$ ,*

$$X_{\eta,a}(t) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \phi_{\eta_t}^{\epsilon,0} a(t) + \frac{d\eta_t}{dt}(\gamma(t)).$$

- (ii)  *$\mathcal{F}(A)$  is a foliation on  $P(A)$  of finite codimension equal to  $n + k$  where  $n = \dim M$  and  $k = \text{rank } A$ .*
- (iii) *Two  $A$ -paths are equivalent (homotopic) if and only if they are in the same leaf of  $\mathcal{F}(A)$ .*
- (iv) *For any (local) connection  $\nabla$  on  $A$ , the exponential map  $\text{Exp}_\nabla : A \rightarrow P(A)$  is transverse to  $\mathcal{F}(A)$ .*

*Proof.* We first assume that  $\eta_t$  is an element of  $P_0\Gamma(A)$  and we will see that it induces a vector field  $X_\eta$  on  $P(A)$  tangent to  $\mathcal{F}(A)$  the flow of which preserves the equivalence of paths. This is only a reformulation of Proposition 1.3. Hence, let  $a_0 \in P(A)$  with base path  $\gamma_0$ , and let  $\xi_0$  be a time-dependent section of  $A$  such that  $\xi_0(t, \gamma_0(t)) = a_0(t)$ . We denote by  $\xi$  the solution of (3) with the initial condition  $\xi(0, t) = \xi_0(t)$ . Then, as in the proof of Proposition 1.3,

$$\xi(\epsilon, t) = \int_0^\epsilon (\phi_{\eta_t}^{\epsilon', \epsilon})^* \frac{d\eta}{dt} d\epsilon' + (\phi_{\eta_t}^{0, \epsilon})^* \xi_0.$$

Now consider the base path

$$\gamma_\epsilon(t) = \Phi_{\#\eta_t}^{\epsilon,0} \gamma_0(t)$$

and the paths above it

$$a_\epsilon(t) = \xi_\epsilon(t, \gamma_\epsilon(t)), \quad b_\epsilon(t) = \eta(t, \gamma_\epsilon(t)).$$

We can view  $\epsilon \mapsto a_\epsilon$  as a curve in  $P(A)$  starting at  $a_0$ , and defining a tangent vector

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} a_\epsilon \in T_{a_0}P(A).$$

Given some connection  $\nabla$ , Proposition 1.3 shows that this tangent vector has vertical component

$$\partial_\epsilon a = \partial_t b - T(a, b) = \overline{\nabla}_a(b)$$

at  $\epsilon = 0$ , while the horizontal component is

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \gamma_\epsilon(t) = \#b_0(t).$$

In other words,

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} a_\epsilon = X_{b_0, a_0} \in \mathcal{F}_{a_0}(A).$$

On the other hand, the formula above describing  $\xi$  shows that

$$a_\epsilon(t) = \int_0^\epsilon \phi_{\eta t}^{\epsilon, \epsilon'} \frac{d\eta t}{dt} (\phi_{\# \eta t}^{\epsilon', 0}(\gamma_0(t))) d\epsilon' + \phi_{\eta t}^{\epsilon, 0}(\gamma_0(t)).$$

The derivative at  $\epsilon = 0$  is precisely the expression given in (i) and this also shows that (iii) holds.

To determine the codimension of  $\mathcal{F}_a(A)$  we note that given  $(u, \phi)$  satisfying  $\#u = \bar{\nabla}_a \phi$  (i.e., a vector tangent to  $P(A)$ ) and lying in  $\mathcal{F}_a(A)$ , we have

(a)  $\phi(0) = 0$ ;

(b) If we consider the solution  $b$  of the equation  $\bar{\nabla}_a(b) = u$  with initial condition  $b(0) = 0$  (which can be expressed in terms of the parallel transport along  $a$  with respect to  $\bar{\nabla}$ ), we must have  $b(1) = 0$ .

Conversely, if (a) and (b) hold, we have that  $\bar{\nabla}_a(\#b - \phi) = 0$  and  $\#b - \phi$  vanishes at  $t = 0$ . It follows that  $\phi = \#b$  and  $u = \bar{\nabla}_a b$ , so  $(u, \phi)$  is a tangent vector in  $\mathcal{F}_a(A)$ . This shows that  $\text{codim } \mathcal{F} = \dim M + \text{rank } A$ .

Finally, to prove (iv), we assume for simplicity that we are in local coordinates and that  $\nabla$  is the trivial flat connection (this is actually all we will use for the proof of the main theorem, and this in turn will imply the full statement of (iv)). Also, all we need to show is that  $\text{Exp}_\nabla(A)$  is transverse to  $\mathcal{F}(A)$  at any trivial  $A$ -path  $a = O_x$  over  $x \in M$ . Now, the equations for the geodesics show that if  $(u, \phi)$  is a tangent vector to  $\text{Exp}_\nabla(A)$  at  $a$  then we must have:

$$\dot{\phi}^i = b_p^i(x)u^p, \quad \dot{u}^p = 0.$$

Therefore, we see that:

$$T_a \text{Exp}_\nabla(A) = \{(u, \phi) \in T_a P(A) : u(t) = u_0, \phi(t) = \phi_0 + t\#u_0\}.$$

Suppose that a tangent vector  $(u, \phi)$  belongs to this  $n + k$ -dimensional space and is also tangent to  $\mathcal{F}(A)$ . Then (a) above implies that  $\phi_0 = 0$ , while (b) says that the solution  $b$  of  $\frac{db}{dt} = u_0$  with  $b(0) = 0$  satisfies  $b(1) = 0$ . Therefore, we must have  $\phi_0 = 0$  and  $u_0 = 0$ , so  $(u, \phi)$  is the null tangent vector. This shows that  $\text{Exp}_\nabla(A)$  is transverse to  $\mathcal{F}(A)$  at  $O_x$ , for any  $x$ . □

*Remark 4.8.* In [4], Cattaneo and Felder obtain the Weinstein groupoid for the special case of Poisson manifolds by a Hamiltonian reduction procedure. The Lie algebraic interpretation given above for the foliation  $\mathcal{F}(A)$  shows that our construction of  $\mathcal{G}(A)$  for general  $A$  is also obtained by a kind of reduction procedure for Lie algebroid actions.

4.3. *Proof of the main theorem.* In this section we prove Theorem 4.1 (for notations, see §3).

To prove that both conditions are necessary, choose some connection  $\nabla$  on  $A$ , and let  $\overline{\text{Exp}}_\nabla : A \rightarrow \mathcal{G}(A)$  be the associated exponential map. Clearly the restriction of  $\overline{\text{Exp}}_\nabla$  to  $\mathfrak{g}_x$  is the composition of the exponential map of  $\mathfrak{g}_x$  with the obvious map  $i : \mathcal{G}(\mathfrak{g}_x) \rightarrow \mathcal{G}(A)_x$ , which shows that  $\overline{\text{Exp}}_\nabla(v_x) = 1_x$  for all  $v_x \in N_x(A)$  in the domain of the exponential map. On the other hand, if  $A$  is integrable, we know that  $\overline{\text{Exp}}_\nabla$  will be a diffeomorphism on a small neighborhood of  $M$  on  $A$ . Hence there must exist an open  $\mathcal{U} \subset A$  such that  $\mathcal{U} \cap N(A) = M$ , where  $N(A) = \cup_x N_x(A)$ . But this is obviously equivalent to the conditions in the statement.

We now show that these conditions also guarantee the integrability of  $A$ . First we prove that the two conditions together imply that  $\mathcal{F}(A)$  is a simple foliation:

LEMMA 4.9. *For each  $a \in P(A)$ , there exists  $S_a \subset P(A)$  transverse to  $\mathcal{F}(A)$ , which intersects each leaf of  $\mathcal{F}(A)$  in at most one point.*

*Proof.* The proof is a sequence of reductions and careful choices, and is divided into several steps. So let  $a \in P(A)$  and denote by  $x$  the initial point of its base path.

*Claim 1.* We may assume that  $a = 0_x$ .

To see this, we choose a compactly supported, time-dependent, section  $\xi$  of  $A$  so that  $\xi(t, \gamma(t)) = a(t)$ . If  $\sigma_\xi$  is as in the definition of admissible section (see §2.3), then left multiplication  $T : P(A) \rightarrow P(A)$ ,  $T(b) = a_\xi(\mathbf{t}(b))b$ , defines a smooth injective map. If there is a section  $S_x$  around  $0_x$ , as in the statement of the lemma, it then follows that  $T : S_x \rightarrow P(A)$  intersects each leaf in at most one point. Since  $S_x$  has the same dimension as the codimension of  $\mathcal{F}(A)$ ,  $S := T(S_x)$  will be a section through  $T(0_x)$  with the desired properties. Since  $T(0_x)$  and  $a$  are in the same leaf, this implies the existence of a similar section  $S_a$  through  $a$  (use the holonomy of the foliation  $\mathcal{F}(A)$  along any path from  $T(0_x)$  to  $a$ ).

From now on we fix  $x \in M$  and we are going to prove the lemma for  $a = 0_x$ . We also fix local coordinates around  $x$ , and let  $\nabla$  be the canonical flat connection on the coordinate neighborhood. We also choose a small neighborhood  $U$  of  $0_x$  in  $A$  so that the exponential map  $\text{Exp}_\nabla : U \rightarrow P(A)$  is defined and is transverse to  $\mathcal{F}(A)$ . We are going to show that it intersects each leaf of  $\mathcal{F}(A)$  in at most one point, provided  $U$  is chosen small enough.

*Claim 2.* We may choose  $U$  such that for any  $v \in U \cap \mathfrak{g}_y$  ( $y \in M$ ) with the property that  $\text{Exp}_\nabla(v)$  is homotopic to  $0_y$ , we must have  $v \in Z(\mathfrak{g}_y)$ .

Given a norm  $|\cdot|$  on  $A$ , the set  $\{|[v, w]| : v, w \in \mathfrak{g}_y \text{ with } |v| = |w| = 1\}$ , where  $y \in M$  varies in a neighborhood of  $x$ , is bounded. Rescaling  $|\cdot|$  if necessary, we find a neighborhood  $D$  of  $x$  in  $M$ , and a norm  $|\cdot|$  on  $A_D = \{v : \pi(v) \in D\}$ , such that  $|[v, w]| \leq |v||w|$  for all  $v, w \in \mathfrak{g}_y$  with  $y \in D$ . We now choose  $U$  so that  $U \subset A_D$  and  $|v| \leq \pi$  for all  $v \in U$ . If  $v$  is as in the claim, it follows from Proposition 1.6 (see also the proof of Lemma 3.3) that parallel transport  $T_v : \mathfrak{g}_y \rightarrow \mathfrak{g}_y$  along the constant  $A$ -path  $v$  is the identity. But  $T_v$  is precisely the exponential of the linear map  $\text{ad}_v : \mathfrak{g}_y \rightarrow \mathfrak{g}_y$ . Since  $|\text{ad}_v| \leq |v| \leq \pi$ , it follows (see e.g. [10]) that  $\text{ad}_v = 0$  so  $v \in Z(\mathfrak{g}_y)$ .

*Claim 3.* We may choose  $U$  such that, if  $v \in U \cap \mathfrak{g}_y$  ( $y \in M$ ) has the property that  $\text{Exp}_\nabla(v)$  is homotopic to  $0_y$  then  $v = 0_y$ .

Obviously this is just a restatement of the obstruction assumptions, combined with the previous claim.

*Claim 4.* We may choose  $U$  such that, if  $v \in U$  has the property that the base path of  $\text{Exp}_\nabla(v)$  is closed, then  $v \in \mathfrak{g}_y$ .

To see this, we note that the equations for the geodesics in local coordinates reduce to:

$$\begin{cases} \dot{x}^i = \sum_p b_p^i(x(t))a^p, \\ \dot{a}^p = 0. \end{cases}$$

By the period bounding lemma ([28]), any nontrivial periodic orbit of this system with initial condition on a open set  $D$  has period

$$T \geq \frac{2\pi}{M_D} \quad \text{where} \quad M_D = \sup_{\substack{1 \leq j, k \leq m \\ x \in D}} \left\| \sum_p \frac{\partial b_p^j}{\partial x^k}(x) a^p \right\|.$$

Hence it suffices to make sure that  $U \subset A_D$  where  $D$  is chosen small enough so that  $M_D < 2\pi$ .

Now, for any open  $O \subset P(A)$ , we consider the plaques in  $O$  of  $\mathcal{F}(A)$ , or, equivalently, the leaves of  $\mathcal{F}(A)|_O$ . For  $a, b \in O$ , we write  $a \sim_O b$  if  $a$  and  $b$  lie in the same plaque. From now on we fix  $U$  satisfying all the conditions above, and we choose an open  $O$  so that  $\text{Exp}_\nabla : U \rightarrow P(A)$  intersects each plaque inside  $O$  exactly in one point. This is possible since  $\text{Exp}_\nabla$  is transversal to  $\mathcal{F}(A)$ . Apart from the pair  $(O, U)$ , we also choose similar pairs  $(O_i, U_i)$ ,  $i = 1, 2$ , such that  $O_1 O_1 \subset O$ ,  $O_2 O_2 \subset O_1$  and  $O_i^{-1} = O_i$ .

*Claim 5.* It is possible to choose a neighborhood  $V$  of  $x$  in  $U_2$  so that, for all  $v \in V$ ,

$$0_y \cdot \text{Exp}_\nabla(v) \sim_O \text{Exp}_\nabla(v).$$

We know that for any  $v$  there is a natural homotopy between the two elements above. This homotopy can be viewed as a smooth map  $h : I \times U \rightarrow P(A)$  with  $h(0, v) = 0 \cdot \text{Exp}_{\nabla}(v)$ ,  $h(1, v) = \text{Exp}_{\nabla}(v)$ ,  $h(t, 0_y) = 0_y$ . Since  $I$  is compact and  $O$  is open, we can find  $V$  around  $x$  such that  $h(I \times V) \subset O$ . Obviously  $V$  has the desired property.

*Claim 6.* It is possible to choose  $V$  so that, for all  $v, w \in V$ ,

$$(\text{Exp}_{\nabla}(v) \cdot \overline{\text{Exp}_{\nabla}(w)}) \cdot \text{Exp}_{\nabla}(w) \sim_O \text{Exp}_{\nabla}(v).$$

This is proved exactly as the previous claim.

*Claim 7.*  $\text{Exp}_{\nabla} : V \rightarrow P(A)$  intersects each leaf of  $\mathcal{F}(A)$  in at most one point.

To see this, let us assume that  $v, w \in V$  have  $\text{Exp}_{\nabla}(v) \sim \text{Exp}_{\nabla}(w)$ . Then  $a_1 := \text{Exp}_{\nabla}(v) \cdot \overline{\text{Exp}_{\nabla}(w)} \in O_1$  will be homotopic to the trivial  $A$ -path  $0_y$ . On the other hand, by the choice of the pair  $(O_1, U_1)$ ,  $a_1 \sim_{O_1} \text{Exp}_{\nabla}(u)$  for a unique  $u \in U_1$ . Since  $\text{Exp}_{\nabla}(u)$  is equivalent to  $0_y$ , its base path must be closed, hence, by Claim 4 above,  $u \in \mathfrak{g}_y$ . Using Claim 3, it follows that  $u = 0$ , hence  $a_1 \sim_{O_1} 0_y$ . Since  $O_1 O_1 \subset O$ , this obviously implies that

$$a_1 \cdot \text{Exp}_{\nabla}(w) \sim_O 0_y \cdot \text{Exp}_{\nabla}(w).$$

Since  $V$  satisfies Claims 5 and 6, we get  $\text{Exp}_{\nabla}(v) \sim_O \text{Exp}_{\nabla}(w)$ . Hence, by the construction of  $O$ ,  $v = w$ . This also concludes the proof of the lemma.  $\square$

Note that the previous lemma implies that  $\mathcal{G}(A)$  has a natural quotient differentiable structure: the charts are just the  $S_a$ 's, and the change of coordinates is smooth since it is just the holonomy of  $\mathcal{F}(A)$ . Hence we can complete the proof of Theorem 4.1 by showing that:

**LEMMA 4.10.** *For the quotient differentiable structure,  $\mathcal{G}(A)$  is a Lie groupoid with Lie algebroid  $A$ .*

*Proof.* It is clear from the definitions that  $A$  can be identified with  $T_M^s \mathcal{G}(A)$  and that under this identification  $\#$  coincides with the differential of the target  $\mathbf{t}$ . So we need only to check that the bracket of right-invariant vector fields on  $\mathcal{G}(A)$  is identified with the bracket of sections of  $A$ . For this we note that on one hand, the bracket is completely determined by the infinitesimal flow of sections through the basic formula (A.2). On the other hand, we now know that the exponential  $\exp : \Gamma(A) \rightarrow \Gamma(\mathcal{G}(A))$  is injective in a neighborhood of the zero section, and so Proposition 2.3 shows that the infinitesimal flow of a section  $\alpha$  is the infinitesimal flow of the right-invariant vector field on  $\mathcal{G}(A)$  determined by  $\alpha$ . Hence, we must have  $A(\mathcal{G}(A)) = A$ .  $\square$

*Remark 4.12.* The proof above (namely an argument similar to Claim 1 above) shows that, in the main theorem, it suffices to require that for each leaf  $L$ , there exists  $x \in L$  satisfying the two obstructions.

## 5. Examples and applications

In this section we review the known integrability criteria, we derive them from Theorem 4.1, and present an application to the theory of transversally parallelizable foliations.

5.1. *Local integrability.* Regarding the local nature of integrability, with respect to the base manifold  $M$ , note that

- From Examples 4.3 and 4.5 we learn that a Lie algebroid can be locally integrable (i.e. each point has a neighborhood  $U$  so that  $A|_U$  is integrable), and not globally integrable. This shows that the integrability problem is not a local one.
- Example 4.4 shows that there are algebroids which are not even locally integrable.

However, a general “local integrability” result has long been assumed to be true, namely the integrability by *local groupoids*. This result was first announced by Pradines, but a proof has never been published. One of the main difficulties is that, if one tries to extend the known result from Lie groups (see e.g. [11]), one faces the problem of finding a CBH-formula. However, with the Weinstein groupoid at hand (and its description as a leaf space) this result can be proved quite easily.

For a *local Lie groupoid* the structure maps are only defined on (and the usual properties only hold for) elements which are close enough to the space  $M$  of units (these are obvious generalizations of Cartan’s local Lie groups, as explained in Section 1.8 of [11]).

**COROLLARY 5.1.** *Any Lie algebroid is integrable by a local Lie groupoid.*

*Proof.* One uses exactly the same arguments as in the proof of Claim 2 and Claims 4 through 6 in Lemma 4.9, namely: We choose a connection  $\nabla$  on  $A$ , a neighborhood  $U$  of  $M$  in  $A$  and an open  $O$  in  $P(A)$ , with  $O = O^{-1}$ , such that  $\text{Exp}_\nabla : U \rightarrow P(A)$  intersects each plaque of  $\mathcal{F}(A)$  in  $O$  in exactly one point. Eventually choosing smaller pairs  $(O, U)$  (similar to the  $(O_i, U_i)$  in the cited proof), the structure of local groupoid will be defined on  $U$ : the inverse of  $v \in U$  is the unique  $\bar{v} \in U$  with the property that  $\overline{\text{Exp}_\nabla(v)} \sim_O \text{Exp}_\nabla(\bar{v})$ ; the multiplication  $v \cdot w$  of  $v, w \in U$  is defined only for pairs  $(u, v)$  for which  $\text{Exp}_\nabla(v)\text{Exp}_\nabla(w) \in O$ , and is the unique element with the property that the

last product of exponentials is  $\sim_{\mathcal{O}} \text{Exp}_{\nabla}(v \cdot w)$ . The associativity around the units is proved exactly as Claims 5 and 6 of the cited lemma, while the fact that the resulting local groupoid integrates  $A$  is a variation of Lemma 4.10.  $\square$

5.2. *Integrability criteria.* We start with the following general integrability criterion, which is an obvious consequence of our main result, and which implies most of the known results (or even much stronger versions of them):

COROLLARY 5.2. *If  $N_x(A)$  is trivial for all  $x \in M$ , then  $A$  is integrable. In particular,  $A$  is integrable if any of the following three conditions holds for all  $x \in M$ :*

- (i) *the Lie algebras  $\mathfrak{g}_x$  are semi-simple (more generally, if they have trivial center);*
- (ii) *the leaves  $L_x$  are 2-connected (more generally, if  $\pi_2(L_x)$  have only elements of finite order);*
- (iii) *there is a splitting  $\sigma : TL_x \rightarrow A|_{L_x}$  of the anchor, which is compatible with the Lie bracket;*

We now briefly deduce the known integrability results.

5.2.1. *Lie algebra bundles.* For Lie algebroids with zero anchor map, the orbits are the points of  $M$ , so the conditions of the main theorem are trivially satisfied and we obtain the following result of Douady and Lazard [10]:

COROLLARY 5.3. *Any Lie algebra bundle is integrable to a Lie group bundle.*

5.2.2. *Transitive algebroids.* In the case of Lie algebroids with surjective anchor the main theorem becomes (see also Remark 4.10):

COROLLARY 5.4. *Let  $A$  be a transitive Lie algebroid over  $M$ . Then  $A$  is integrable if and only if  $N_x(A)$  is discrete in  $A_x$  for one (or, equivalently, all)  $x \in M$ .*

We mention in passing that this is strongly related to Mackenzie's criteria [17], and the interested reader will be able to provide the precise relation with his cohomological obstruction.

There are some obvious consequences of this result. For example:

COROLLARY 5.5. *Every transitive Lie algebroid  $A$  over a 2-connected base  $M$  is integrable.*

Note also that since  $\mathfrak{s}^{-1}(x)$  is a principal  $\mathcal{G}(A)_x$ -bundle over  $M$ , it follows that if  $M$  is contractible then  $A$  is in fact isomorphic to a direct sum  $TM \oplus \mathfrak{g}$  (compatible with the Lie brackets), where  $\mathfrak{g} = \mathfrak{g}_x$ . Hence:

**COROLLARY 5.6.** *Any transitive Lie algebroid over a contractible base  $M$  is isomorphic to  $TM \oplus \mathfrak{g}$  for some Lie algebra  $\mathfrak{g}$ .*

In Mackenzie's approach this result is first obtained in order to to construct his obstruction.

**5.2.3. Regular Lie algebroids.** Although many of the known integrability criteria require regular algebroids, it turns out that regularity is superfluous (see below). This is the case, for example, with the Dazord-Hector ([8]) integrability criteria for totally aspherical regular Poisson manifolds, and with Nistor's results [22] on the integrability of regular algebroids with anchor possessing a splitting compatible with the Lie bracket, or with semi-simple kernels.

Let us mentioned, however, a result which fails in the nonregular case as shown by Example 4.4:

**COROLLARY 5.7.** *Any regular Lie algebroid is locally integrable.*

This follows because regular foliations are locally trivial. As in the transitive case, it is possible to describe explicitly the local structure of regular algebroids. Choosing local coordinates in  $M$  so that the foliation becomes the obvious  $p$ -dimensional foliation on  $\mathbb{R}^p \times \mathbb{R}^q$ , we then obtain that, locally, the algebroid is  $T\mathbb{R}^p \times \mathfrak{g}$ , where  $\mathfrak{g}$  is a bundle of Lie algebras over  $\mathbb{R}^q$ .

**5.2.4. Semi-direct products.** Closely related to Palais' integrability [24] of infinitesimal actions of Lie algebras  $\mathfrak{g}$  on manifolds  $M$  is the integrability of the transformation Lie algebroid  $A = \mathfrak{g} \times M$ . Recall that, as a vector bundle,  $A$  is just the trivial vector bundle with fiber  $\mathfrak{g}$ , the anchor is the infinitesimal action, while the bracket on  $\Gamma(A) = C^\infty(M; \mathfrak{g})$  is uniquely determined by the Leibniz rule and the Lie bracket of  $\mathfrak{g}$ .

If  $G_x \subset \mathcal{G}(\mathfrak{g})$  is the connected Lie group with Lie algebra  $\mathfrak{g}_x$ ,  $N_x(A)$  sits inside  $\pi_1(G_x)$ , so the conditions of the main theorem are satisfied; hence

**COROLLARY 5.8.** *For any infinitesimal action of the Lie algebra  $\mathfrak{g}$  on  $M$ , the transformation Lie algebroid  $\mathfrak{g} \times M$  is integrable.*

This is known as Dazord's criterion (cf. [3]), but it also appears implicitly in Palais' work [24]. Implicit in Palais' work is also the precise relation between this result and the integrability of infinitesimal actions. This relation has been clearly explained by Moerdijk-Mrčun in [20], where the reader can find various extensions to semi-direct products of algebroids. Let us point out that exactly

the same argument as above shows that the semi-direct product of an integrable algebroid by a regular foliation is integrable, and this is one of the main results of [20].

5.2.5. *Algebras of vector fields and quasi-foliations.* For a Lie algebroid  $A$  over  $M$  we say that the anchor is *almost injective* at  $x_0 \in M$  if there is a neighborhood  $U$  of  $x_0$  in  $M$  and an open dense subset  $O$  of  $U$  such that  $\#_x$  is injective for all  $x \in O$ . Note that if the anchor is injective at  $x_0$  then it is almost injective at  $x_0$ . We say that the anchor is almost injective if it is almost injective at every point.

Any Lie subalgebra  $\Gamma$  of  $\mathcal{X}(M)$  which is a finitely generated projective  $C^\infty(M)$ -module is the space of sections of an algebroid with almost injective anchor. This produces a large class of examples of Lie algebroids, including all regular foliations. As explained in [22], such  $\Gamma$ 's arise naturally in the analysis on manifolds with corners as algebras of vector fields with a certain behavior on the faces of  $M$ . Their integrability is relevant to various aspects of analysis and quantization (see [23] for details). Such algebroids were also studied by Claire Debord in her Ph.D. thesis ([9]), and they give rise to *quasi-foliations* of  $M$ .

Our main result implies the following integrability criterion due to Debord:

**COROLLARY 5.9.** *A Lie algebroid with injective anchor on a dense open set is integrable.*

This result follows from the following general continuity type property of the monodromy groups, combined with our main result.

**PROPOSITION 5.10.** *Let  $A$  be a Lie algebroid over  $M$ . Then, for every  $x \in M$ , there exists an open set  $U \subset A$  containing  $0_x$  such that  $N(A) \cap U$  has enough smooth local sections, i.e., for all  $v \in N(A) \cap U$ , there is a section  $\alpha : V \rightarrow A$  defined in a neighborhood  $V$  of  $\pi(v) \in M$  such that  $\alpha(\pi(v)) = v$  and  $\alpha(y) \in N_y(A) \cap U$  for all  $y \in V$ .*

*Proof.* Using Claim 2 and Claim 4 in the proof of the main theorem (which are independent of any integrability conditions!), we find a neighborhood  $U$  of  $0_x$  in  $A$  such that:

- (a)  $\text{Exp}_\nabla : U \rightarrow P(A)$  is well defined and transversal to the foliation;
- (b) if  $w \in U$  has  $\text{Exp}_\nabla(w) \sim 0$ , then  $w \in Z(\mathfrak{g})$ .

Here, as in the proof of the main theorem,  $\nabla$  is a fixed local connection around  $x$ .

We claim that  $U$  does satisfy the desired property. Assume  $v \in N_z(A) \cap U$ . Then a homotopy between the zero path and  $v$  can be viewed as a leafwise path in  $P(A)$ , connecting the constant path  $0_z$  with the constant path  $v$ . We denote

by  $H$  the induced holonomy, where we use  $(\text{Exp}_\nabla, 0_z)$  as transversal through  $0_z$ , and  $(\text{Exp}_\nabla, v)$  as transversal through  $v$ . Hence,  $H : (U_0, 0_z) \rightarrow (U_1, v)$  for some neighborhoods  $U_0$  and  $U_1$  in  $U$ , and  $\text{Exp}_\nabla(H(w)) \sim \text{Exp}_\nabla(w)$  for all  $w \in U_0$ . From condition (b) it follows that  $H(0_y)$  is a constant  $A$ -path in  $Z(\mathfrak{g}_y)$  for all  $y$  in  $V = \{y : 0_y \in U_0\}$ . Therefore we can choose  $\alpha : V \rightarrow A$  to be the section defined by  $\alpha(y) \equiv H(0_y)$ .  $\square$

Let us point out also the following improvement of Corollary 5.9:

**COROLLARY 5.11.** *A Lie algebroid  $A$  such that the monodromy groups  $N_x(A)$  are trivial for  $x$  on a dense open subset of the base manifold  $M$  is integrable.*

5.2.6. *Poisson manifolds.* The Weinstein groupoid of the algebroid associated to a Poisson manifold (the cotangent bundle  $T^*M$ ) is precisely the phase space  $\mathcal{G}$  of the Poisson sigma-model studied by Cattaneo and Felder in [4]. Our constructions explain the constructions in [4], while our main result clarifies the smoothness of the Poisson-sigma model  $\mathcal{G}$ .

The following obvious application of our general criteria, is the main positive result of [4].

**COROLLARY 5.12.** *Any Poisson structure on a domain in  $\mathbb{R}^2$  is integrable.*

The result is certainly not true in higher dimension, as shown by Weinstein’s example of a nonintegrable regular Poisson structure in  $\mathbb{R}^3 - 0$  (Example 4.5).

In general, our main result applied to this context describes the precise obstructions for the integrability of Poisson manifolds. Let us point out the following simple integrability result:

**COROLLARY 5.13.** *Any Poisson manifold where the symplectic leaves have vanishing second homotopy groups is integrable.*

The integrability criterion of Dazord and Hector ([8]) is in fact this result specialized to the case of a regular Poisson manifold for which the foliation has no vanishing cycles.

Note also that the monodromy groups of the regular symplectic leaves  $L$  (i.e. around which the rank is locally maximal) of a Poisson manifold  $M$  are particularly simple, as it is the associated monodromy map

$$\partial : \pi_2(L, x) \rightarrow \mathcal{N}_x^*(L).$$

Indeed, since the kernel  $\mathcal{N}^*(L)$  of  $\#$  over  $L$  is abelian, by Lemma 3.6 we can use any linear splitting  $\sigma$ . The resulting cohomology class

$$\Omega_L = [\Omega_\sigma] \in H^2(L; \mathcal{N}^*(L))$$

is independent of the splitting  $\sigma$ ,  $\partial$  is just the integration of  $\Omega_L$  over elements in  $\pi_2(L, x)$ , and its image defines the monodromy groups

$$N_x \subset \mathcal{N}_x^*(L).$$

These groups appeared already in the work of Alcalde-Cuesta and Hector ([1]), who also recognized the two obstructions to integrability (in an equivalent form). To our knowledge, their work contained the most detailed investigation up to this date of the integrability problem for Poisson manifolds. Unfortunately, their methods do not seem to apply beyond the regular case.

Notice also that if  $M$  is regular and  $\mathcal{F}$  is its symplectic foliation, then using a global splitting  $\sigma$  for  $\#$  one gets a globally defined cohomology class  $\Omega \in H^2(\mathcal{F}; \mathcal{N}^*)$  which lies in the foliated cohomology with coefficients in the kernel of  $\#$ . Clearly,  $\Omega_L = \Omega|_L$  for each  $L$ . For more on integrability of Poisson manifolds we refer the reader for the upcoming article [7].

5.2.7. *Van Est's argument.* Probably the most elegant proof of the integrability of Lie algebras is Van Est's cohomological argument which we briefly recall. Given a Lie algebra  $\mathfrak{g}$ , we form the exact sequence  $0 \rightarrow Z(\mathfrak{g}) \rightarrow \mathfrak{g} \rightarrow \text{ad}(\mathfrak{g}) \rightarrow 0$ . Here  $\text{ad}(\mathfrak{g})$  is easily seen to be integrable (it is a Lie sub-algebra of  $\mathfrak{gl}(\mathfrak{g})$ ). Also recall that simply connected Lie groups are automatically 2-connected. The core of Van Est's argument is then the following result for the particular case of Lie algebras

COROLLARY 5.14. *If  $B$  fits into an exact sequence of Lie algebroids*

$$0 \rightarrow E \rightarrow B \xrightarrow{\pi} A \rightarrow 0$$

*with  $E$  abelian, and  $A$  integrable by a groupoid with 2-connected  $\mathfrak{s}$ -fibers, then  $B$  is integrable.*

This result for Lie algebroids is Theorem 5 of [5]. Interestingly enough, it shows that the integrability criterion of Dazord and Hector [8] mentioned above is actually Van Est's argument applied to regular Poisson manifolds.

The proof in [5] is an extension of Van Est's cohomological methods. Using a splitting  $\sigma$  of  $\pi$  we obtain an action of  $A$  on  $E$ , and a 2-cocycle  $\Omega_\sigma$  on  $A$  with values on  $E$ . This is well known (see e.g. [17]), and can also be viewed as an extension of the constructions in Section 3.3. We can then form the group of periods  $\text{Per}_x \subset E_x$  of  $\Omega_\sigma$ . The cohomological proof actually shows that  $B$  is integrable provided  $A$  is, and the groups  $P_x$  vanish (cf. Remark 5 and Corollary 2 in [5]).

Let us briefly point out how our result implies (and further clarifies) the previous corollary. Let  $x \in M$  sitting in a singular leaf  $L$ . The necessary information is organized in the following diagram

$$\begin{array}{ccccccc}
 \pi_2(\mathfrak{s}_A^{-1}(x)) & \xrightarrow{\mathfrak{t}_A} & \pi_2(L) & \xrightarrow{\partial} & \pi_1(\mathcal{G}(A)_x) & & \\
 \downarrow \partial_E & & \swarrow \partial_B & & \searrow \partial_A & & \swarrow j \\
 0 & \longrightarrow & E_x & \longrightarrow & \mathcal{G}(\mathfrak{g}_x(B)) & \longrightarrow & \mathcal{G}(\mathfrak{g}_x(A)) \longrightarrow 0.
 \end{array}$$

Here  $\mathfrak{s}_A$ ,  $\mathfrak{t}_A$ ,  $\mathfrak{g}(A)$ ,  $\partial_A$  are respectively the source and target map, the kernel of the anchor, and the monodromy map of  $A$ , and we use analogous notations for  $B$ . Also,  $\partial$  is the boundary map in homotopy associated to  $\mathfrak{s}_A^{-1}(x) \rightarrow L$  with fiber  $\mathcal{G}(A)_x$  and  $j$  is the obvious inclusion for which the image is precisely  $\tilde{N}_x(A)$ . Finally,  $\partial_E$  denotes the monodromy map associated to the exact sequence in the corollary (constructed exactly as the monodromy map of §3.2), and its image is precisely the group of periods  $\text{Per}_x$ .

LEMMA 5.15. *There is a short exact sequence of abelian groups:*

$$0 \rightarrow \text{Per}_x \rightarrow \tilde{N}_x(B) \rightarrow \tilde{N}_x(A) \rightarrow 0.$$

*Proof.* These follows by diagram chasing since the two horizontal sequences above are exact. □

Therefore  $\tilde{N}_x(B)$  appears as a twisted semi-direct product of  $\tilde{N}_x(A)$  and  $\text{Per}_x$ . The simplest case where our main theorem applies is when  $\text{Per}_x$  vanishes. This gives precisely the corollary (and its stronger version) above.

5.3. *Transversally parallelizable foliations.* Historically, the first examples of nonintegrable Lie algebroids [2] came from Molino’s treatment (see [21]) of transversally parallelizable foliations which we now briefly recall.

Given a foliation  $\mathcal{F}$  of  $M$ , let us denote by  $l(M, \mathcal{F})$  the algebra of transversal vector fields, i.e. sections of the normal bundle which can be locally projected along submersions which locally define the foliation. Then  $(M, \mathcal{F})$  is *transversally parallelizable* if its normal bundle admits a global frame consisting of transversal vector fields. In this case the Lie algebra  $l(M, \mathcal{F})$  is free as a module over the space  $\Omega_b^0(M, \mathcal{F})$  of basic functions, on which it acts by derivations.

Let us show that the Lie bracket on  $l(M, \mathcal{F})$  is of the type studied in this paper. We assume for simplicity that  $M$  is compact. Then the closures of the leaves of  $\mathcal{F}$  form a new foliation  $\bar{\mathcal{F}}$  on  $M$ , with leaf space a smooth (Hausdorff) manifold  $W = M/\bar{\mathcal{F}}$ , called the *basic manifold* of the foliation. Since  $\mathcal{F}$  and  $\bar{\mathcal{F}}$  have the same basic functions,  $l(M, \mathcal{F})$  is the space of sections of a transitive Lie algebroid over  $W$ , which we denote by  $A(M, \mathcal{F})$ . Its anchor  $\#$  is just the action of  $l(M, \mathcal{F})$  on  $\Omega_b^0(M, \mathcal{F}) \cong C^\infty(W)$ , and the kernel of  $\#$  has the

following geometric interpretation. For each leaf  $L$  of  $\mathcal{F}$ , the foliation  $(\bar{L}, \mathcal{F}|_{\bar{L}})$  is transversally parallelizable with dense leaves. It follows that  $l(\bar{L}, \mathcal{F}|_{\bar{L}})$  is a finite-dimensional Lie algebra, and moreover,  $(\bar{L}, \mathcal{F}|_{\bar{L}})$  is a Lie foliation induced by a canonical  $l(\bar{L}, \mathcal{F}|_{\bar{L}})$ -valued Maurer-Cartan form. Denoting by  $w \in W$  the point defined by  $\bar{L}$ , we see that  $l(\bar{L}, \mathcal{F}|_{\bar{L}})$  is canonically isomorphic to  $\text{Ker}(\#_w)$ . This shows that all the Lie algebras  $l(\bar{L}, \mathcal{F}|_{\bar{L}})$  are isomorphic. The resulting Lie algebra  $\mathfrak{g}(M, \mathcal{F})$  (defined up to isomorphisms) is usually called the *structural Lie algebra* of the foliation.

The main result of Almeida and Molino in [2] says that  $(M, \mathcal{F})$  is developable (i.e. its lift to the universal cover of  $M$  is simple) if and only if the Lie algebroid  $A(M, \mathcal{F})$  is integrable. This discussion extends to transversally complete foliations  $(M, \mathcal{F})$  without any compactness assumption on  $M$  (see [21]).

Now, our constructions produce a monodromy map  $\partial : \pi_2(W) \rightarrow G(M, \mathcal{F})$  with values in the simply connected Lie group integrating the structural Lie algebra  $\mathfrak{g}(M, \mathcal{F})$ , which controls the developability of the foliation:

**COROLLARY 5.16.** *A transversally parallelizable foliation  $(M, \mathcal{F})$  on a compact manifold  $M$  is developable if and only if the image of the monodromy map*

$$\partial : \pi_2(W) \rightarrow G(M, \mathcal{F})$$

*is discrete.*

A simple consequence of this result is:

**COROLLARY 5.17.** *Let  $(M, \mathcal{F})$  be a transversally parallelizable manifold on a compact manifold  $M$ . Then  $(M, \mathcal{F})$  is developable in any of the following cases:*

- (i) *the structural Lie algebra  $\mathfrak{g}(M, \mathcal{F})$  has trivial center;*
- (ii)  *$\pi_2(W)$  has only elements of finite order.*

This result should be compared with Corollary 1, p. 301, and Corollary 1 p. 303 in [21].

## Appendix A. Flows

In this appendix we discuss the flows associated to sections of Lie algebroids, which generalize the ordinary flows of vector fields (sections of  $A = TM$ ). This is used throughout the paper, most notably for defining the equivalence relation on  $A$ -paths (§1.3). As in the main body of the paper,  $A$  denotes a Lie algebroid over  $M$ ,  $\# : A \rightarrow TM$  denotes its anchor and  $\pi : A \rightarrow M$  the projection.

A.1. *Flows and infinitesimal flows.* Given a time-dependent vector field  $X$  on  $M$ , we denote by  $\Phi_X^{t,s}$  its flow from time  $s$  to time  $t$ . Hence

$$\frac{d}{dt}\Phi_X^{t,s}(x) = X(t, \Phi_X^{t,s}(x)), \quad \Phi_X^{s,s}(x) = x.$$

We have  $\Phi^{t,s}\Phi^{s,u} = \Phi^{t,u}$  and, when  $X$  is autonomous,  $\Phi_X^{t,s} = \Phi_X^{t-s}$  only depends on  $t - s$ . Differentiating, we obtain the infinitesimal flow of  $X$ :

$$\phi_X^{t,s}(x) \equiv (d\Phi_X^{t,s})_x : T_xM \rightarrow T_{\Phi_X^{t,s}(x)}M.$$

Let us assume now that  $\mathcal{G}$  is a Lie groupoid integrating the algebroid  $A$ . Given a time-dependent section  $\alpha$  of  $A$ , we denote by the same letter the right invariant (time-dependent) vector field on  $\mathcal{G}$  induced by  $\alpha$ , and by  $\varphi_\alpha^{t,s} : \mathcal{G} \rightarrow \mathcal{G}$  its flow. If  $x = \mathbf{s}(g)$  and  $y = \mathbf{t}(g)$ , then  $\varphi_\alpha^{t,s}(g)$  is the arrow

$$\varphi_\alpha^{t,s}(g) : x \longrightarrow \Phi_{\#\alpha}^{t,s}(y)$$

and also satisfies the right-invariance property:

$$\varphi_\alpha^{t,s}(g) = \varphi_\alpha^{t,s}(y)g.$$

The *infinitesimal flow* of  $\alpha$ ,

$$\phi_\alpha^{t,s}(x) : A_x \rightarrow A_{\Phi_{\#\alpha}^{t,s}(y)},$$

is defined as

$$(A.1) \quad \phi_\alpha^{t,s}(x) \equiv (dR_{\varphi_\alpha^{s,t}(x)})_{\varphi_\alpha^{t,s}(x)}(d\varphi_\alpha^{t,s})_x.$$

The classical relation between Lie brackets and flows translates at this level to

$$(A.2) \quad \left. \frac{d}{dt} \right|_{t=s} (\phi_\alpha^{t,s})^*\beta = [\alpha^s, \beta],$$

where we have set

$$(\phi_\alpha^{t,s})^*(\beta)(x) = \phi_\alpha^{s,t}\beta(\phi_{\#\alpha}^{t,s}(x)).$$

We wish to extend the infinitesimal flow to sections of general Lie algebroids, not necessarily integrable. For this we can use the following general construction of (infinitesimal) flows. Let us assume that  $E$  is a vector bundle over  $M$ . A derivation on  $E$  is a pair  $(D, X)$  where  $D : \Gamma(E) \rightarrow \Gamma(E)$  is a differential operator,  $X$  is a vector field on  $M$ , satisfying the Leibniz rule

$$D(f\alpha) = fD(\alpha) + X(f)\alpha, \quad \text{for all } f \in C^\infty(M), \alpha \in \Gamma(E).$$

Now, any time-dependent derivation  $(D, X)$  on  $E$  has an associated (infinitesimal) flow: a standard argument shows that there is a family of linear isomorphisms

$$\phi_D^{t,s}(x) : E_x \rightarrow E_{\Phi_X^{t,s}(x)},$$

which is characterized uniquely by the properties

- (a)  $\phi_D^{t,s} \phi_D^{s,u} = \phi_D^{t,u}, \phi_D^{t,t} = \text{Id};$
- (b)  $\left. \frac{d}{dt} \right|_{t=s} (\phi_D^{t,s})^* \beta = D^s(\beta),$  for all sections  $\beta \in \Gamma(E).$

Here  $D^t$  is  $D$  at the fixed time  $t$ , and  $(\phi_D^{t,s})^* \beta = \phi_D^{s,t} \beta \Phi_X^{t,s}.$

Alternatively, one can use the groupoid  $\text{Aut}(E)$  over  $E$ , for which the arrows from  $x$  to  $y$  are all linear isomorphisms  $E_x \rightarrow E_y.$  Its Lie algebroid is usually denoted by  $DO(E),$  and its sections are precisely derivations of  $E$  (cf. [15], [17]). Hence  $(D, X)$  can be viewed as a time-dependent section of  $DO(E),$  and then  $\phi_D^{t,s}$  is just the associated flow on  $\text{Aut}(E).$  Both definitions of  $\phi_D^{t,s}(x)$  show they are defined whenever  $\Phi_X^{t,s}(x)$  is defined.

Most flows in differential geometry (e.g. the flows of vector fields, parallel transport, etc.) are obtained in this way.

*A.2. The infinitesimal flow of a section.* We apply the previous construction to a time-dependent section  $\alpha$  of the Lie algebroid  $A,$  where  $X = \#\alpha$  and  $D = [\alpha, -] : \Gamma(A) \rightarrow \Gamma(A).$  The resulting flow

$$\phi_\alpha^{t,s}(x) : A_x \rightarrow A_{\Phi_{\#\alpha}^{t,s}(x)}$$

is uniquely determined by  $\phi_\alpha^{t,s} \phi_\alpha^{s,u} = \phi_\alpha^{t,u}, \phi_\alpha^{t,t} = \text{Id},$  and the formula (A.2) above. In particular, if  $A$  is integrable, then  $\phi_\alpha^{t,s}$  coincides with (A.1) above. As in the case of vector fields, if  $\alpha$  is autonomous, then  $\phi_\alpha^{t,s} = \phi_\alpha^{t-s}$  only depends on  $t - s.$

Let us indicate an alternative description. Recall that on  $A^*$  one has a Poisson bracket  $\{ , \}_A$  which is linear on the fibers. A section  $\alpha$  of  $A$  defines in a natural way a function  $f_\alpha : A^* \rightarrow \mathbb{R}$  which is linear on the fibers (“evaluation”), and we denote by  $X_\alpha$  the Hamiltonian vector field associated with  $f_\alpha.$  It is easy to see (cf. [13]) that:

- (a) The assignment  $\alpha \mapsto f_\alpha$  defines a Lie algebra homomorphism  $(\Gamma(A), [ , ]) \rightarrow (C^\infty(A^*), \{ , \}_A);$
- (b)  $X_\alpha$  is  $\pi$ -related to  $\#\alpha: \pi_* X_\alpha = \#\alpha,$  where  $\pi : A^* \rightarrow M$  is the natural projection;

For each  $t,$  the flow  $\Phi^{s,t}$  of  $X_\alpha$  defines a Poisson automorphism of  $A^*$  (wherever defined), which maps linearly fibers to fibers of  $A^*.$  So, in fact,  $\Phi^{s,t} : A^* \rightarrow A^*$  is a bundle map and from (b) we have that it covers  $\Phi_{\#\alpha}^{s,t},$  the flow of  $\#\alpha.$  By transposition we obtain the infinitesimal flow  $\phi_\alpha^{t,s}(x) : A_x \rightarrow A_{\Phi_{\#\alpha}^{t,s}(x)}.$

*Example A.1.* As a simple example, consider a Lie algebra  $A = \mathfrak{g}$  as a Lie algebroid over a point, and  $\alpha \in \mathfrak{g}$  (a constant section). The Poisson structure on the dual  $\mathfrak{g}^*$  is the Kirillov-Kostant bracket and so the hamiltonian flow on  $\mathfrak{g}^*$  of the evaluation function  $f_\alpha$  is given by the co-adjoint action. It follows that the infinitesimal flow of  $\alpha$  is then  $\phi_\alpha^t = \text{Ad}(\exp(t\alpha))$ .

This example shows that one can think of the infinitesimal flow of a section as a generalization of the adjoint action, although for a general Lie algebroid it does not make sense to speak of the adjoint representation!

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(Received May 18, 2001)