

Lectures on Integrability of Lie Brackets

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This research was supported in part by NWO, FCT/POCTI/FEDER, and by grants POCI/MAT/55958/2004 and POCI/MAT/57888/2004.

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Preface

The subject of these lecture notes is the problem of integrating infinitesimal geometric structures to global geometric structures. We follow the categorical approach to differential geometry, where the infinitesimal geometric structures are called *Lie algebroids* and the global geometric structures are called *Lie groupoids*. It is also one of our aims to convince you of the advantages of this approach.

You may not be familiar with the language of Lie algebroids or Lie groupoids, but you have already seen several instances of the integrability problem. For example, the problem of integrating a vector field to a flow, the problem of integrating an involutive distribution to a foliation, the problem of integrating a Lie algebra to a Lie group, or the problem of integrating a Lie algebra action to a Lie group action. In all these special cases, the integrability problem always has a solution. However, in general, this need not be the case and this new aspect of the problem, the *obstructions to integrability*, is one of the main topics of these notes. One such example, that you may have seen before, is the problem of geometric quantization of a presymplectic manifold, where the so-called prequantization condition appears.

These notes are made up of five lectures. In the first lecture, we introduce Lie groupoids and their infinitesimal versions. In the second lecture, we introduce the abstract notion of a Lie algebroid and discuss how many notions of differential geometry can be described in this new language. These first two lectures should help you in becoming familiar with the language of Lie groupoids and Lie algebroids. The third and fourth lectures are concerned with the various aspects of the integrability problem. These two lectures form the core material of this course, and contain a detailed description of the integrability obstructions. In the last lecture we consider, as an example, aspects of integrability related to Poisson geometry. At the end of each lecture, we have included a few notes with some references to the literature. We warn you that these notes are not meant to be complete; they are simply a way to provide you some historical background as well as further material for you to read and discover. There are also around 100 exercises which are an integral part of the lectures: you will need to solve the vast majority of the exercises to get a good feeling of what integrability is all about!

A version of these lecture notes were used for a course we gave at the Summer School in Poisson Geometry, held at ICTP, Trieste, in the summer of 2005. Due to a lack of time and space, we were not able to include in these notes all the topics discussed in the course. Topics left out include aspects of integrability related to cohomology, quantization and homotopy theory. We have plans to write a book on the *Geometry of Lie Brackets*, where all those topics that were left out (and more!) will be included.

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Utrecht and Lisbon, November 2006

LECTURE 1

Lie Groupoids

1.1. Why groupoids?

These lectures are centered around the notion of a *groupoid*. What are groupoids? What are they good for? These are basic questions that we will be addressing and we hope that, by the end of these lectures, we will have convinced you that groupoids are worth studying.

It maybe a good idea, even before we start with any formal definitions, to look at an example. You may wish to keep this kind of example in mind, since it illustrates very nicely many of the basic abstract concepts we will be introducing in this lecture.

Let N be a manifold, and suppose that we want to classify the set of Riemannian metrics on N (this is obviously too ambitious, but keep on reading!). This means that our space of objects, which we will denote by M , is the space of metrics on N :

$$M = \{g : g \text{ is a Riemannian metric on } N\}.$$

This space is quite large. In the classification problem it is natural not to distinguish two metrics that are related by a diffeomorphism. So let us consider the triples (g_1, ϕ, g_2) , where g_i are metrics on N and ϕ is a diffeomorphism of N which relates the two metrics:

$$\mathcal{G} = \{(g_2, \phi, g_1) : g_2 = \phi_* g_1\}.$$

Now \mathcal{G} is precisely the prototype of a groupoid. The word “groupoid” is meant to be suggestive of the notion of group. You will notice that we cannot always compose two elements of \mathcal{G} , but that sometimes we can do it: if (g_2, ϕ, g_1) and (h_2, ψ, h_1) are two elements of \mathcal{G} , then we can compose them obtaining the new element $(g_2, \phi \circ \psi, h_1)$, provided $g_1 = h_2$. Also, there are elements which behave like units under this multiplication, namely (g, I, g) , where $I : N \rightarrow N$ is the identity map. Finally, each element (g_2, ϕ, g_1) has an inverse, namely (g_1, ϕ^{-1}, g_2) .

As we have mentioned above, in the classification problem, we identify two metrics g_1 and g_2 that differ by a diffeomorphism ϕ (i.e., such that

$g_2 = \phi_*g_1$). In other words, we are interested in understanding the quotient space M/\mathcal{G} , which we may call the *moduli space* of Riemann metrics on M . On the other hand, if we want to pay attention to a fixed metric g , then we recognize immediately that the triples (g, ϕ, g) (i.e., the set of diffeomorphisms preserving this metric) is just the group of isometries of (N, g) . Hence, our groupoid encodes all the relevant data of our original problem of studying and classifying metrics on N .

There is nothing special about this example, where we have chosen to look at metrics on a manifold. In fact, anytime one tries to study and classify some class of structures on a space there will be a groupoid around.

1.2. Groupoids

Here is the shortest definition of a groupoid:

Definition 1.1. A **groupoid** \mathcal{G} is a (small) category in which every arrow is invertible.

The set of morphisms (arrows) of the groupoid will be denoted by the same letter \mathcal{G} , while the set of objects is denoted by $M_{\mathcal{G}}$, or even by M , provided it is clear from the context what the groupoid is. We call M the base of the groupoid, and we say that \mathcal{G} is a groupoid over M .

From its very definition, a groupoid \mathcal{G} over M has certain underlying **structure maps**:

- the **source** and the **target** maps

$$\mathbf{s}, \mathbf{t} : \mathcal{G} \longrightarrow M,$$

associating to each arrow g its source object $\mathbf{s}(g)$ and its target object $\mathbf{t}(g)$. Given $g \in \mathcal{G}$, we write $g : x \longrightarrow y$, or $x \xrightarrow{g} y$, or $y \xleftarrow{g} x$ to indicate that g is an arrow from x to y .

- the **composition map**

$$m : \mathcal{G}_2 \longrightarrow \mathcal{G},$$

is defined on the set \mathcal{G}_2 of composable arrows:

$$\mathcal{G}_2 = \{(g, h) \in \mathcal{G} \times \mathcal{G} : \mathbf{s}(g) = \mathbf{t}(h)\}.$$

For a pair (g, h) of composable arrows, $m(g, h)$ is the composition $g \circ h$. We also use the notation $m(g, h) = gh$, and sometimes we call gh the multiplication of g and h .

- the **unit map**

$$u : M \longrightarrow \mathcal{G},$$

which sends $x \in M_{\mathcal{G}}$ to the identity arrow $1_x \in \mathcal{G}$ at x . We will often identify 1_x and x .

- the **inverse map**

$$i : \mathcal{G} \longrightarrow \mathcal{G},$$

which sends an arrow g to its inverse g^{-1} .

Of course, these structure maps also satisfy some identities, similar to the group case, which again are consequences of Definition 1.1:

- **law of composition:** if $x \xleftarrow{g} y \xleftarrow{h} z$, then $x \xleftarrow{gh} z$.
- **law of associativity:** if $x \xleftarrow{g} y \xleftarrow{h} z \xleftarrow{k} u$, then $g(hk) = (gh)k$.
- **law of units:** $x \xleftarrow{1_x} x$ and, for all $x \xleftarrow{g} y$, $1_x g = g 1_y = g$.
- **law of inverses:** if $x \xleftarrow{g} y$, then $y \xrightarrow{g^{-1}} x$ and $g g^{-1} = 1_y$, $g^{-1} g = 1_x$.

Hence, in a more explicit form, here is the long definition of a groupoid:

Definition 1.2. A **groupoid** consists of a set \mathcal{G} (of arrows), a set $M_{\mathcal{G}}$ (of objects), and maps $\mathbf{s}, \mathbf{t}, u, m$, and i as above, satisfying the laws of composition, associativity, units and inverses.

We will be using the following notation for a groupoid \mathcal{G} over M : if $x \in M$, then the sets

$$\mathcal{G}(x, -) = \mathbf{s}^{-1}(x), \quad \mathcal{G}(-, x) = \mathbf{t}^{-1}(x)$$

are called **the s -fiber at x** , and **the t -fiber at x** , respectively. The inverse map induces a natural bijection between these two sets:

$$i : \mathcal{G}(x, -) \longrightarrow \mathcal{G}(-, x).$$

Given $g : x \longrightarrow y$, the **right multiplication by g** is only defined on the s -fiber at y , and induces a bijection

$$R_g : \mathcal{G}(y, -) \longrightarrow \mathcal{G}(x, -).$$

Similarly, the **left multiplication by g** induces a map from the t -fiber at x to the t -fiber at y .

Next, the intersection of the s and t -fiber at $x \in M$,

$$\mathcal{G}_x = \mathbf{s}^{-1}(x) \cap \mathbf{t}^{-1}(x) = \mathcal{G}(x, -) \cap \mathcal{G}(-, x)$$

together with the restriction of the groupoid multiplication, is a group called **the isotropy group at x** .

On the other hand, at the level of the base M , one has an equivalence relation $\sim_{\mathcal{G}}$: two objects $x, y \in M$ are said to be equivalent if there exists an arrow $g \in \mathcal{G}$ whose source is x and whose target is y . The equivalence class of $x \in M$ is called **the orbit through x** :

$$\mathcal{O}_x = \{\mathbf{t}(g) : g \in \mathbf{s}^{-1}(x)\},$$

and the quotient set

$$M/\mathcal{G} := M/\sim_{\mathcal{G}} = \{\mathcal{O}_x : x \in M\}$$

is called **the orbit set of \mathcal{G}** .

The following exercise shows that for a groupoid \mathcal{G} there is still an underlying group around.

Exercise 1. Let \mathcal{G} be a groupoid over M . Define a **bisection** of \mathcal{G} to be a map $b : M \rightarrow \mathcal{G}$ such that $\mathbf{s} \circ b$ and $\mathbf{t} \circ b$ are bijections. Show that any two bisections can be multiplied, so that the set of bisections form a group, denoted $\Gamma(\mathcal{G})$.

The groupoids we will be interested in are not just algebraic objects. Usually we will be interested in comparing two arrows, looking at neighborhoods of an arrow, etc.

Definition 1.3. A **topological groupoid** is a groupoid \mathcal{G} whose set of arrows and set of objects are both topological spaces, whose structure maps $\mathbf{s}, \mathbf{t}, u, m, i$ are all continuous, and such that \mathbf{s} and \mathbf{t} are open maps.

Example 1.4. Consider the groupoid \mathcal{G} formed by triples (g_2, ϕ, g_1) where g_i are metrics on N and ϕ is a diffeomorphism taking g_1 to g_2 . The compact-open topology on the space of diffeomorphisms of N and on the space of metrics, induces a natural topology on \mathcal{G} so that it becomes a topological groupoid.

Note that for a topological groupoid all the s and t -fibers are topological spaces, the isotropy groups are topological groups, and the orbit set of \mathcal{G} has an induced quotient topology.

Exercise 2. For a topological groupoid \mathcal{G} prove that the unit map

$$u : M \rightarrow \mathcal{G}$$

is a topological embedding, i.e., it is a homeomorphism onto its image (furnished with the relative topology).

Obviously, one can go one step further and set:

Definition 1.5. A **Lie groupoid** is a groupoid \mathcal{G} whose set of arrows and set of objects are both manifolds, whose structure maps $\mathbf{s}, \mathbf{t}, u, m, i$ are all smooth maps and such that \mathbf{s} and \mathbf{t} are submersions.

Note that the condition that \mathbf{s} and \mathbf{t} are submersions ensure that the \mathbf{s} and \mathbf{t} -fibers are manifolds. They also ensure that the space \mathcal{G}_2 of composable arrows is a submanifold of $\mathcal{G} \times \mathcal{G}$, and the smoothness of the multiplication map m is to be understood with respect to the induced smooth structure on \mathcal{G}_2 .

Convention 1.6. Unless otherwise stated, all our manifolds are second countable and Hausdorff. An exception to this convention is the total space of a Lie groupoid \mathcal{G} which is allowed to be non-Hausdorff (to understand why, look at Example 8). But we will assume that the base manifold M as well as all the s -fibers $\mathcal{G}(x, -)$ (and, hence, the t -fibers $\mathcal{G}(-, x)$) are Hausdorff.

Exercise 3. Given a Lie groupoid \mathcal{G} over M and $x \in M$, prove that:

- (a) the isotropy groups \mathcal{G}_x are Lie groups;
- (b) the orbits \mathcal{O}_x are (regular immersed) submanifolds in M ⁽¹⁾;
- (c) the unit map $u : M \rightarrow \mathcal{G}$ is an embedding;
- (d) $\mathbf{t} : \mathcal{G}(x, -) \rightarrow \mathcal{O}_x$ is a principal \mathcal{G}_x -bundle.

Remark 1.7. Let \mathcal{G} be a Lie groupoid over M . Define a smooth bisection to be a smooth map $b : M \rightarrow \mathcal{G}$ such that $\mathbf{s} \circ b$ and $\mathbf{t} \circ b$ are diffeomorphisms. If one furnishes the group $\Gamma(\mathcal{G})$ of smooth bisections with the compact-open topology, one can show that $\Gamma(\mathcal{G})$ is a Fréchet Lie group. However, very little is known about these infinite dimensional Lie groups, so one prefers to study the Lie groupoid \mathcal{G} , which is a finite dimensional object.

¹An immersion $i : N \rightarrow M$ is called *regular* if for any map $f : P \rightarrow N$ the composition $i \circ f : P \rightarrow M$ is smooth iff $f : P \rightarrow N$ is smooth.

Since groupoids are categories, a morphism $\mathcal{G} \rightarrow \mathcal{H}$ between two groupoids is a functor: to each arrow and each object in \mathcal{G} we associate an arrow and an object in \mathcal{H} , and these two assignments have to be compatible with the various structure maps. Later we will see more general notions of morphisms, but for now this suffices:

Definition 1.8. Given a groupoid \mathcal{G} over M and a groupoid \mathcal{H} over N , a **morphism** from \mathcal{G} to \mathcal{H} , consists of a map $\mathcal{F} : \mathcal{G} \rightarrow \mathcal{H}$ between the sets of arrows, and a map $f : M \rightarrow N$ between the sets of objects, which are compatible with all the structure maps. A morphism between two topological (respectively, Lie) groupoids is a morphism whose components are continuous (respectively, smooth).

The compatibility of \mathcal{F} with the structure maps translate into the following explicit conditions:

- if $g : x \rightarrow y$ is in \mathcal{G} , then $\mathcal{F}(g) : f(x) \rightarrow f(y)$ in \mathcal{H} .
- if $g, h \in \mathcal{G}$ are composable, then $\mathcal{F}(gh) = \mathcal{F}(g)\mathcal{F}(h)$.
- if $x \in M$, then $\mathcal{F}(1_x) = 1_{f(x)}$.
- if $g : x \rightarrow y$, then $\mathcal{F}(g^{-1}) = \mathcal{F}(g)^{-1}$.

Notice that the last property actually follows from the first three.

Exercise 4. If $\mathcal{F} : \mathcal{G} \rightarrow \mathcal{H}$ is a morphism of Lie groupoids, show that the restrictions $\mathcal{F}|_x : \mathcal{G}_x \rightarrow \mathcal{H}_{f(x)}$ are morphisms of Lie groups.

If \mathcal{G} is a groupoid, a **subgroupoid** of \mathcal{G} is a pair (\mathcal{H}, i) , where \mathcal{H} is a groupoid and $i : \mathcal{H} \rightarrow \mathcal{G}$ is an injective groupoid homomorphism. In the case of topological (respectively, Lie) groupoids we require i to be a topological (respectively, smooth) immersion. A **wide subgroupoid** is a subgroupoid $\mathcal{H} \subset \mathcal{G}$ which has the same space of units as \mathcal{G} .

The following exercise shows that this notion is more subtle for Lie groupoids than it is for Lie groups.

Exercise 5. Give an example of a Lie groupoid \mathcal{G} and a wide Lie subgroupoid (\mathcal{H}, i) , such that $i : \mathcal{H} \rightarrow \mathcal{G}$ is an embedding, but its image is not a closed submanifold of \mathcal{G} .

(Hint: Try the action groupoids, introduced later in this lecture.)

Next we observe that groupoids act on fiber spaces:

Definition 1.9. Given a groupoid \mathcal{G} over M , a **\mathcal{G} -space** E is defined by a map $\mu : E \rightarrow M$, called the **moment map**, together with a map

$$\mathcal{G} \times_M E = \{(g, e) : \mathbf{s}(g) = \mu(e)\} \rightarrow E, \quad (g, e) \mapsto ge,$$

such that the following action identities are satisfied:

- (i) $\mu(ge) = \mathbf{t}(g)$.
- (ii) $g(he) = (gh)e$, for all $g, h \in \mathcal{G}$ and $e \in E$ for which it makes sense;
- (iii) $1_{\mu(e)}e = e$, for all $e \in E$.

Note that an action of \mathcal{G} on E , with moment map $\mu : E \rightarrow M$, really means that to each arrow $g : x \rightarrow y$ one associates an isomorphism

$$E_x \rightarrow E_y, \quad e \mapsto ge,$$

where $E_x = \mu^{-1}(x)$, such that the action identities are satisfied.

Obviously, if \mathcal{G} is a topological (respectively, a Lie groupoid) and E is a topological space (respectively, a manifold), then we have the notion of a continuous (respectively, smooth) action.

What we have defined, is actually the notion of a *left* \mathcal{G} -space. We leave it to reader the task of defining the notion of a *right* \mathcal{G} -space.

Example 1.10. A groupoid \mathcal{G} acts on itself by left and right multiplication. For the left action, we let $\mu = \mathbf{t} : \mathcal{G} \rightarrow M$ be the moment map, while for the right action, we let $\mu = \mathbf{s} : \mathcal{G} \rightarrow M$. These are continuous (respectively, smooth) actions if \mathcal{G} is a topological (respectively, Lie) groupoid. We will see more examples of actions below.

We mention here one more important notion, which is a simple extension of the notion of representations of groups, and which corresponds to linear groupoid actions.

Definition 1.11. Given a (topological, Lie) groupoid \mathcal{G} over M , a **representation of \mathcal{G}** consists of a vector bundle $\mu : E \rightarrow M$, together with a (continuous, smooth) linear action of \mathcal{G} on $E \rightarrow M$: for any arrow of \mathcal{G} , $g : x \rightarrow y$, one has an induced linear isomorphism $E_x \rightarrow E_y$, denoted $v \mapsto gv$, such that the action identities are satisfied.

Note that, in particular, each fiber E_x is a representation of the isotropy Lie group \mathcal{G}_x .

The isomorphism classes of representations of a groupoid \mathcal{G} form a semi-ring $(\text{Rep}(\mathcal{G}), \oplus, \otimes)$, where, for two representations E_1 and E_2 of \mathcal{G} , the induced actions on $E_1 \oplus E_2$ and $E_1 \otimes E_2$ are the diagonal ones. The unit element is given by the trivial representation, i.e., the trivial line bundle \mathbb{L}_M over M with $g \cdot 1 = 1$.

1.3. First examples of Lie groupoids

Let us list some examples of groupoids. We start with two extreme classes of groupoids which, in some sense, are of opposite nature.

Example 1.12 (Groups). A group is the same thing as a groupoid for which the set of objects contains a single element. So groups are very particular instances of groupoids. Obviously, topological (respectively, Lie) groups are examples of topological (respectively, Lie) groupoids.

Example 1.13 (The pair groupoid). At the other extreme, let M be any set. The Cartesian product $M \times M$ is a groupoid over M if we think of a pair (y, x) as an arrow $x \rightarrow y$. Composition is defined by:

$$(z, y)(y, x) = (z, x).$$

If M is a topological space (respectively, a manifold), then the pair groupoid is a topological (respectively, Lie) groupoid.

Note that each representation of the pair groupoid is isomorphic to a trivial vector bundle with the tautological action. Hence

$$\text{Rep}(M \times M) = \mathbb{N},$$

the semi-ring of non-negative integers.

Our next examples are genuine examples of groupoids, which already exhibit some distinct features.

Example 1.14 (General linear groupoids). If E is a vector bundle over M , there is an associated general linear groupoid, denoted by $GL(E)$, which is similar to the general linear group $GL(V)$ associated to a vector space V . $GL(E)$ is a groupoid over M whose arrows between two points x and y consist of linear isomorphisms $E_x \rightarrow E_y$, and the multiplication of arrows is given by the composition of maps.

Note that, given a general Lie groupoid \mathcal{G} and a vector bundle E over M , a linear action of \mathcal{G} on E (making E into a representation) is the same thing as a homomorphism of Lie groupoids $\mathcal{G} \rightarrow GL(E)$.

Example 1.15 (The action groupoid). Let $G \times M \rightarrow M$ be an action of a group on a set M . The corresponding action groupoid over M , denoted $G \ltimes M$, has as space of arrows the Cartesian product:

$$\mathcal{G} = G \times M.$$

For an arrow $(g, x) \in \mathcal{G}$ its source and target are given by:

$$\mathbf{s}(g, x) = x, \quad \mathbf{t}(g, x) = g \cdot x,$$

while the composition of two arrows is given by:

$$(h, y)(g, x) = (hg, x).$$

Notice that the orbits and isotropy groups of \mathcal{G} coincide with the usual notions of orbits and isotropy groups of the action $G \times M \rightarrow M$. Also,

$$\text{Rep}(G \ltimes M) = \text{Vect}_G(M),$$

the semi-ring of equivariant vector bundles over M .

Example 1.16 (The gauge groupoid). If G is a Lie group and $P \rightarrow M$ is a principal G -bundle, then the quotient of the pair groupoid $P \times P$ by the (diagonal) action of G is a groupoid over $P/G = M$, denoted $P \otimes_G P$, and called the gauge groupoid of P . Note that all isotropy groups are isomorphic to G . Also, fixing a point $x \in M$, a representation E of $P \otimes_G P$ is uniquely determined by $E_x \in \text{Rep}(G)$, and this defines an isomorphism of semi-rings:

$$\text{Rep}(P \otimes_G P) \sim \text{Rep}(G).$$

A particular case is when $G = GL_n$ and P is the frame bundle of a vector bundle E over M of rank n . Then the resulting gauge groupoid coincides with $GL(E)$ mentioned above.

Example 1.17 (The fundamental groupoid of a manifold). Closely related to the pair groupoid of a manifold M is the fundamental groupoid of M , denoted $\Pi_1(M)$, which consists of homotopy classes of paths with fixed end points (we assume that M is connected). In the light of Example 1.16, this is the gauge groupoid associated to the universal cover of M , viewed as a principal bundle over M with structural group the fundamental group of M - and this point of view also provides us with a smooth structure on $\Pi_1(M)$, making it into a Lie groupoid. Note also that, when M is simply-connected, $\Pi_1(M)$ is isomorphic to the pair groupoid $M \times M$ (a homotopy class of a path is determined by its end points). In general, there is an

obvious homomorphism of Lie groupoids $\Pi_1(M) \longrightarrow M \times M$, which is a local diffeomorphism.

Exercise 6. Show that representations of $\Pi_1(M)$ correspond to vector bundles over M endowed with a flat connection.

Example 1.18 (The fundamental groupoid of a foliation). More generally, let \mathcal{F} be a foliation of a space M of class C^r ($0 \leq r \leq \infty$). The fundamental groupoid of \mathcal{F} , denoted by $\Pi_1(\mathcal{F})$, consists of the leafwise homotopy classes of paths (relative to the end points):

$$\Pi_1(\mathcal{F}) = \{[\gamma] : \gamma : [0, 1] \rightarrow M \text{ a path lying in a leaf}\}.$$

For an arrow $[\gamma] \in \Pi_1(\mathcal{F})$ its source and target are given by:

$$\mathbf{s}([\gamma]) = \gamma(0), \quad \mathbf{t}([\gamma]) = \gamma(1),$$

while the composition of two arrows is just concatenation of paths:

$$[\gamma_1][\gamma_2] = [\gamma_1 \cdot \gamma_2].$$

In this example, the orbit O_x coincides with the leaf L through x , while the isotropy group $\Pi_1(\mathcal{F})_x$ coincides with the fundamental group $\pi_1(L, x)$.

Exercise 7. Verify that if \mathcal{F} is a foliation of class C^r ($0 \leq r \leq \infty$) then $\Pi_1(\mathcal{F})$ is a groupoid of class C^r .

The next exercise gives a simple example of a non-Hausdorff Lie groupoid.

Exercise 8. Let \mathcal{F} be the smooth foliation of $\mathbb{R}^3 - \{0\}$ by horizontal planes. Check that the Lie groupoid $\Pi_1(\mathcal{F})$ is not Hausdorff.

1.4. The Lie algebroid of a Lie groupoid

Motivated by what we know about Lie groups, it is natural to wonder what are the infinitesimal objects that correspond to Lie groupoids, and these ought to be called *Lie algebroids*.

Let us recall how we construct a Lie algebra out of a Lie group: if G is a Lie group, then its Lie algebra $\text{Lie}(G)$ consists of:

- an underlying vector space, which is just the tangent space to G at the unit element.
- A Lie bracket on $\text{Lie}(G)$, which comes from the identification of $\text{Lie}(G)$ with the space $\mathfrak{X}_{\text{inv}}(G)$ of right invariant vector fields on G , together with the fact that the space of right invariant vector fields is closed under the usual Lie bracket of vector fields:

$$[\mathfrak{X}_{\text{inv}}(G), \mathfrak{X}_{\text{inv}}(G)] \subset \mathfrak{X}_{\text{inv}}(G).$$

Now, back to general Lie groupoids. One remarks two novelties when comparing with the Lie group case. First of all, there is more than one unit element. In fact, there is one unit for each point in M , hence we expect a vector bundle over M , instead of a vector space. Secondly, the right multiplication by elements of \mathcal{G} is only defined on the s -fibers. Hence, to talk about right invariant vector fields on \mathcal{G} , we have to restrict attention to those vector fields which are tangent to the s -fibers, i.e., to the sections of the sub-bundle $T^s\mathcal{G}$ of $T\mathcal{G}$ defined by

$$T^s\mathcal{G} = \text{Ker}(ds) \subset T\mathcal{G}.$$

Having in mind the discussion above, we first define $\text{Lie}(\mathcal{G})$ as a vector bundle over M .

Definition 1.19. Given a Lie groupoid \mathcal{G} over M , we define the vector bundle $A = \text{Lie}(\mathcal{G})$ whose fiber at $x \in M$ coincides with the tangent space at the unit 1_x of the s -fiber at x . In other words:

$$A := T^s\mathcal{G}|_M.$$

Now we describe the Lie bracket of the Lie algebroid A . This is, in fact, a bracket on the space of sections $\Gamma(A)$, and to deduce it we will identify the space $\Gamma(A)$ with the space of right invariant vector fields on \mathcal{G} . To see this, we observe that the fiber of $T^s(\mathcal{G})$ at an arrow $h : y \rightarrow z$ is

$$T_h^s\mathcal{G} = T_h\mathcal{G}(y, -),$$

so, for any arrow $g : x \rightarrow y$, the differential of the right multiplication by g induces a map

$$R_g : T_h^s\mathcal{G} \rightarrow T_{hg}^s\mathcal{G}.$$

Hence, we can describe the space of right invariant vector fields on \mathcal{G} as:

$$\mathfrak{X}_{\text{inv}}^s(\mathcal{G}) = \{X \in \Gamma(T^s\mathcal{G}) : X_{hg} = R_g(X_h), \forall (h, g) \in \mathcal{G}_2\}.$$

Now, given $\alpha \in \Gamma(A)$, the formula

$$\tilde{\alpha}_g = R_g(\alpha_{t(g)})$$

clearly defines a right invariant vector field. Conversely, any vector field $X \in \mathfrak{X}_{\text{inv}}^s(\mathcal{G})$ arises in this way: the invariance of X shows that X is determined by its values at the points in M :

$$X_g = R_g(X_y), \quad \text{for all } g : x \rightarrow y,$$

i.e., $X = \tilde{\alpha}$ where $\alpha := X|_M \in \Gamma(A)$. Hence, we have shown that there exists an isomorphism

$$(1.20) \quad \Gamma(A) \xrightarrow{\sim} \mathfrak{X}_{\text{inv}}^s(\mathcal{G}), \quad \alpha \mapsto \tilde{\alpha}.$$

On the other hand, the space $\mathfrak{X}_{\text{inv}}^s(\mathcal{G})$ is a Lie subalgebra of the Lie algebra $\mathfrak{X}(\mathcal{G})$ of vector fields on \mathcal{G} with respect to the usual Lie bracket of vector fields. This is clear since the pull-back of vector fields on the s -fibers, along R_g , preserves brackets.

Definition 1.21. The **Lie bracket** on A is the Lie bracket on $\Gamma(A)$ obtained from the Lie bracket on $\mathfrak{X}_{\text{inv}}^s(\mathcal{G})$ under the isomorphism (1.20).

Hence this new bracket on $\Gamma(A)$, which we denote by $[\cdot, \cdot]_A$ (or simply $[\cdot, \cdot]$, when there is no danger of confusion) is uniquely determined by the formula:

$$(1.22) \quad [\widetilde{\alpha}, \widetilde{\beta}]_A = [\tilde{\alpha}, \tilde{\beta}].$$

To describe the entire structure underlying A , we need one more piece.

Definition 1.23. The **anchor map** of A is the bundle map

$$\rho_A : A \rightarrow TM$$

obtained by restricting $\text{dt} : T\mathcal{G} \rightarrow TM$ to $A \subset T\mathcal{G}$.

Again, when there is no danger of confusion (e.g., in the discussion below), we simply write ρ instead of ρ_A . The next proposition shows that the bracket and the anchor are related by a Leibniz-type identity. We use the notation \mathcal{L}_X for the Lie derivative along a vector field.

Proposition 1.24. *For all $\alpha, \beta \in \Gamma(A)$ and all $f \in C^\infty(M)$,*

$$[\alpha, f\beta] = f[\alpha, \beta] + \mathcal{L}_{\rho(\alpha)}(f)\beta.$$

Proof. We use the characterization of the bracket (1.22). First note that $\widetilde{f\beta} = (f \circ \mathbf{t})\tilde{\beta}$. From the usual Leibniz identity of vector fields (on \mathcal{G}),

$$\begin{aligned} [\widetilde{\alpha}, \widetilde{f\beta}] &= [\tilde{\alpha}, \widetilde{f\beta}] \\ &= [\tilde{\alpha}, (f \circ \mathbf{t})\tilde{\beta}] \\ &= (f \circ \mathbf{t})[\tilde{\alpha}, \tilde{\beta}] + \mathcal{L}_{\tilde{\alpha}}(f \circ \mathbf{t})\tilde{\beta}. \end{aligned}$$

But, at a point $g : x \rightarrow y$ in \mathcal{G} , we find:

$$\begin{aligned} \mathcal{L}_{\tilde{\alpha}}(f \circ \mathbf{t})(g) &= d_g(f \circ \mathbf{t})(\tilde{\alpha}(g)) \\ &= d_{\mathbf{t}(g)}f \circ d_g\mathbf{t}(\tilde{\alpha}(g)) = \mathcal{L}_{\rho(\alpha)}(f)(\mathbf{t}(g)). \end{aligned}$$

Hence, we conclude that:

$$[\widetilde{\alpha}, \widetilde{f\beta}] = \widetilde{f[\alpha, \beta]} + \mathcal{L}_{\rho(\alpha)}(\widetilde{f})\beta,$$

and the result follows. \square

We summarize this discussion in the following definition:

Definition 1.25. The **Lie algebroid of the Lie groupoid \mathcal{G}** is the vector bundle $A = \text{Lie}(\mathcal{G})$, together with the anchor $\rho_A : A \rightarrow TM$ and the Lie bracket $[\cdot, \cdot]_A$ on $\Gamma(A)$.

Exercise 9. For each example of a Lie groupoid furnished in the paragraph above, determine its Lie algebroid.

To complete our analogy with the Lie algebra of a Lie group, and introduce the exponential map of Lie groupoid, we set:

Definition 1.26. For $x \in M$, we put

$$\phi_\alpha^t(x) := \phi_{\tilde{\alpha}}^t(1_x) \in \mathcal{G}$$

where $\phi_{\tilde{\alpha}}^t$ is the flow of the right invariant vector field $\tilde{\alpha}$ induced by α . We call ϕ_α^t **the flow** of α .

Remark 1.27. The relevance of the flow comes from the fact that it provides the bridge between sections of A (infinitesimal data) and elements of \mathcal{G} (global data). In fact, building on the analogy with the exponential map of a Lie group and Remark 1.7, we see that the flow of sections can be interpreted as defining an **exponential map** $\exp : \Gamma(A) \rightarrow \Gamma(\mathcal{G})$ to the group of bisections of \mathcal{G} :

$$\exp(\alpha)(x) = \phi_\alpha^1(x).$$

This is defined provided α behaves well enough (e.g., if it has compact support). This shows that, in some sense, $\Gamma(A)$ is the Lie algebra of $\Gamma(\mathcal{G})$.

Using the exponential notation, we can write the flow as $\phi_\alpha^t = \exp(t\alpha)$. The next exercise will show that, just as for Lie groups, $\exp(t\alpha)$ contains all the information needed to recover the entire flow of $\tilde{\alpha}$.

Exercise 10. For $\alpha \in \Gamma(A)$, show that:

- (a) For all $y \in M$, $\phi_\alpha^t(y) : y \longrightarrow \phi_{\rho(\alpha)}^t(y)$.
 - (b) For all $g : x \longrightarrow y$ in \mathcal{G} , $\phi_\alpha^t(g) : x \longrightarrow \phi_{\rho(\alpha)}^t(y)$, and $\phi_\alpha^t(g) = \phi_\alpha^t(y)g$.
- (here, $\phi_{\rho(\alpha)}^t$ is the flow of the vector field $\rho(\alpha)$ on M).

1.5. Particular classes of groupoids

There are several particular classes of groupoids that deserve special attention, due to their relevance both in the theory and in various applications.

First of all, recall that a topological space X is called k -connected (where $k \geq 0$ is an integer) if $\pi_i(X)$ is trivial for all $0 \leq i \leq k$ and all base points.

Definition 1.28. A topological groupoid \mathcal{G} over a space M is called **source k -connected** if the s -fibers $\mathbf{s}^{-1}(x)$ are k -connected for every $x \in M$. When $k = 0$ we say that \mathcal{G} is a **s -connected groupoid**, and when $k = 1$ we say that \mathcal{G} is a **s -simply connected groupoid**.

Exercise 11. Show that a source n -connected Lie groupoid \mathcal{G} over a n -connected base M , has a space of arrows which is n -connected.

(Hint: Recall that for a Lie groupoid the source map $\mathbf{s} : \mathcal{G} \rightarrow M$ is a submersion.)

Most of the groupoids that we will meet in these lectures are source-connected. Any Lie groupoid \mathcal{G} has an associated source-connected groupoid \mathcal{G}^0 - the s -connected component of the identities. More precisely, $\mathcal{G}^0 \subset \mathcal{G}$ consists of those arrows $g : x \longrightarrow y$ of \mathcal{G} which are in the connected component of $G(x, -)$ containing 1_x .

Proposition 1.29. *For any Lie groupoid, \mathcal{G}^0 is an open subgroupoid of \mathcal{G} . Hence, \mathcal{G}^0 is a s -connected Lie groupoid that has the same Lie algebroid as \mathcal{G} .*

Proof. Note that right multiplication by an arrow $g : x \longrightarrow y$ is a homeomorphism from $\mathbf{s}^{-1}(y)$ to $\mathbf{s}^{-1}(x)$. Therefore, it maps connected components to connected components. If g belongs to the connected component of $\mathbf{s}^{-1}(x)$ containing 1_x , then right multiplication by g maps 1_y to g , so it maps the connected component of 1_y to the connected component of 1_x . Hence \mathcal{G}^0 is closed under multiplication. Moreover, g^{-1} is mapped to 1_x , so that g^{-1} belongs to the connected component of 1_y , and hence \mathcal{G}^0 is closed under inversion. This shows that \mathcal{G}^0 is a subgroupoid.

To check that \mathcal{G}^0 is open, it is enough to observe that there exists an open neighborhood U of the identity section $M \subset \mathcal{G}$ which is contained in \mathcal{G}^0 . To see that, observe that by the local normal form of submersions, each $1_x \in \mathcal{G}$ has an open neighborhood U_x which intersects each s -fiber in a connected set, i.e., $U_x \subset \mathcal{G}^0$. Then $U = \cup_{x \in M} U_x$ is the desired neighborhood. □

Exercise 12. Is this proposition still true for topological groupoids?

Analogous to simply connected Lie groups and their role in standard Lie theory, groupoids which are s -simply connected play a fundamental role in the Lie theory of Lie groupoids.

Theorem 1.30. *Let \mathcal{G} be a s -connected Lie groupoid. There exist a Lie groupoid $\tilde{\mathcal{G}}$ and homomorphism $\mathcal{F} : \tilde{\mathcal{G}} \rightarrow \mathcal{G}$ such that:*

- (i) $\tilde{\mathcal{G}}$ is s -simply connected.
- (ii) $\tilde{\mathcal{G}}$ and \mathcal{G} have the same Lie algebroid.
- (iii) \mathcal{F} is a local diffeomorphism.

Moreover, $\tilde{\mathcal{G}}$ is unique up to isomorphism.

Proof. We define the new groupoid $\tilde{\mathcal{G}}$ by letting $\tilde{\mathcal{G}}(x, -)$ be the universal cover of $\mathcal{G}(x, -)$ consisting of homotopy classes (with fixed end points) of paths starting at 1_x . Given $g : x \rightarrow y$, representing a homotopy class $[g] \in \tilde{\mathcal{G}}$, we set $\mathbf{s}([g]) = x$ and $\mathbf{t}([g]) = y$. Given $[g_1], [g_2] \in \tilde{\mathcal{G}}$ composable, we define $[g_1] \cdot [g_2]$ as the homotopy class of the concatenation of g_2 with $R_{g_2(1)} \circ g_1$ (draw a picture!).

It is easy to check that $\tilde{\mathcal{G}}$ is a groupoid. To describe its smooth structure, one remarks that $\tilde{\mathcal{G}} = p^{-1}(M)$, where $p : \Pi_1(\mathcal{F}(\mathbf{s})) \rightarrow \mathcal{G}$ is the source map of the fundamental groupoid of the foliation on \mathcal{G} by the fibers of \mathbf{s} . Since p is a submersion, $\tilde{\mathcal{G}}$ will be smooth and of the same dimension as \mathcal{G} . It is not difficult to check that the structure maps are also smooth. Moreover, the projection $\tilde{\mathcal{G}} \rightarrow \mathcal{G}$ which sends $[g]$ to $g(1)$ is clearly a surjective groupoid morphism (\mathcal{G} is \mathbf{s} -connected), which is easily seen to be a local diffeomorphism. Hence, it induces an isomorphism at the level of algebroids. We leave the proof of uniqueness as an exercise. \square

At the opposite extreme of s -connectedness there are the groupoids that model the leaf spaces of foliations, known as étale groupoids.

Definition 1.31. A Lie groupoid \mathcal{G} is called **étale** if its source map \mathbf{s} is a local diffeomorphism.

Example 1.32. Here are a few simple examples of étale groupoids:

- (1) The fundamental groupoid $\Pi_1(M)$ of any manifold M is always étale.
- (2) The fundamental groupoid of a foliation is not étale in general. However, let T be a complete transversal, i.e., an immersed submanifold which intersects every leaf, and is transversal at each intersection point (note that T does not have to be connected). Then, each holonomy transformation determines an arrow between points of T , and so they form a groupoid $\text{Hol}(T) \rightrightarrows T$, which is étale ⁽²⁾.
- (3) An action groupoid $\mathcal{G} = G \ltimes M$ is étale if and only if G is discrete. To see this, just notice that $\mathbf{s} : \mathcal{G} \rightarrow M$ is the projection $\mathbf{s}(g, m) = m$, so it is a local diffeomorphism iff G is discrete.

Exercise 13. Show that a Lie groupoid is étale iff its source fibers are discrete.

² The resulting groupoid is equivalent to the fundamental groupoid of the foliation, in a certain sense that can be made precise.

Analogous to compact Lie groups are proper Lie groupoids, which define another important class.

Definition 1.33. A Lie groupoid \mathcal{G} over M is called **proper** if the map $(s, t) : \mathcal{G} \rightarrow M \times M$ is a proper map.

We will see that, for proper groupoids, any representation admits an invariant metric. The following exercise asks you to prove some other basic properties of proper Lie groupoids.

Exercise 14. Let \mathcal{G} be a proper Lie groupoid over M . Show that:

- (a) All isotropy groups of \mathcal{G} are compact.
- (b) All orbits of \mathcal{G} are closed submanifolds.
- (c) The orbit space M/\mathcal{G} is Hausdorff.

Example 1.34. Let us list some examples of proper groupoids:

- (1) For a manifold M , the pair groupoid $M \times M$ is always proper. On the other hand, the fundamental groupoid $\Pi_1(M)$ is proper iff the fundamental groups of M are finite.
- (2) A Lie group G is proper (as a groupoid) if and only if it is compact.
- (3) An action Lie groupoid $G \ltimes M$ is proper if and only if the action of G on M is a proper action. Actually, this is just a matter of definitions, since a proper action is usually defined as one for which the map $G \times M \rightarrow M \times M$, $(g, m) \mapsto (m, gm)$ is a proper map.

Exercise 15. Check that groupoids of type $\text{GL}(E)$ are never proper. Choose a metric on the vector bundle E and define $O(E) \subset \text{GL}(E)$ to be the subgroupoid of isometries of the fibers. Show that $O(E)$ is proper.

We will say that a Lie groupoid \mathcal{G} is **source locally trivial** if the source map $s : \mathcal{G} \rightarrow M$ is a locally trivial fibration. This implies that the target map is also a locally trivial fibration. Note that there are examples of proper groupoids for which the source (or target) map is not a locally trivial fibration.

Exercise 16. Consider the foliation \mathcal{F} of $\mathbb{R}^2 - \{0\}$ given by horizontal lines. Show that $\Pi_1(\mathcal{F})$ is neither proper nor source locally trivial.

Exercise 17. Consider the foliation \mathcal{F} of $\mathbb{R}^3 - \{(x, 0, 0) : x \in [0, 1]\}$ given by spheres around the origin. Show that $\Pi_1(\mathcal{F})$ is proper but not source locally trivial.

Exercise 18. Show that a Lie groupoid which is both proper and étale is locally trivial.

Groupoids that are both proper and étale form a very important class of groupoids, since they serve as models for orbifolds.

1.6. Notes

According to Weinstein, the notion of a groupoid was discovered by Brandt [5] in the early twenty century, while studying quadratic forms over the integers. Groupoids were introduced into differential geometry by Ehresmann in the 1950's, and he also considered more general "structured categories"

(see the comments on his work in [22]). Already in Ehresmann's work one can find applications to foliations, fibered spaces, geometry of p.d.e.'s, etc. More or less at the same time, Grothendieck [28] advocated the use of groupoids in algebraic geometry as the right notion to understand moduli spaces (in the spirit of the introductory section in this lecture), and from that the theory of stacks emerged [3]. Important sources of examples, which strongly influenced the theory of Lie groupoids, comes from Haefliger's approach to transversal geometry of foliations, from Connes' noncommutative geometry [11] and from Poisson geometry (see the announcement [62] and the first systematic exposition of symplectic groupoids in [12]), with independent contributions from others (notably Karasëv [34] and Zakrzewski [72]).

The book of MacKenzie [37] contains a nice introduction to the theory of transitive and locally trivial groupoids, which has now been superseded by his new book [39], where he also treats non-transitive groupoids and double structures. The modern approach to groupoids can also be found in recent monographs, such as the book by Cannas da Silva and Weinstein [7] devoted to the geometry of noncommutative objects, the book by Moerdijk and Mrčun [44] on Lie groupoids and foliations, and the book by Dufour and Zung [20] on Poisson geometry.

Several important aspects of Lie groupoid theory have not made it yet to expository books. Let us mention as examples the theory of proper Lie groupoids (see, e.g., the papers by Crainic [13], Weinstein [65] and Zung [73]), the theory of differentiable stacks, gerbes and non-abelian cohomology (see, e.g., the preprints by Behrend and Xu [4] and Moerdijk [46]).

As time progresses, there are also a few aspects of Lie groupoid theory that seem to have lost interest or simply disappeared. We feel that some of these are worth recovering and we mentioned two examples. A prime example is provided by Haefliger's approach to integrability and homotopy theory (see the beautiful article [30]). Another remarkable example is the geometric approach to the theory of p.d.e.'s sketched by Ehresmann's, with roots in E. Cartan's pioneer works, and which essentially came to an halt with the monograph by Kumpera and Spencer [35].

LECTURE 2

Lie Algebroids

2.1. Why algebroids?

We saw in Lecture 1 that Lie groupoids are natural objects to study in differential geometry. Moreover, to every Lie groupoid \mathcal{G} over M there is associated a certain vector bundle $A \rightarrow M$, which carries additional structure, namely a Lie bracket on the sections and a bundle map $A \rightarrow TM$. If one axiomatizes these properties, one obtains the abstract notion of a Lie algebroid:

Definition 2.1. A **Lie algebroid** over a manifold M consists of a vector bundle A together with a bundle map $\rho_A : A \rightarrow TM$ and a Lie bracket $[\cdot, \cdot]_A$ on the space of sections $\Gamma(A)$, satisfying the Leibniz identity

$$[\alpha, f\beta]_A = f[\alpha, \beta]_A + \mathcal{L}_{\rho_A(\alpha)}(f)\beta,$$

for all $\alpha, \beta \in \Gamma(A)$ and all $f \in C^\infty(M)$.

Exercise 19. Prove that the induced map $\rho_A : \Gamma(A) \rightarrow \mathfrak{X}(M)$ is a Lie algebra homomorphism.

Before we plug into the study of Lie algebroids, we would like to show that Lie algebroids are interesting by themselves, independently of Lie groupoids. In fact, in geometry one is led naturally to study Lie algebroids and, in their study, it is useful to integrate them to Lie groupoids!

In order to illustrate this point, just like we did at the beginning of Lecture 1, we look again at equivalence problems in geometry. Élie Cartan observed that many equivalence problems in geometry can be best formulated in terms of coframes. Working out the coframe formulation, he was able to come up with a method, now called Cartan's equivalence method, to deal with such problems.

A local version of Cartan's formulation of equivalence problems can be described as follows: one is given a family of functions $c_{j,k}^i, b_i^a$ defined on some open set $U \subset \mathbb{R}^n$, where the indices satisfy $1 \leq i, j, k \leq r, 1 \leq a \leq n$ (n, r positive integers). Then the problem is:

Cartan's problem: *find a manifold N , a coframe $\{\eta^i\}$ on N , and a function $h : N \rightarrow U$, satisfying the equations:*

$$(2.2) \quad d\eta^i = \sum_{j,k} c_{j,k}^i(h) \eta^j \wedge \eta^k,$$

$$(2.3) \quad dh^a = \sum_i b_i^a(h) \eta^i.$$

As part of this problem we should be able to answer the following questions:

- When does Cartan's problem have a solution?
- What are the possible solutions to Cartan's problem?

Here is a simple, but interesting, illustrative example:

Example 2.4. Let us consider the problem of equivalence of metrics in \mathbb{R}^2 with constant Gaussian curvature. If ds^2 is a metric in \mathbb{R}^2 then there exists a diagonalizing coframe $\{\eta^1, \eta^2\}$, so that:

$$ds^2 = (\eta^1)^2 + (\eta^2)^2.$$

In terms of Cartesian coordinates, this coframe can be written as:

$$\eta^1 = A dx + B dy, \quad \eta^2 = C dx + D dy,$$

where A, B, C, D are smooth functions of (x, y) satisfying $AD - BC \neq 0$. Taking exterior derivatives, we obtain:

$$(2.5) \quad d\eta^1 = J \eta^1 \wedge \eta^2, \quad d\eta^2 = K \eta^1 \wedge \eta^2,$$

where:

$$J = \frac{B_x - A_y}{AD - BC}, \quad K = \frac{D_x - C_y}{AD - BC}.$$

If f is any smooth function in \mathbb{R}^2 , the coframe derivatives of f are defined to be the coefficients of the differential df when expressed in terms of the coframe:

$$df = \frac{\partial f}{\partial \eta^1} \eta^1 + \frac{\partial f}{\partial \eta^2} \eta^2.$$

For example, the Gaussian curvature of ds^2 is given in terms of the coframe derivatives of the structure functions by:

$$\kappa = \frac{\partial J}{\partial \eta^2} - \frac{\partial K}{\partial \eta^1} - J^2 - K^2$$

So we see that for metrics of constant Gaussian curvature, the two structure functions J and K are not independent, and we can choose one of them as the independent one, say J . Then we can write:

$$(2.6) \quad dJ = L \eta^1 \wedge \eta^2.$$

Equations (2.5) and (2.6) form the structure equations of the Cartan problem of classifying metrics of constant Gaussian curvature. Using these equations one can find normal forms for all metrics of constant Gaussian curvature.

Exercise 20. Show that the metrics of zero Gaussian curvature ($\kappa = 0$) can be reduced to the following form:

$$ds^2 = dx^2 + dy^2.$$

Obvious *necessary conditions* to solve Cartan's problem can be obtained as immediate consequences of the fact that $d^2 = 0$ and that $\{\eta^i\}$ is a coframe. In fact, a simple computation gives⁽¹⁾:

$$(2.7) \quad F_i^b(h) \frac{\partial F_j^a}{\partial h^b}(h) - F_j^b(h) \frac{\partial F_i^a}{\partial h^b}(h) = -c_{i,j}^l(h) F_l^a(h),$$

$$(2.8) \quad F_j^a(h) \frac{\partial c_{k,l}^i}{\partial h^a}(h) + F_k^a(h) \frac{\partial c_{l,j}^i}{\partial h^a}(h) + F_l^a(h) \frac{\partial c_{j,k}^i}{\partial h^a}(h) = \\ = -(c_{m,j}^i(h) c_{k,l}^m(h) + c_{mk}^i(h) c_{l,j}^m(h) + c_{m,l}^i(h) c_{j,k}^m(h)).$$

Exercise 21. Take exterior derivatives of the structure equations (2.2) and (2.3), and deduce the relations (2.7) and (2.8).

Cartan's problem is extremely relevant and suggestive already in the case where the structure functions $c_{j,k}^i$ are constants (hence no h 's appear in the problem). In this case, condition (2.7) is vacuous, while condition (2.8) is the usual Jacobi identity. Therefore, these conditions precisely mean that the $c_{j,k}^i$'s define an (r -dimensional) Lie algebra. Let us call this Lie algebra \mathfrak{g} , so that on a preferred basis $\{e_i\}$ we have:

$$(2.9) \quad [e_j, e_k] = c_{j,k}^i e_i$$

Moreover, a family $\{\eta^i\}$ of one-forms on N can be viewed as a \mathfrak{g} -valued form on N , $\eta = \eta^i e_i \in \Omega^1(N; \mathfrak{g})$, and the equations take the following global form:

$$(2.10) \quad d\eta + \frac{1}{2}[\eta, \eta] = 0.$$

This is just the well known Maurer-Cartan equation! It is the basic equation satisfied by the Maurer-Cartan form of a Lie group. In other words, if G is the (unique) simply connected Lie group integrating \mathfrak{g} , then the “tautological” one-form $\eta_{MC} \in \Omega^1(G; \mathfrak{g})$ is a solution of the Maurer-Cartan equation. And it is a very special (“universal”) one since, by a well known result in differential geometry, it classifies all the solutions to Cartan's problem:

Lemma 2.11. *Any one-form $\eta \in \Omega^1(N; \mathfrak{g})$ on a simply connected manifold N , satisfying the Maurer-Cartan equation (2.10) is of type $\eta = f^* \eta_{MC}$ for some smooth map $f : N \rightarrow G$, which is unique up to conjugation by an element in G .*

It is worth keeping in mind what we have actually done to “solve” the original equations: the obvious necessary conditions (equation (2.8), in this case) put us into the context of Lie algebras, then we *integrate* the Lie algebra \mathfrak{g} to G , and finally we pick out the Maurer Cartan form as the solution we were looking for. Moreover, to prove its universality (see the Lemma), we had to produce a map $f : N \rightarrow G$ out of a form $\eta \in \Omega^1(N; \mathfrak{g})$. Viewing η as a map $\eta : TN \rightarrow \mathfrak{g}$, we may say that we have “integrated η ”.

Now what if we allow the h into the picture, and try to extend this piece of basic geometry to the general equations? This leads us immediately to the world of algebroids! Since the structure functions $c_{j,k}^i$ are no longer

¹We will be using the Einstein's summation convention without further notice.

constant (they depend on $h \in U$), the brackets they define (2.9) make sense provided $\{e_i\}$ depend themselves on h . Of course, $\{e_i\}$ can then be viewed as trivializing sections of an r -dimensional vector bundle A over U . On the other hand, the functions b_i^a are the components of a map:

$$\Gamma(A) \ni e_i \longmapsto b_i^a \frac{\partial}{\partial x^a} \in \mathfrak{X}(U).$$

The necessary conditions (2.7) and (2.8) are just the conditions that appear in the definition of a Lie algebroid. This is the content of the next exercise.

Exercise 22. Let A be a Lie algebroid over a manifold M . Pick a contractible open coordinate neighborhood $U \subset M$, with coordinates (x^a) and a basis of sections $\{e_i\}$ that trivialize the bundle $A|_U$. Also, define structure functions $c_{j,k}^i$ and b_i^a by:

$$[e_j, e_k] = c_{j,k}^i e_i, \quad \rho(e_i) = b_i^a \frac{\partial}{\partial x^a}.$$

Check that equation (2.7) is equivalent to the condition that ρ is a Lie algebra homomorphism:

$$[\rho(e_j), \rho(e_k)] = \rho([e_j, e_k]),$$

and that equation (2.8) is equivalent to the Jacobi identity:

$$[e_i, [e_j, e_k]] + [e_j, [e_k, e_i]] + [e_k, [e_i, e_j]] = 0.$$

What about the associated global objects and the associated Maurer Cartan forms? Our objective here is not to give a detailed discussion of Cartan's problem, but it is not hard to guess that, in the end, you will find Lie algebroid valued forms and that you will need to integrate the Lie algebroid, eventually rediscovering the notion of a *Lie groupoid*.

After this motivating example, it is now time to start our study of Lie algebroids.

2.2. Lie algebroids

Let A be a Lie algebroid over M . We start by looking at the kernel and at the image of the anchor $\rho : A \rightarrow TM$.

If we fix $x \in M$, the kernel $\text{Ker}(\rho_x)$ is naturally a Lie algebra: if $\alpha, \beta \in \Gamma(A)$ lie in $\text{Ker}(\rho_x)$ when evaluated at x , the Leibniz identity implies that $[\alpha, f\beta](x) = f(x)[\alpha, \beta](x)$. Hence, there is a well defined bracket on $\text{Ker}(\rho_x)$ such that

$$[\alpha, \beta](x) = [\alpha(x), \beta(x)],$$

for α and β as above.

Definition 2.12. At any point $x \in M$ the Lie algebra

$$\mathfrak{g}_x(A) := \text{Ker}(\rho_x),$$

is called the **isotropy Lie algebra** at x .

Exercise 23. Let \mathcal{G} be a s -connected Lie groupoid over M and let A be its Lie algebroid. Show that the isotropy Lie algebra $\mathfrak{g}_x(A)$ is isomorphic to the Lie algebra of the isotropy Lie group \mathcal{G}_x , for all $x \in M$.

Let us now look at the image of the anchor. This gives a distribution

$$M \ni x \mapsto \text{Im}(\rho_x) \subset T_x M,$$

of subspaces whose dimension, in general, will vary from point to point. If the rank of ρ is constant we say that A is a *regular Lie algebroid*.

Exercise 24. If A is a regular Lie algebroid show that the resulting distribution is integrable, so that M is foliated by immersed submanifolds \mathcal{O} 's, called *orbits*, satisfying $T_x \mathcal{O} = \text{Im}(\rho_x)$, for all $x \in \mathcal{O}$.

For a general Lie algebroid A there is still an induced partition of M by immersed submanifolds, called the *orbits* of A . As in the regular case, one looks for maximal immersed submanifolds \mathcal{O} 's satisfying:

$$T_x \mathcal{O} = \text{Im}(\rho_x),$$

for all $x \in \mathcal{O}$. Their existence is a bit more delicate in the non-regular case. One possibility is to use a Frobenius type theorem for singular foliations. However, there is a simple way of describing the orbits, at least set theoretically. This uses the notion of A -path, which will play a central role in the integrability problem to be studied in the next lecture.

Definition 2.13. Given a Lie algebroid A over M , an A -**path** consists of a pair (a, γ) where $\gamma : I \rightarrow M$ is a path in M , $a : I \rightarrow A$ is a path in $A^{(2)}$, such that

- (i) a is a path above γ , i.e., $a(t) \in A_{\gamma(t)}$ for all $t \in I$.
- (ii) $\rho(a(t)) = \frac{d\gamma}{dt}(t)$, for all $t \in I$.

Of course, a determines the base path γ , hence when talking about an A -path we will only refer to a . The reason we mention γ in the previous definition is to emphasize the way we think of A -paths: a should be interpreted as an “ A -derivative of γ ”, and the last condition in the definition should be read: “the usual derivative of γ is related to the A -derivative by the anchor map”.

We can now define an equivalence relation on M , denoted \sim_A , as follows. We say that $x, y \in M$ are equivalent if there exists an A -path a , with base path γ , such that $\gamma(0) = x$ and $\gamma(1) = y$. An equivalence class of this relation will be called an **orbit of A** . When ρ is surjective we say that A is a **transitive Lie algebroid**. In this case, each connected component of M is an orbit of A .

Remark 2.14. It remains to show that each orbit \mathcal{O} is an immersed submanifold of M , which integrates $\text{Im}(\rho)$, i.e., $T_x \mathcal{O} = \text{Im}(\rho_x)$ for all $x \in \mathcal{O}$. This can be proved using a local normal form theorem for Lie algebroids.

Exercise 25. Let \mathcal{G} be a s -connected Lie groupoid over M and let A be its Lie algebroid. Show that the orbits of \mathcal{G} in M coincide with the orbits of A in M .

(Hint: Check that if $g(t) : I \rightarrow \mathcal{G}$ is a path that stays in a s -fiber and starts at 1_x , then $a(t) = d_{g(t)} R_{g(t)^{-1}} \cdot \dot{g}(t)$ is an A -path.)

²Here and below, $I = [0, 1]$ will always denote the unit interval.

As we have seen in the previous lecture, any Lie groupoid has an associated Lie algebroid. For future reference, we introduce the following terminology:

Definition 2.15. A Lie algebroid A is called **integrable** if it is isomorphic to the Lie algebroid of a Lie groupoid \mathcal{G} . For such a \mathcal{G} , we say that \mathcal{G} integrates A .

Similar to Lie's first theorem for Lie algebras (which asserts that there is at most one simply-connected Lie group integrating a given Lie algebra), we have the following theorem:

Theorem 2.16 (Lie I). *If A is integrable, then there exists a unique (up to isomorphism) s -simply connected Lie groupoid \mathcal{G} integrating A .*

Proof. This follows at once from the previous lecture, where we have shown that for any Lie groupoid \mathcal{G} there exists a unique (up to isomorphism) Lie groupoid $\tilde{\mathcal{G}}$ which is s -simply connected and which has the same Lie algebroid as \mathcal{G} (see Theorem 1.30). \square

Let us turn now to morphisms of Lie algebroids.

Definition 2.17. Let $A_1 \rightarrow M_1$ and $A_2 \rightarrow M_2$ be Lie algebroids. A **morphism of Lie algebroids** is a vector bundle map

$$\begin{array}{ccc} A_1 & \xrightarrow{F} & A_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

which is compatible with the anchors and the brackets.

Let us explain what we mean by *compatible*. First of all, we say that the map $F : A_1 \rightarrow A_2$ is compatible with the anchors if

$$df(\rho_{A_1}(a)) = \rho_{A_2}(F(a)).$$

This can be expressed by the commutativity of the diagram:

$$\begin{array}{ccc} A_1 & \xrightarrow{F} & A_2 \\ \rho_1 \downarrow & & \downarrow \rho_2 \\ TM_1 & \xrightarrow{df} & TM_2 \end{array}$$

Secondly, we would like to say what we mean by compatibility with the brackets. The difficulty here is that, in general, sections of A_1 cannot be pushed forward to sections of A_2 . Instead we have to work at the level of the pull-back bundle f^*A_2 ⁽³⁾. First note that from sections α of A_1 or α' of A_2 , we can produce new sections $F(\alpha)$ and $f^*(\alpha')$ of f^*A_2 by:

$$F(\alpha) = F \circ \alpha, \quad f^*(\alpha') = \alpha' \circ f.$$

³an alternative, more intrinsic definition, will be described in Exercise 48

Now, given any section $\alpha \in \Gamma(A_1)$, we can express its image under F as a (non-unique) finite combination

$$F(\alpha) = \sum_i c_i f^*(\alpha_i),$$

where $c_i \in C^\infty(M_1)$ and $\alpha_i \in \Gamma(A_2)$. By compatibility with the brackets we mean that, if $\alpha, \beta \in \Gamma(A_1)$ are sections such that their images are expressed as finite combinations as above, then their bracket is a section whose image can be expressed as:

$$(2.18) \quad F([\alpha, \beta]_{A_1}) = \sum_{i,j} c_i c'_j f^*[\alpha_i, \beta_j]_{A_2} + \sum_j \mathcal{L}_{\rho(\alpha)}(c'_j) f^*(\beta_j) - \sum_i \mathcal{L}_{\rho(\beta)}(c_i) f^*(\alpha_i).$$

Notice that, in the case where the sections $\alpha, \beta \in \Gamma(A_1)$ can be pushed forward to sections $\alpha', \beta' \in \Gamma(A_2)$, so that $F(\alpha) = \alpha' \circ f$ and $F(\beta) = \beta' \circ f$, this just means that:

$$F([\alpha, \beta]_{A_1}) = [\alpha', \beta']_{A_2} \circ f.$$

The following exercises should help make you familiar with the notion of a Lie algebroid morphism.

Exercise 26. Check that condition (2.18) is independent of the way one expresses the image of the sections under F as finite combinations.

Exercise 27. Let A be a Lie algebroid. Show that a path $a : [0, 1] \rightarrow A$ is an A -path iff the map $adt : TI \rightarrow A$ is a morphism of Lie algebroids.

Exercise 28. Show that if $\mathcal{F} : \mathcal{G} \rightarrow \mathcal{H}$ is a homomorphism of Lie groupoids, then it induces a Lie algebroid homomorphism $F : A \rightarrow B$ of their Lie algebroids.

Again, just like in the case of Lie algebras, under a suitable assumption, we can integrate morphisms of Lie algebroids to morphisms of Lie groupoids:

Theorem 2.19 (Lie II). *Let $F : A \rightarrow B$ be a morphism of integrable Lie algebroids, and let \mathcal{G} and \mathcal{H} be integrations of A and B . If \mathcal{G} is \mathfrak{s} -simply connected, then there exists a (unique) morphism of Lie groupoids $\mathcal{F} : \mathcal{G} \rightarrow \mathcal{H}$ integrating F .*

You may try to reproduce the proof that you know for the Lie algebra case. You will also find a proof in the next lecture (see Exercise 63). The next exercise shows how to define the exponential map for Lie algebroids/groupoids.

Exercise 29. Let A be the Lie algebroid of a Lie groupoid \mathcal{G} . Use Lie II to define the exponential map $\exp : \Gamma_c(A) \rightarrow \Gamma(\mathcal{G})$, taking sections of compact support to bisections of \mathcal{G} . How does this relate to the construction hinted at in Remark 1.27?

2.3. First examples of Lie algebroids

Let us present now a few basic examples of Lie algebroids.

Example 2.20 (tangent bundles). One of the extreme examples of a Lie algebroid over M is the tangent bundle $A = TM$, with the identity map as anchor, and the usual Lie bracket of vector fields. Here the isotropy Lie algebras are trivial and the Lie algebroid is transitive.

Exercise 30. Prove that the Lie algebroids of the pair groupoid $M \times M$ and of the fundamental groupoid $\Pi_1(M)$ are both isomorphic to TM .

Example 2.21 (Lie algebras). At the other extreme, any Lie algebra \mathfrak{g} is a Lie algebroid over a singleton. Here there is only one isotropy Lie algebra which coincides with \mathfrak{g} . Obviously, any Lie group with Lie algebra \mathfrak{g} gives a Lie groupoid integrating A .

Both these examples can be slightly generalized. For example, the tangent bundle can be generalized as follows:

Example 2.22 (foliations). Let $A \subset TM$ be an involutive subbundle, i.e., constant rank smooth distribution which is closed for the usual Lie bracket. This gives a Lie algebroid (in fact a Lie subalgebroid of TM) over M , with anchor map the inclusion, and the Lie bracket the restriction of the usual Lie bracket of vector fields.

Recall that, by the Frobenius Integrability Theorem, A determines a foliation \mathcal{F} of a manifold M (and conversely, every foliation determines a Lie algebroid $T\mathcal{F} \subset TM$). In fact, \mathcal{F} is just the orbit foliation of A (see the discussion above). On the other hand, since the anchor is injective, the isotropy Lie algebras \mathfrak{g}_x are all trivial.

Exercise 31. Prove that the Lie algebroid of the fundamental groupoid $\Pi_1(\mathcal{F})$ is isomorphic to $T\mathcal{F}$. Can you give another example of a groupoid integrating $T\mathcal{F}$?

On the other hand, Lie algebras can be generalized as follows:

Example 2.23 (bundles of Lie algebras). A bundle of Lie algebras over M is a vector bundle A over M together with a Lie algebra bracket $[\cdot, \cdot]_x$ on each fiber A_x , which varies smoothly with respect to x in the sense that if $\alpha, \beta \in \Gamma(A)$, then $[\alpha, \beta]$ defined by

$$[\alpha, \beta](x) = [\alpha(x), \beta(x)]_x$$

is a smooth section of A . Note that this notion is weaker than that of Lie algebra bundle, when one requires that A is locally trivial as a bundle of Lie algebras (in particular, all the Lie algebras A_x should be isomorphic).

Since the anchor is identically zero, the orbits of A are the points of M , while the isotropy Lie algebras are the fibers A_x . It is easy to see that a bundle of Lie algebras over M is precisely the same thing as a Lie algebroid over M with zero anchor map.

A general Lie algebroid can be seen as combining aspects from both the previous two examples. In fact, take a Lie algebroid A over M and fix an orbit $i : \mathcal{O} \hookrightarrow M$. In the following exercise we ask you to check that the bracket restricts to a bracket on $A_{\mathcal{O}} = A|_{\mathcal{O}} := i^*A$.

Exercise 32. Let $\alpha, \beta \in \Gamma(A_{\mathcal{O}})$ be local sections, and $\tilde{\alpha}, \tilde{\beta} \in \Gamma(A)$ be any choice of local extensions. Show that:

$$[\alpha, \beta] := [\tilde{\alpha}, \tilde{\beta}]|_{\mathcal{O}},$$

is well-defined, i.e., it is independent of the choice of extensions.

(Hint: Since \mathcal{O} is a leaf, the vector fields $\rho(\alpha)$ and $\rho(\beta)$ are tangent to \mathcal{O} .)

Since the restriction of the anchor gives a map $\rho_{\mathcal{O}} : A_{\mathcal{O}} \rightarrow T\mathcal{O}$, we obtain a Lie algebroid structure on $A_{\mathcal{O}}$ over \mathcal{O} . On the other hand, one has an induced bundle of Lie algebras:

$$\mathfrak{g}_{\mathcal{O}}(A) = \text{Ker}(\rho_{\mathcal{O}}),$$

whose fiber at x is the isotropy Lie algebra $\mathfrak{g}_x(A)$. All these fit into a short exact sequence of algebroids over \mathcal{O} :

$$0 \longrightarrow \mathfrak{g}_{\mathcal{O}}(A) \longrightarrow A_{\mathcal{O}} \longrightarrow T\mathcal{O} \longrightarrow 0.$$

The isotropy bundle $\mathfrak{g}_{\mathcal{O}}(A)$ is a Lie algebra bundle, as indicated in the following exercise.

Exercise 33. (a) Show that a bundle of Lie algebras $\mathfrak{g}_M \rightarrow M$ is a Lie algebra bundle iff there exists a connection $\nabla : \mathfrak{X}(M) \otimes \Gamma(\mathfrak{g}_M) \rightarrow \Gamma(\mathfrak{g}_M)$ satisfying

$$\nabla_X([\alpha, \beta]) = [\nabla_X(\alpha), \beta] + [\alpha, \nabla_X(\beta)]$$

for all $\alpha, \beta \in \Gamma(\mathfrak{g}_M)$.

(b) Deduce that each isotropy bundle $\mathfrak{g}_{\mathcal{O}}(A)$ is a Lie algebra bundle. In particular, all the isotropy Lie algebras $\mathfrak{g}_x(A)$, with $x \in \mathcal{O}$, are isomorphic. (Hint: for the first part, use parallel transport. For the second part, use a splitting of $\rho_{\mathcal{O}} : A|_{\mathcal{O}} \rightarrow T\mathcal{O}$ and the bracket of $A_{\mathcal{O}}$ to produce a connection).

Example 2.24 (vector fields). It is not difficult to see that Lie algebroid structures on the trivial line bundle over M are in 1-1 correspondence with vector fields on M . Given a vector field X , we denote by A_X the induced Lie algebroid. Explicitly, as a vector bundle, $A_X = \mathbb{L} = M \times \mathbb{R}$ is the trivial line bundle, while the anchor is given by multiplication by X , and the Lie bracket of two section $f, g \in \Gamma(A_X) = C^\infty(M)$ is defined by:

$$[f, g] = f\mathcal{L}_X(g) - \mathcal{L}_X(f)g.$$

Exercise 34. Find out how the flow of X defines an integration of A_X .

Example 2.25 (action Lie algebroid). Generalizing the Lie algebroid of a vector field, consider an infinitesimal action of a Lie algebra \mathfrak{g} on a manifold M , i.e., a Lie algebra homomorphism $\rho : \mathfrak{g} \rightarrow \mathfrak{X}(M)$. The standard situation is when \mathfrak{g} is the Lie algebra of a Lie group G that acts on M . Then

$$\rho(v)(x) := \frac{d}{dt} \exp(tv)x, \quad (v \in \mathfrak{g}, x \in M)$$

defines an infinitesimal action of \mathfrak{g} on M . We say that an infinitesimal action of \mathfrak{g} on M is integrable if it comes from a Lie group action.

Given an infinitesimal action of \mathfrak{g} on M , we form a Lie algebroid $\mathfrak{g} \times M$, called the action Lie algebroid, as follows. As a vector bundle, it is the trivial vector bundle $M \times \mathfrak{g}$ over M with fiber \mathfrak{g} , the anchor is given by

the infinitesimal action, while the Lie bracket is uniquely determined by the Leibniz identity and the condition that

$$[c_v, c_w] = c_{[v,w]},$$

for all $v, w \in \mathfrak{g}$, where c_v denotes the constant section of \mathfrak{g} .

Exercise 35. This exercise discusses the relationship between the integrability of Lie algebra actions and the corresponding action Lie algebroid:

- (a) Given an action of a Lie group G on M , show that the corresponding action Lie groupoid $G \ltimes M$ over M has Lie algebroid the action Lie algebroid $\mathfrak{g} \ltimes M$. Hence, if an infinitesimal action of \mathfrak{g} on M is integrable, then the Lie algebroid $\mathfrak{g} \ltimes M$ is integrable.
- (b) Find an infinitesimal action of a Lie algebra \mathfrak{g} which is not integrable but which has the property that the Lie algebroid $\mathfrak{g} \ltimes M$ is integrable. (Hint: think of vector fields!)

Exercise 36. Show that *all* action Lie algebroids $\mathfrak{g} \ltimes M$, arising from infinitesimal Lie algebra actions are integrable.

(Hint: think again of what happens for vector fields!).

Example 2.26 (Two forms). Any closed 2-form ω on a manifold M has an associated Lie algebroid, denoted A_ω , and defined as follows. As a vector bundle,

$$A = TM \oplus \mathbb{L},$$

the anchor is the projection on the first component, while the bracket on sections $\Gamma(A_\omega) \simeq \mathfrak{X}(M) \times C^\infty(M)$ is defined by:

$$[(X, f), (Y, g)] = ([X, Y], \mathcal{L}_X(g) - \mathcal{L}_Y(f) + \omega(X, Y)).$$

Exercise 37. Given a 2-form on M , check that the previous formulas make A_ω into a Lie algebroid if and only if ω is closed.

Example 2.27 (Atiyah sequences). Let G be a Lie group. To any principal G -bundle P over M there is an associated Lie algebroid over M , denoted $A(P)$, and defined as follows. As a vector bundle, $A(P) := TP/G$ (over $P/G = M$). The anchor is induced by the differential of the projection from P to M . Also, since the sections of $A(P)$ correspond to G -invariant vector fields on P , we see that there is a canonical Lie bracket on $\Gamma(A(P))$. With these, $A(P)$ becomes a Lie algebroid.

Exercise 38. Show that $A(P)$ is just the Lie algebroid of the gauge groupoid $P \otimes_G P$.

Note that $A(P)$ is transitive, i.e., the anchor map is surjective. We denote by $P[\mathfrak{g}]$ the kernel of the anchor map. Hence we have a short exact sequence

$$0 \longrightarrow P[\mathfrak{g}] \longrightarrow A(P) \longrightarrow TM \longrightarrow 0,$$

known as the Atiyah sequence associated to P . Of course, this is obtained by dividing out the action of G on the exact sequence of vector bundles over P : $P \times \mathfrak{g} \longrightarrow TP \longrightarrow TM$. In particular, $P[\mathfrak{g}]$ is the Lie algebra bundle obtained by attaching to P the adjoint representation of G .

One of the interesting things about the Atiyah sequence is its relation with connections and their curvatures. First of all, connections on the principal bundle P are the same thing as splittings of the Atiyah sequence. Indeed, a left splitting of the sequence is the same thing as a bundle map $TP \rightarrow P \times \mathfrak{g}$ which is G -invariant, and which is left inverse to the infinitesimal action $\mathfrak{g} \rightarrow TP$, in other words, a connection 1-form. By standard linear algebra, left splittings of a short exact sequence are in 1-1 correspondence with right splittings. Henceforth, we will identify connections with left/right splittings.

Next, given a connection whose associated right splitting is denoted by $\sigma : TM \rightarrow A(P)$, the curvature of the connection is a 2-form on M with values in $P[\mathfrak{g}]$, and can be described using σ and the Lie bracket on $A(P)$:

$$\Omega_\sigma(X, Y) = [\sigma(X), \sigma(Y)] - \sigma([X, Y]).$$

Using these as motivations, transitive (i.e., with surjective anchor) Lie algebroids A are also called abstract Atiyah sequence, while splittings of their anchor map are called connections. The sequence associated to A is, of course:

$$0 \longrightarrow \text{Ker}(\rho) \longrightarrow A \longrightarrow TM \longrightarrow 0.$$

Note that not all abstract Atiyah sequences come from principal bundles.

Exercise 39. Give an example of a transitive Lie algebroid which is not the Atiyah sequence of some principal bundle.

Example 2.28 (Poisson manifolds). Any Poisson structure on a manifold M induces a Lie algebroid structure on T^*M as follows. Let π be the Poisson bivector on M , which is related to the Poisson bracket by $\{f, g\} = \pi(df, dg)$. Also, the hamiltonian vector field X_f associated to a smooth function f on M is given by $X_f(g) = \{f, g\}$. We use the notation

$$\pi^\sharp : T^*M \rightarrow TM$$

for the map defined by $\beta(\pi^\sharp(\alpha)) = \pi(\alpha, \beta)$. The Lie algebroid structure on T^*M has π^\sharp as anchor map, and the Lie bracket is defined by

$$[\alpha, \beta] = \mathcal{L}_{\pi^\sharp(\alpha)}(\beta) - \mathcal{L}_{\pi^\sharp(\beta)}(\alpha) - d(\pi(\alpha, \beta)).$$

We call this the **cotangent Lie algebroid** of the Poisson manifold (M, π) .

Exercise 40. Show that this Lie algebroid structure on T^*M is the unique one with the property that the anchor maps df to X_f and $[df, dg] = d\{f, g\}$, for all $f, g \in C^\infty(M)$.

Example 2.29 (Nijenhuis tensors). Given a bundle map $\mathcal{N} : TM \rightarrow TM$, recall that its Nijenhuis torsion, denoted $T_{\mathcal{N}} \in \Gamma(\wedge^2 T^*M \otimes TM)$, is defined by

$$T_{\mathcal{N}}(X, Y) = [\mathcal{N}X, \mathcal{N}Y] - \mathcal{N}[\mathcal{N}X, Y] - \mathcal{N}[X, \mathcal{N}Y] + \mathcal{N}^2[X, Y],$$

for $X, Y \in \mathfrak{X}(M)$. When $T_{\mathcal{N}} = 0$ we call \mathcal{N} a **Nijenhuis tensor**. To any Nijenhuis tensor \mathcal{N} , there is associated a new Lie algebroid structure on TM : the anchor is given by $\rho(X) = \mathcal{N}(X)$, while the Lie bracket is defined by

$$[X, Y]_{\mathcal{N}} := [\mathcal{N}X, Y] + [X, \mathcal{N}Y] - \mathcal{N}([X, Y]).$$

This kind of Lie algebroid plays an important role in the theory of integrable systems, in the study of (generalized) complex structures, etc.

Exercise 41. Thinking of a Lie algebroid as a generalized tangent bundle, extend this construction to any Lie algebroid $A \rightarrow M$: given a bundle map $\mathcal{N} : A \rightarrow A$ over the identity, such that its Nijenhuis torsion vanishes, i.e.,

$$[\mathcal{N}\alpha, \mathcal{N}\beta]_A - \mathcal{N}[\mathcal{N}\alpha, \beta]_A - \mathcal{N}[\alpha, \mathcal{N}\beta]_A + \mathcal{N}^2[\alpha, \beta]_A = 0,$$

for all $\alpha, \beta \in \Gamma(A)$, show that there exists a new Lie algebroid $A_{\mathcal{N}}$ associated with \mathcal{N} .

2.4. Connections

Concepts in Lie algebroid theory arise often as generalizations of standard notions both of Lie theory and/or of differential geometry. This is related with the two extreme examples of Lie algebroids: Lie algebras and tangent bundles. This dichotomy affects also notation and terminology as we will see now.

Having the example of the tangent bundle in mind, let us interpret a general Lie algebroid as describing a space of “generalized vector fields on M ” (A -fields), so that the anchor map relates these generalized vector fields to the usual vector fields. This leads immediately to the following notion of connection:

Definition 2.30. Given a Lie algebroid A over M and a vector bundle E over M , an A -**connection** on E is a bilinear map

$$\begin{aligned} \Gamma(A) \times \Gamma(E) &\rightarrow \Gamma(E), \\ (\alpha, s) &\mapsto \nabla_{\alpha}(s), \end{aligned}$$

which is $C^{\infty}(M)$ linear on α , and which satisfies the following Leibniz rule with respect to s :

$$\nabla_{\alpha}(fs) = f\nabla_{\alpha}(s) + \mathcal{L}_{\rho(\alpha)}(f)s.$$

The **curvature** of the connection is the map:

$$\begin{aligned} R_{\nabla} : \Gamma(A) \times \Gamma(A) &\rightarrow \text{Hom}(\Gamma(E), \Gamma(E)), \\ R_{\nabla}(\alpha, \beta)(X) &= \nabla_{\alpha}\nabla_{\beta}X - \nabla_{\beta}\nabla_{\alpha}X - \nabla_{[\alpha, \beta]}X. \end{aligned}$$

The connection is called **flat** if $R_{\nabla} = 0$.

Exercise 42. Check that $R_{\nabla}(\alpha, \beta)(X)$ is $C^{\infty}(M)$ -linear in α , β and X . In other words,

$$R_{\nabla} \in \Gamma(\wedge^2 A^* \otimes \text{End}(E)).$$

Note that a TM -connection is just an ordinary connection, and all notions we have introduced (and that we will introduce!) reduce to well-known notions of ordinary connection theory.

Given an A -path a with base path $\gamma : I \rightarrow M$, and $u : I \rightarrow E$ a path in E above γ , then the derivative of u along a , denoted $\nabla_a u$, is defined as usual: choose a time dependent section ξ of E such that $\xi(t, \gamma(t)) = u(t)$, then

$$\nabla_a u(t) = \nabla_a \xi^t(x) + \frac{d\xi^t}{dt}(x), \text{ at } x = \gamma(t).$$

Exercise 43. For an A -connection ∇ on a vector bundle E define parallel transport along A -paths. When is a connection complete (i.e., when is parallel transport defined for every A -path)?

The following exercise gives an alternative approach to connections in terms of the principal frame bundle. It also suggests how to define Ehresmann A -connections on any principal G -bundle.

Exercise 44. Let ∇ be an A -connection on a vector bundle E of rank r . If $P := \text{GL}(E) \rightarrow M$ is the bundle of linear frames on E , show that ∇ induces a smooth bundle map $h : p^*A \rightarrow TP$, such that:

(i) h is horizontal, i. e., the following diagram commutes:

$$\begin{array}{ccc} p^*A & \xrightarrow{h} & TP \\ \hat{p} \downarrow & & \downarrow p^* \\ A & \xrightarrow{\#} & TM \end{array}$$

(ii) h is $\text{GL}(r)$ -invariant, i. e., we have

$$h(ug, a) = (R_g)_*h(u, a), \quad \text{for all } g \in \text{GL}(r);$$

Conversely, show that every smooth bundle map $h : p^*A \rightarrow TP$ satisfying (i) and (ii) induces a connection ∇ on E .

Of special importance are the A -connections on A . If ∇ is an A -connection on A , we can define its **torsion** to be the $C^\infty(M)$ -bilinear map:

$$\begin{aligned} T_\nabla &: \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A), \\ T_\nabla(\alpha, \beta) &= \nabla_\alpha\beta - \nabla_\beta\alpha - [\alpha, \beta]. \end{aligned}$$

We say that ∇ is torsion free if $T_\nabla = 0$.

Exercise 45. Check that $T_\nabla(\alpha, \beta)$ is $C^\infty(M)$ -linear in α and β . In other words,

$$T_\nabla \in \Gamma(\wedge^2 A^* \otimes A).$$

We will use later some special connections A -connections which arise once an ordinary (TM -) connection on A is fixed. These constructions are explained in the following two exercises:

Exercise 46. Let A be a Lie algebroid and let ∇ be a TM -connection on the vector bundle A . Show that:

(a) The following formula defines an A -connection on the vector bundle A :

$$\nabla_\alpha\beta \equiv \nabla_{\rho(\alpha)}\beta.$$

(b) The following formula defines an A -connection on the vector bundle A :

$$\bar{\nabla}_\alpha\beta \equiv \nabla_{\rho(\beta)}\alpha + [\alpha, \beta].$$

(c) The following formula defines an A -connection on the vector bundle TM :

$$\bar{\nabla}_\alpha X \equiv \rho(\nabla_X\alpha) + [\rho(\alpha), X].$$

Note that the last two connections are compatible with the anchor:

$$\bar{\nabla}_\alpha\rho(\beta) = \rho(\bar{\nabla}_\alpha\beta).$$

Exercise 47 (Levi-Civita connections). Let $\langle \cdot, \cdot \rangle$ be a metric on A . We say that an A -connection ∇ on A is compatible with the metric if

$$\mathcal{L}_{\rho(\alpha)}(\langle \beta, \gamma \rangle) = \langle \nabla_\alpha \beta, \gamma \rangle + \langle \beta, \nabla_\alpha \gamma \rangle,$$

for all sections α, β, γ of A . Show that A admits a unique torsion free A -connection compatible with the metric.

Finally we remark that one can give a more intrinsic description of algebroid homomorphisms by using A -connections.

Exercise 48. Let A_1 and A_2 be Lie algebroids over M_1 , and M_2 , respectively, and let $F : A_1 \rightarrow A_2$ be a bundle map covering $f : M_1 \rightarrow M_2$ which is compatible with the anchor.

(a) For a connection ∇ on A_2 , we denote by the same letter the pull-back of ∇ via F (a connection in f^*A_2). Show that the expression

$$R_F(\alpha, \beta) := \nabla_\alpha(F(\beta)) - \nabla_\beta(F(\alpha)) - F([\alpha, \beta]) - T_\nabla(F(\alpha), F(\beta))$$

defines an element

$$R_F \in \Gamma(\wedge^2 A_1^* \otimes f^*A_2)$$

which is independent of the choice of ∇ .

- (b) Show that F is a Lie algebroid homomorphism if and only if $R_F = 0$.
(c) Deduce that, given a manifold M and a Lie algebra \mathfrak{g} , an algebroid morphism $TM \rightarrow \mathfrak{g}$ is the same thing as a 1-form $\omega \in \Omega^1(M; \mathfrak{g})$ satisfying the Maurer-Cartan equation $d\omega + \frac{1}{2}[\omega, \omega] = 0$. What does ω integrate to?
(d) Back to the general context, make sense of a similar Maurer-Cartan formula

$$R_F = d_\nabla(F) + \frac{1}{2}[F, F]_\nabla.$$

2.5. Representations

While the notion of connection is motivated by the tangent bundle, the following notion is motivated by Lie theory:

Definition 2.31. A **representation** of a Lie algebroid A over M consists of a vector bundle E over M together with a flat connection ∇ , i.e., a connection such that

$$\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha = \nabla_{[\alpha, \beta]},$$

for all $\alpha, \beta \in \Gamma(A)$.

The isomorphism classes of representations of a Lie algebroid A form a semiring $(\text{Rep}(A), \oplus, \otimes)$, where we use the direct sum and tensor product of connections to define the addition and the multiplication: given connections ∇^i on E^i , $i = 1, 2$, $E^1 \oplus E^2$ and $E^1 \otimes E^2$ have induced connections $\nabla^\oplus, \nabla^\otimes$, given by

$$\begin{aligned} \nabla_\alpha^\oplus(s_1, s_2) &= (\nabla_\alpha^1(s_1), \nabla_\alpha^2(s_2)), \\ \nabla_\alpha^\otimes(s_1 \otimes s_2) &= \nabla_\alpha^1(s_1) \otimes s_2 + s_1 \otimes \nabla_\alpha^2(s_2). \end{aligned}$$

The identity element is the trivial line bundle \mathbb{L} over M with the flat connection $\nabla_\alpha := \mathcal{L}_\alpha$.

The notion of a Lie algebroid representation is the infinitesimal counterpart of a representation of a Lie groupoid. To make this more precise, we denote by $\mathfrak{gl}(E)$ the Lie algebroid of $\mathrm{GL}(E)$. This Lie algebroid is related to the Lie algebra $\mathrm{Der}(E)$ of derivations on E .

Definition 2.32. Given a vector bundle E over M , a derivation of E is a pair (D, X) consisting of a vector field X on M and a linear map

$$D : \Gamma(E) \rightarrow \Gamma(E),$$

satisfying the Leibniz identity

$$D(fs) = fDs + \mathcal{L}_X(f)s,$$

for all $f \in C^\infty(M)$, $s \in \Gamma(E)$. We denote by $\mathrm{Der}(E)$ the Lie algebra of derivations of E , where the Lie bracket is given by

$$[(D, X), (D', X')] = (DD' - D'D, [X, X']).$$

Lemma 2.33. *Given a vector bundle E over M , the Lie algebra of sections of the algebroid $\mathfrak{gl}(E)$ is isomorphic to the Lie algebra $\mathrm{Der}(E)$ of derivations on E , and the anchor of $\mathfrak{gl}(E)$ is identified with the projection $(D, X) \mapsto X$.*

Proof. Given a section α of $\mathfrak{gl}(E)$, its flow (see Definition 1.26) gives linear maps

$$\phi_\alpha^t(x) : E_x \rightarrow E_{\phi_\alpha^t(x)}.$$

In particular, we obtain a map at the level of sections,

$$\begin{aligned} (\phi_\alpha^t)^* : \Gamma(E) &\rightarrow \Gamma(E), \\ (\phi_\alpha^t)^*(s) &= \phi_\alpha^t(x)^{-1}(s(\phi_\alpha^t(x))). \end{aligned}$$

The derivative of this map induces a derivation D_α on $\Gamma(E)$:

$$D_\alpha s = \left. \frac{d}{dt} \right|_{t=0} (\phi_\alpha^t)^*(s).$$

Conversely, given a derivation D , we find a 1-parameter group ϕ_D^t of automorphisms of E , sitting over the flow ϕ_X^t of the vector field X associated to D , as the solution of the equation

$$Ds = \left. \frac{d}{dt} \right|_{t=0} (\phi_D^t)^*(s).$$

Viewing $\phi_D^t(x)$ as an element of \mathcal{G} , and differentiating with respect to t at $t = 0$, we obtain a section α_D of $\mathfrak{gl}(E)$. Clearly, the correspondences $\alpha \mapsto D_\alpha$ and $D \mapsto \alpha_D$ are inverse to each other, hence we have an isomorphism between $\Gamma(\mathfrak{gl}(E))$ and $\mathrm{Der}(E)$. One way to see that this preserves the bracket is by using the description of Lie brackets in terms of flows. Alternatively, one remarks that the correspondences we have defined are local. Hence we may assume that E is trivial as a vector bundle, in which case the computation is simple and can be left to the reader. \square

Exercise 49. Show that a representation (E, ∇) of a Lie algebroid A is the same thing as a Lie algebroid homomorphism $\nabla : A \rightarrow \mathfrak{gl}(E)$.

The infinitesimal version of a Lie groupoid homomorphism $\mathcal{G} \rightarrow \mathrm{GL}(E)$ is a Lie algebroid homomorphism $A \rightarrow \mathfrak{gl}(E)$ (Exercise 28). By the previous exercise, such a homomorphism is the same thing as a flat A -connection on E , so we deduce the following.

Corollary 2.34. *Let \mathcal{G} be a Lie groupoid over M and let A be its Lie algebroid. Any representation E of \mathcal{G} can be made into a representation of A with corresponding A -connection defined by*

$$\nabla_\alpha s(x) = \left. \frac{d}{dt} \phi_\alpha^t(x)^{-1} s(\phi_{\rho(\alpha)}^t(x)) \right|_{t=0}.$$

Moreover, this construction defines a homomorphism of semi-rings

$$\Phi : \mathrm{Rep}(\mathcal{G}) \rightarrow \mathrm{Rep}(A),$$

which is an isomorphism if \mathcal{G} is \mathfrak{s} -simply connected.

Proof. We first interpret the groupoid action of \mathcal{G} on E as a groupoid morphism $\mathcal{F} : \mathcal{G} \rightarrow \mathrm{GL}(E)$. Passing to algebroids, we obtain an algebroid homomorphism $F : A \rightarrow \mathfrak{gl}(E)$. By the exercise above, this is the same thing as a flat A -connection ∇ on E : $\nabla_\alpha = D_{F(\alpha)}$. The actual formula for ∇ follows from the fact that the flow of $F(\alpha)$ is obtained by applying F to the flow of α . Using the definition of the derivation induced by a section of $\mathfrak{gl}(E)$ (see the previous proof), we conclude that

$$\nabla_\alpha s = \left. \frac{d}{dt} \right|_{t=0} (F \circ \phi_\alpha^t)^*(s),$$

which is a compact version of the formula in the statement. \square

A notion which actually generalizes both notions from geometry and Lie theory, is the notion of **Lie algebroid cohomology with coefficients** in a representation E . Given a representation E , defined by a flat A -connection ∇ , we introduce the de Rham complex of A with coefficients in E as follows. A p -differential A -form with values in E is an alternating $C^\infty(M)$ -multilinear map

$$\omega : \underbrace{\Gamma(A) \times \cdots \times \Gamma(A)}_{p \text{ times}} \rightarrow \Gamma(E).$$

We denote by

$$\Omega^p(A; E) = \Gamma(\wedge^p A^* \otimes E)$$

the set of all p -differential A -forms with values in E . The differential

$$d : \Omega^p(A; E) \rightarrow \Omega^{p+1}(A; E)$$

is defined by the usual Koszul-type formula

$$(2.35) \quad d\omega(\alpha_0, \dots, \alpha_p) = \sum_{k=0}^p (-1)^k \nabla_{\alpha_k} (\omega(\alpha_0, \dots, \hat{\alpha}_k, \dots, \alpha_p)) \\ + \sum_{k < l} (-1)^{k+l+1} \omega([\alpha_k, \alpha_l], \alpha_0, \dots, \hat{\alpha}_k, \dots, \hat{\alpha}_l, \dots, \alpha_p),$$

where $\alpha_0, \dots, \alpha_p \in \Gamma(A)$. You should check that $d^2 = 0$. The resulting cohomology is denoted $H^\bullet(A; E)$ and is called the **Lie algebroid cohomology with coefficients** in the representation E . In the particular case

of trivial coefficients, i.e., where E is the trivial line bundle over M with $\nabla_\alpha = \mathcal{L}_{\rho(\alpha)}$, we talk about the de Rham complex $\Omega^\bullet(A)$ of A and the Lie algebroid cohomology $H^\bullet(A)$ of A .

Exercise 50. The notion of Lie algebroid cohomology generalizes many well-known cohomology theories. In particular:

- (a) When $A = TM$, check that $H(TM; E) = H_{\text{dR}}(M; E)$ is the usual de Rham cohomology;
- (b) When $A = \mathfrak{g}$ is a Lie algebra, check that $H(A; E) = H(\mathfrak{g}; E)$ is the usual Chevalley-Eilenberg Lie algebra cohomology;
- (c) When $A = T\mathcal{F}$ is the Lie algebroid of a foliation, check that $H(A; E) = H(\mathcal{F}; E)$ is the usual foliated cohomology.

The last example in the previous exercise shows that Lie algebroid cohomology is, in general, infinite dimensional, and quite hard to compute. With the exception of transitive Lie algebroids, there is no known effective method to compute Lie algebroid cohomology.

Just like in the case of Lie groups, we can relate the Lie algebroid cohomology of a Lie algebroid A , associated with a Lie groupoid \mathcal{G} , with the de Rham cohomology of invariant forms on the groupoid.

Exercise 51. Let \mathcal{G} be an s -connected Lie groupoid with Lie algebroid A . Define a right-invariant differential form on \mathcal{G} to be a s -foliated differential form ω which satisfies:

$$R_g^* \omega = \omega, \quad \forall g \in \mathcal{G}.$$

We denote by $\Omega_{\text{inv}}^\bullet(\mathcal{G})$ the space of right invariant forms on \mathcal{G} .

- (a) Show that there is a natural isomorphism between $\Omega_{\text{inv}}^\bullet(\mathcal{G})$ and the de Rham complex $\Omega^\bullet(A)$ of A . Conclude that the Lie algebroid cohomology of A is isomorphic to the invariant de Rham cohomology of \mathcal{G} ;
- (b) Generalize this isomorphism to any coefficients.

2.6. Notes

The concept of Lie algebroid was introduced by Jean Pradines in 1966–68 who, in a series of notes [51, 52, 53, 54], introduced a Lie theory for Lie groupoids. Related purely algebraic notions, in particular the notion of a Lie pseudo-algebra, were introduced much before by many authors, and under different names (see the historical notes in [38] and [39, Chapter 3.8]).

The theory of Lie algebroids only took off in the late 1980's with the works of Almeida and Molino [2] on developable foliations and the works of Mackenzie on connection theory (see the account in his first book [37]). These works were devoted almost exclusively to transitive Lie algebroids, and it was Weinstein [62] and Karasëv [34], who understood first the need to study non-transitive algebroids, namely from the appearance of the cotangent Lie algebroid of a Poisson manifold ([12]).

The notion of morphism between two Lie algebroids is first appropriately understood in the work of Higgins and Mackenzie [31], where one can also find the description in terms of connections. An alternative treatment using the supermanifold formalism is due to Vaintrob [58]. The theory of connections played a large motivation in Mackenzie's approach to Lie groupoid

and algebroid theory. A geometric approach to the theory of connections on Lie algebroids was given by Fernandes in [24, 25].

Representations of Lie algebroids were introduced first for transitive Lie algebroids by Mackenzie [37], and they appear in various different contexts such as in the study of cohomological invariants attached to Lie algebroids (see, e.g., [23, 13, 25]). The cohomology theory of Lie algebroids was started by Mackenzie [37] (transitive case) inspired by the cohomology theory of Lie pseudoalgebras. Since then, many authors have developed many aspects of the theory (see, e.g., [27, 32, 66, 69]) though computations are in general quite difficult.

The connection between Lie algebroids and Cartan's equivalence method is not well-known, and deserves to be explored. We have learned it from Bryant [6, Appendix A].

LECTURE 3

Integrability: Topological Theory

3.1. What is integrability?

In the previous lecture we saw many examples of Lie algebroids. Some of these were *integrable* Lie algebroids, i.e., they were isomorphic to the Lie algebroid of some Lie groupoid. We are naturally led to ask:

- Is every Lie algebroid integrable?

It may come as a surprise to you, based on your experience with Lie algebras or with the Frobenius integrability theorem, that the answer is *no*. This is a subtle (but important) phenomenon, which was overlooked for sometime (see the notes at the end of this lecture).

Let us start by giving a few examples of non-integrable Lie algebroids, though at this point we cannot fully justify why they are not integrable (this will be clear later on).

Our first source of examples comes from the Lie algebroids associated with closed 2-forms.

Example 3.1. Recall from the previous lecture (see Example 2.26) that every closed 2-form $\omega \in \Omega^2(M)$ determines a Lie algebroid structure on $A_\omega = TM \oplus \mathbb{L}$. If M is simply-connected and the cohomology class of ω is integral, we have a prequantization principal \mathbb{S}^1 -bundle which gives rise to a Lie groupoid \mathcal{G}_ω integrating A_ω . The groupoid \mathcal{G}_ω being transitive, all its isotropy Lie groups are isomorphic, and one can see that they are in fact isomorphic to \mathbb{R}/Γ_ω , where Γ_ω is the group of spherical periods of ω :

$$\Gamma_\omega = \left\{ \int_\gamma \omega : \gamma \in \pi_2(M) \right\} \subset \mathbb{R}.$$

When ω is not integral, we don't have a prequantization bundle, and the construction of \mathcal{G}_ω fails. However, we will see later in this lecture that, if A_ω integrates to a source simply-connected Lie groupoid \mathcal{G}_ω , then its isotropy groups must still be isomorphic to \mathbb{R}/Γ_ω . One can then show that A_ω is integrable iff the group of spherical periods $\Gamma_\omega \subset \mathbb{R}$ is a discrete subgroup.

Let us take, for example, $M = \mathbb{S}^2 \times \mathbb{S}^2$ with $\omega = dS \oplus \lambda dS$, where dS is the standard area form on \mathbb{S}^2 . If we choose $\lambda \in \mathbb{R} - \mathbb{Q}$, then the group of

spherical periods is $\Gamma_\omega = \mathbb{Z} \oplus \lambda\mathbb{Z}$, so that \mathbb{R}/Γ_ω is non-discrete. Therefore, the corresponding Lie algebroid is non-integrable.

Another source of examples of non-integrable Lie algebroids is Poisson geometry, since there are Poisson manifolds whose cotangent Lie algebroids (see Example 2.28) are not integrable.

Let us recall that the dual \mathfrak{g}^* of a finite dimensional Lie algebra has a natural linear Poisson structure, namely the Kostant-Kirillov-Souriau Poisson structure, which is defined by:

$$\{f_1, f_2\}(\xi) = \langle \xi, [d_\xi f_1, d_\xi f_2] \rangle, \quad f_1, f_2 \in C^\infty(\mathfrak{g}^*), \quad \xi \in \mathfrak{g}^*.$$

The corresponding cotangent Lie algebroid integrates to the Lie groupoid $\mathcal{G} = T^*G$, where G is a Lie group with Lie algebra \mathfrak{g} . The source and target maps $\mathbf{s}, \mathbf{t} : T^*G \rightarrow \mathfrak{g}^*$ are the right and left translation maps trivializing the cotangent bundle:

$$\mathbf{s}(\omega_g) = (d_e R_g)^* \omega_g, \quad \mathbf{t}(\omega_g) = (d_e L_g)^* \omega_g,$$

while composition is given by

$$\omega_g \cdot \eta_h = (d_{hg} R_{g^{-1}})^* \eta_h.$$

Exercise 52. Consider the trivialization $T^*G = G \times \mathfrak{g}^*$ obtained by right translations. Determine the new expressions of \mathbf{s}, \mathbf{t} and the product under this isomorphism, and check that the orbits of \mathcal{G} are precisely the coadjoint orbits. Note that these coincide with the symplectic leaves of \mathfrak{g}^* .

Though linear Poisson structures are always integrable, one can easily produce non-integrable Poisson structures by slightly modifying them. Here are two examples.

Example 3.2. Let us take $\mathfrak{g} = \mathfrak{su}(n)$, with $n > 2$. We use the Ad-invariant inner product on \mathfrak{g} defined by:

$$\langle X, Y \rangle = \text{Tr}(XY^*),$$

to identify $\mathfrak{g}^* \simeq \mathfrak{g}$. Observe that the spheres $\mathbb{S}_r = \{X \in \mathfrak{g} : \|X\| = r\}$ are collections of (co)adjoint orbits, and hence are Poisson submanifolds of \mathfrak{g}^* , with the KKS Poisson structure. As we will see later, it turns out that (the cotangent Lie algebroids of) the Poisson manifolds \mathbb{S}_r are not integrable.

In the case $n = 2$, the spheres are symplectic submanifolds and so they turn out to be integrable. However, we can still change slightly the Poisson bracket and obtain another kind of non-integrable Poisson structure.

Example 3.3. Let us take then $\mathfrak{g} = \mathfrak{su}(2)$. If we identify $\mathfrak{su}(2)^* \simeq \mathbb{R}^3$, the Poisson bracket in euclidean coordinates (x, y, z) is determined by:

$$\{x, y\} = z, \quad \{y, z\} = x, \quad \{z, x\} = y.$$

The symplectic leaves of this Poisson structure (the coadjoint orbits) are the spheres $r^2 = x^2 + y^2 + z^2 = C$ and the origin. In fact, any smooth function $f = f(r)$ is a Casimir (Poisson commutes with any other function).

Let us choose a smooth function of the radius $a(r)$, such that $a(r) > 0$ for $r > 0$. Then we can define a new rescaled Poisson bracket:

$$\{f, g\}_a := a\{f, g\}.$$

This bracket clearly has the same symplectic foliation, while the symplectic area of each leaf $x^2 + y^2 + z^2 = r^2$ is rescaled by a factor of $1/a(r)$. We will see later that this new Poisson structure is integrable iff the symplectic area function $A(r) = 4\pi r/a(r)$ has no critical points. For example, if $a(r) = e^{r^2/2}$ then $A(r)$ has a critical point at $r = 1$, and the corresponding Poisson manifold is not integrable.

As the previous examples illustrate, it is not at all obvious when a Lie algebroid is integrable, and even “reduction” may turn an integrable Lie algebroid to a non-integrable one. Hence, it is quite important to find the answer to the following question:

- What are the precise obstructions to integrate a Lie algebroid?

This is the problem we shall address in the remainder of this lecture and in the next lecture.

3.2. Integrating Lie algebras

Before we look at the general integrability problem for Lie algebroids, it is worth to consider the special case of Lie algebras:

Theorem 3.4 (Lie III). *Every finite dimensional Lie algebra is isomorphic to the Lie algebra of some Lie group.*

In most texts in Lie theory, the proof of Lie’s third theorem uses the structure theory of Lie algebras. The usual strategy is to prove first Ado’s Theorem, stating that every Lie algebra has a faithful finite dimensional representation, from which it follows that there exists a matrix Lie group (not necessarily simply connected) with the given Lie algebra. Ado’s Theorem, in turn, can be easily proved for semi-simple Lie algebras, follows from some simple structure theory for solvable Lie algebras, and then extends to any Lie algebra using the Levi decomposition. There is, however, a much more direct geometric approach to Lie’s third theorem, which we will sketch now, and which will be extended to Lie algebroids in later sections.

The main idea is as follows: Suppose \mathfrak{g} is a finite Lie algebra which integrates to a connected Lie group G . Denote by $P(G)$ the space of paths in G starting at the identity $e \in G$, with the C^2 -topology:

$$P(G) = \{g : [0, 1] \rightarrow G \mid g \in C^2, g(0) = e\}.$$

Also, denote by \sim the equivalence relation defined by C^1 -homotopies in $P(G)$ with fixed end-points. Then we have a standard description of the simply-connected Lie group integrating \mathfrak{g} as

$$\tilde{G} = P(G)/\sim.$$

Let us be more specific about the product in \tilde{G} : given two paths $g_1, g_2 \in P(G)$ we define

$$g_1 \cdot g_2(t) \equiv \begin{cases} g_2(2t), & 0 \leq t \leq \frac{1}{2}, \\ g_1(2t-1)g_2(1), & \frac{1}{2} < t \leq 1. \end{cases}$$

Note, however, that this multiplication can take one out of $P(G)$, since the composition will be a path which is only *piecewise* C^1 , so we need:

Exercise 53. Show that any element in $P(G)$ is equivalent to some $g(t)$ with derivatives vanishing at the end-points, and if g_1 and g_2 have this property, then $g_1 \cdot g_2 \in P(G)$.

This will give us a multiplication in $P(G)$, which is associative up to homotopy, so we get the desired multiplication on the quotient space which makes \tilde{G} into a (topological) group. Since \tilde{G} is the universal covering space of the manifold G , there is also a smooth structure in \tilde{G} which makes it into a Lie group. In this way, we have recovered the simply-connected Lie group integrating \mathfrak{g} *assuming* that \mathfrak{g} is integrable.

Now, any G -path g defines a path $a : I \rightarrow \mathfrak{g}$ by differentiation and right translations:

$$(3.5) \quad a(t) = \left. \frac{d}{ds} g(s)g(t)^{-1} \right|_{s=t}.$$

Let us denote by $P(\mathfrak{g})$ the space of paths $a : [0, 1] \rightarrow \mathfrak{g}$ with the C^1 -topology.

Exercise 54. Show that the map $P(G) \rightarrow P(\mathfrak{g})$ just defined is a homeomorphism.

Using this bijection, we can transport our equivalence relation and our product in $P(G)$ to an equivalence relation and a product in $P(\mathfrak{g})$. The next lemma gives an explicit expression for the equivalence relation in $P(\mathfrak{g})$:

Lemma 3.6. *Two paths $a_0, a_1 \in P(\mathfrak{g})$ are equivalent iff there exists a homotopy $a_\epsilon \in P(\mathfrak{g})$, $\epsilon \in [0, 1]$, joining a_0 to a_1 , such that*

$$(3.7) \quad \int_0^1 B_\epsilon(s) \cdot \frac{da_\epsilon}{d\epsilon}(s) ds = 0, \quad \forall \epsilon \in [0, 1],$$

where $B_\epsilon(t) \in GL(\mathfrak{g})$, for each ϵ , is the solution of the initial value problem:

$$\begin{cases} \left. \frac{d}{ds} B_\epsilon(s)B_\epsilon(t)^{-1} \right|_{s=t} = \text{ad}(a_\epsilon(t)), \\ B_\epsilon(0) = I. \end{cases}$$

Exercise 55. Prove this lemma.

(Hint: see the proof of Proposition 3.15 below, which generalizes this to any Lie algebroid.)

Exercise 56. Let $g_1(t)$ and $g_2(t)$ be G -paths and denote by $a_1(t)$ and $a_2(t)$ the corresponding \mathfrak{g} -paths defined by (3.5). Show that \mathfrak{g} -path associated with the concatenation $g_1 \cdot g_2$ is given by:

$$(3.8) \quad a_1 \cdot a_2(t) \equiv \begin{cases} 2a_2(2t), & 0 \leq t \leq \frac{1}{2}, \\ 2a_1(2t - 1), & \frac{1}{2} < t \leq 1. \end{cases}$$

The Lemma and the previous exercise show that the equivalence relation and the product in $P(\mathfrak{g})$ can be defined exclusively in terms of data in \mathfrak{g} , and involve no reference at all to the Lie group G . Hence, we can make a fresh start, *without* assuming that \mathfrak{g} is the Lie algebra of some G :

Definition 3.9. Let \mathfrak{g} be a finite dimensional Lie algebra. Then we define the topological group

$$\mathcal{G}(\mathfrak{g}) = P(\mathfrak{g}) / \sim,$$

where:

- (i) Two paths $a_0, a_1 \in P(\mathfrak{g})$ are equivalent iff there exists a homotopy $a_\epsilon \in P(\mathfrak{g})$, $\epsilon \in [0, 1]$, joining a_0 to a_1 , such that relation (3.7) is satisfied.
- (ii) The product is $[a_1][a_2] = [a_1 \cdot a_2]$, where the dot denotes composition of paths, which is defined by (3.8).
- (iii) The topology on $\mathcal{G}(\mathfrak{g})$ is the quotient topology.

From what we saw above, if \mathfrak{g} is integrable, then $\mathcal{G}(\mathfrak{g})$ is the unique simply-connected Lie group with Lie algebra \mathfrak{g} . Of course, we know that all finite dimensional Lie algebras are integrable. But if we want to prove this, what is left is to prove is to show that the smooth Banach structure on $P(\mathfrak{g})$ descends to a smooth structure on the quotient:

Theorem 3.10. *Let \mathfrak{g} be a finite dimensional Lie algebra. Then $\mathcal{G}(\mathfrak{g})$ is a simply-connected Lie group integrating \mathfrak{g} .*

We will not give a proof here, since we will eventually prove a much more general version of this result, valid for Lie algebroids.

Exercise 57. Use the description of $\mathcal{G}(\mathfrak{g})$ to prove Lie's 2nd theorem for Lie algebras: if $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ is a Lie algebra homomorphism, then there exists a unique Lie group homomorphism $\Phi : \mathcal{G}(\mathfrak{g}) \rightarrow H$, with $d\Phi = \phi$, for any Lie group H integrating \mathfrak{h} .

3.3. Integrating Lie algebroids

Motivated by what we just did for Lie algebras/Lie groups we set:

Definition 3.11. Let \mathcal{G} be a Lie groupoid. A \mathcal{G} -path is a path $g : [0, 1] \rightarrow \mathcal{G}$ such that $\mathbf{s}(g(t)) = x$, for all t , and $g(0) = 1_x$ (i.e., a path lying in a \mathbf{s} -fiber of \mathcal{G} and starting at the unit of the fiber). We denote by $P(\mathcal{G})$ the space of \mathcal{G} -paths, furnished with the C^2 -topology.

Let A be a Lie algebroid which is integrable to a Lie groupoid \mathcal{G} . Let us denote by \sim the equivalence relation defined by C^1 -homotopies in $P(\mathcal{G})$ with fixed end-points. Then we can describe the s -simply connected Lie groupoid integrating A :

$$\tilde{\mathcal{G}} = P(\mathcal{G}) / \sim.$$

The source and target maps are the obvious ones⁽¹⁾:

$$\mathbf{s}([g]) = \mathbf{s}(g(0)), \quad \mathbf{t}([g]) = \mathbf{t}(g(1)),$$

and for two paths $g_1, g_2 \in P(\mathcal{G})$ which are composable, i.e., such that $\mathbf{s}(g_1(1)) = \mathbf{t}(g_2(0))$, we define

$$g_1 \cdot g_2(t) \equiv \begin{cases} g_2(2t), & 0 \leq t \leq \frac{1}{2}, \\ g_1(2t-1)g_2(1), & \frac{1}{2} < t \leq 1. \end{cases}$$

¹Note that the symbols \mathbf{s} and \mathbf{t} have different meanings on different sides of these relations!

Note that we are just following the same strategy as we did for the integrability of Lie algebras. Of course, the same comments about concatenation taking us out of $P(\mathcal{G})$ apply, but:

- (a) any element in $P(\mathcal{G})$ is equivalent to some $g(t)$ with derivatives vanishing at the end-points, and
- (b) if g_1 and g_2 have this property, then $g_1 \cdot g_2 \in P(\mathcal{G})$.

Therefore, the multiplication will be well defined and associative up to homotopy, so we get the desired multiplication on the quotient space which makes $\tilde{\mathcal{G}}$ into a (topological) groupoid. The construction of the smooth structure on $\tilde{\mathcal{G}}$ is similar to the construction of the smooth structure on the universal cover of a manifold (see, also, the proof of Theorem 1.30).

Now, just like in the case of Lie algebras, we have:

Proposition 3.12. *If \mathcal{G} integrates the Lie algebroid A , then there is a homeomorphism $D^R : P(\mathcal{G}) \rightarrow P(A)$ between the space of \mathcal{G} -paths, and the space of A -paths.*

The map D^R will be called the **differentiation of \mathcal{G} -paths**, and its inverse will be called the **integration of A -paths**.

Proof. Any \mathcal{G} -path $g : I \rightarrow \mathcal{G}$ defines an A -path $D^R(g) : I \rightarrow A$, where

$$(3.13) \quad (D^R g)(t) = (dR_{g(t)^{-1}})_{g(t)} \dot{g}(t) ,$$

(here, for $h : x \rightarrow y$ an arrow in \mathcal{G} , $R_h : \mathfrak{s}^{-1}(y) \rightarrow \mathfrak{s}^{-1}(x)$ is the right multiplication by h). Conversely, any A -path a arises in this way: we first integrate (using LIE II) the Lie algebroid morphism $TI \rightarrow A$ defined by a , and then we notice that any Lie groupoid homomorphism $\phi : I \times I \rightarrow \mathcal{G}$, from the pair groupoid into \mathcal{G} , is of the form $\phi(s, t) = g(s)g^{-1}(t)$, for some \mathcal{G} -path g . \square

Our next task is to transport the equivalence relation from \mathcal{G} -paths to A -paths. For that we will need the concept of *flow of a section* of a Lie algebroid: if $\alpha \in \Gamma(A)$ then its flow is the unique 1-parameter group of Lie algebroid automorphisms $\phi_\alpha^t : A \rightarrow A$ such that:

$$(3.14) \quad \left. \frac{d}{dt} \right|_{t=0} \phi_\alpha^t(\beta) = [\alpha, \beta].$$

Exercise 58. Give an explicit construction of the flow $\phi_\alpha^t : A \rightarrow A$ of a section $\alpha \in \Gamma(A)$, and show that they are Lie algebroid automorphisms that cover the flow $\phi_{\rho(\alpha)}^t : M \rightarrow M$ of the vector field $\rho(\alpha)$.

(Hint: Consider the derivation $\text{ad}_\alpha \in \text{Der}(A)$ defined by $\text{ad}_\alpha(\beta) = [\alpha, \beta]$. Identifying ad_α with a section of $\mathfrak{gl}(A)$, take ϕ_α^t to be the flow of ad_α in the sense of the last lecture.)

Exercise 59. Define the flow $\phi_{\alpha_s}^{t,s}$ of a *time-dependent* section $\alpha_s \in \Gamma(A)$.

Henceforth, by a *variation of A -paths* we mean a map

$$a_\epsilon(t) = a(\epsilon, t) : I \times I \rightarrow A$$

such that a_ϵ is a family of A -paths of class C^2 on ϵ , with the property that the base paths $\gamma_\epsilon(t) = \gamma(\epsilon, t) : I \times I \rightarrow M$ have fixed end points. When

$a_\epsilon = D^R g_\epsilon$, the family g_ϵ does not necessarily give a homotopy between g_0 and g_1 , because the end points $g_\epsilon(1)$ may depend on ϵ . The following proposition describes two distinct ways of controlling the variation $\frac{d}{d\epsilon} g_\epsilon(1)$, both depending only on infinitesimal data (i.e., Lie algebroid data).

Proposition 3.15. *Let A be any Lie algebroid and $a = a_\epsilon \in P(A)$ a variation of A -paths.*

(i) *If ∇ is an TM-connection on A , the solution $b = b(\epsilon, t)$ of the differential equation*

$$(3.16) \quad \partial_t b - \partial_\epsilon a = T_\nabla(a, b), \quad b(\epsilon, 0) = 0,$$

does not depend on ∇ . Moreover, $\rho(b) = \frac{d}{d\epsilon} \gamma$.

(ii) *If ξ_ϵ are time depending sections of A such that $\xi_\epsilon(t, \gamma_\epsilon(t)) = a_\epsilon(t)$, then $b(\epsilon, t)$ is given by*

$$(3.17) \quad b(\epsilon, t) = \int_0^t \phi_{\xi_\epsilon}^{t,s} \frac{d\xi_\epsilon}{d\epsilon}(s, \gamma_\epsilon(s)) ds,$$

where $\phi_{\xi_\epsilon}^{t,s}$ denotes the flow of the time-dependent section ξ_ϵ .

(iii) *If \mathcal{G} integrates A and g_ϵ are the \mathcal{G} -paths satisfying $D^R(g_\epsilon) = a_\epsilon$, then $b = D^R(g^t)$, where g^t are the paths in \mathcal{G} : $\epsilon \rightarrow g^t(\epsilon) = g(\epsilon, t)$.*

This motivates the following definition:

Definition 3.18. We say that two A -paths a_0 and a_1 are **A -homotopic** and we write $a_0 \sim a_1$, if there exists a variation a_ϵ with the property that b insured by Proposition 3.15 satisfies $b(\epsilon, 1) = 0$ for all $\epsilon \in I$.

If A admits an integration \mathcal{G} , then the isomorphism $D^R : P(\mathcal{G}) \rightarrow P(A)$ transforms the usual homotopy into A -homotopy. Also, since A -paths should be viewed as algebroid morphisms, the pair (a, b) defining the A -homotopy should be viewed as a true homotopy

$$adt + bde : TI \times TI \rightarrow A.$$

in the world of Lie algebroids.

Exercise 60. Show that equation (3.16) is equivalent to the condition that $adt + bde : TI \times TI \rightarrow A$ is a morphism of Lie algebroids.

(Hint: See Exercise 48.)

Proof of Proposition 3.15. Obviously, (i) follows from (ii). To prove (ii), let ξ_ϵ be as in the statement, and let η be given by

$$\eta(\epsilon, t, x) = \int_0^t \phi_{\xi_\epsilon}^{t,s} \frac{d\xi_\epsilon}{d\epsilon}(s, \phi_{\rho(\xi_\epsilon)}^{s,t}(x)) ds \in A_x.$$

We may assume that ξ_ϵ has compact support. We note that η coincides with the solution of the equation

$$(3.19) \quad \frac{d\eta}{dt} - \frac{d\xi}{d\epsilon} = [\eta, \xi].$$

with $\eta(\epsilon, 0) = 0$. Indeed, since

$$\eta(\epsilon, t, -) = \int_0^t (\phi_{\xi_\epsilon}^{s,t})^* \left(\frac{d\xi_\epsilon^s}{d\epsilon} \right) ds \in \Gamma(A),$$

equation (3.19) immediately follows from the basic formula (3.14) for flows. Also, $X = \rho(\xi)$ and $Y = \rho(\eta)$ satisfy a similar equation on M , and since we have $X(\epsilon, t, \gamma_\epsilon(t)) = \frac{d\gamma}{dt}$, it follows that $Y(\epsilon, t, \gamma_\epsilon(t)) = \frac{d\gamma}{d\epsilon}$. In other words, $b(\epsilon, t) = \eta(\epsilon, t, \gamma(\epsilon, t))$ satisfies $\rho(b) = \frac{d\gamma}{d\epsilon}$. We now have

$$\partial_t b = \nabla_{\frac{d\gamma}{dt}} \eta + \frac{d\eta}{dt} = \nabla_{\rho(\xi)} \eta + \frac{d\eta}{dt}.$$

Subtracting from this the similar formula for $\partial_\epsilon a$ and using (3.19) we arrive at

$$\partial_t b - \partial_\epsilon a = \nabla_{\rho(\xi)} \eta - \nabla_{\rho(\eta)} \xi + [\eta, \xi] = T_{\nabla}(\xi, \eta).$$

We are now left with proving (iii). Assume that \mathcal{G} integrates A and g_ϵ are the \mathcal{G} -paths satisfying $D^R(g_\epsilon) = a_\epsilon$. The formula of variation of parameters applied to the right-invariant vector field ξ_ϵ shows that

$$\begin{aligned} \frac{\partial g(\epsilon, t)}{\partial \epsilon} &= \int_0^t (d\varphi_{\xi_\epsilon}^{t,s})_{g(\epsilon, s)} \frac{d\xi_\epsilon^s}{d\epsilon} (g(\epsilon, s)) ds \\ &= (dR_{g(\epsilon, t)})_{\gamma_\epsilon(t)} \int_0^t \phi_{\xi_\epsilon}^{t,s} \frac{d\xi_\epsilon^s}{d\epsilon} (\gamma_\epsilon(s)) ds. \end{aligned}$$

But then:

$$D^R(g^t) = \int_0^t \phi_{\xi_\epsilon}^{t,s} \frac{d\xi_\epsilon^s}{d\epsilon} (\gamma_\epsilon(s)) ds = b(\epsilon, t).$$

□

Let us now turn to the task of transporting the composition from \mathcal{G} -paths to A -paths. This is an easy task if we use the properties of A -homotopies given in the following exercise:

Exercise 61. Let A be a Lie algebroid. Show that:

- (a) If $\tau : I \rightarrow I$ is a smooth change of parameter, then any A -path a is A -homotopic to its re-parameterization $a^\tau(t) \equiv \tau'(t)a(\tau(t))$.
- (b) Any A -path a_0 is A -homotopic to a smooth (i.e., of class C^∞) A -path.
- (c) If two smooth A -paths a_0 and a_1 are A -homotopic, then there exists a smooth A -homotopy between them.

For a Lie algebroid $\pi : A \rightarrow M$ we say that two A -paths a_0 and a_1 are composable if they have the same end-points, i.e., $\pi(a_0(1)) = \pi(a_1(0))$. In this case, we define their concatenation by

$$a_1 \odot a_0(t) \equiv \begin{cases} 2a_0(2t), & 0 \leq t \leq \frac{1}{2}, \\ 2a_1(2t-1), & \frac{1}{2} < t \leq 1. \end{cases}$$

This is essentially the multiplication that we need. However, $a_1 \odot a_0$ is only *piecewise* smooth. One way around this difficulty is allowing for A -paths which are *piecewise* smooth. Instead, we choose a cutoff function $\tau \in C^\infty(\mathbb{R})$ with the following properties:

- (a) $\tau(t) = 1$ for $t \geq 1$ and $\tau(t) = 0$ for $t \leq 0$;
- (b) $\tau'(t) > 0$ for $t \in]0, 1[$.

For an A -path a we denote by a^τ its re-parameterization $a^\tau(t) := \tau'(t)a(\tau(t))$. We now define the multiplication of composable A -paths by

$$a_1 \cdot a_0 \equiv a_1^\tau \odot a_0^\tau \in P(A).$$

According to Exercise 61 (a), the product $a_0 \cdot a_1$ is equivalent to $a_0 \odot a_1$ whenever $a_0(1) = a_1(0)$. It follows that the quotient $\mathcal{G}(A)$ is a groupoid with this product together with the natural structure maps:

- the source and target $\mathbf{s}, \mathbf{t} : \mathcal{G}(A) \rightarrow M$ map a class $[a]$ to its endpoints $\pi(a(0))$ and $\pi(a(1))$, respectively;
- the unit section $u : M \rightarrow P(A)$ maps x to the class $[0_x]$ of the constant trivial path above x ;
- the inverse $i : P(A) \rightarrow P(A)$ maps a class $[a]$ to the class $[\bar{a}]$ of its opposite path, which is defined by $\bar{a}(t) = -a(1-t)$.

The groupoid $\mathcal{G}(A)$ will be called the **Weinstein groupoid** of the Lie algebroid A .

Theorem 3.20. *Let A be a Lie algebroid over M . Then the quotient*

$$\mathcal{G}(A) \equiv P(A) / \sim$$

is an \mathbf{s} -simply connected topological groupoid independent of the choice of cutoff function. Moreover, whenever A is integrable, $\mathcal{G}(A)$ admits a smooth structure which makes it into the unique \mathbf{s} -simply connected Lie groupoid integrating A .

Proof. If we take the maps on the quotient induced from the structure maps defined above, then $\mathcal{G}(A)$ is clearly a groupoid. Note that the multiplication on $P(A)$ was defined so that, whenever \mathcal{G} integrates A , the map D^R of Proposition 3.12 preserves multiplications. Hence the only thing we still have to prove is that $\mathbf{s}, \mathbf{t} : \mathcal{G}(A) \rightarrow M$ are open maps.

To prove this we show that for any two A -homotopic A -paths a_0 and a_1 , there exists a homeomorphism $T : P(A) \rightarrow P(A)$ such that $T(a) \sim a$ for all a 's, and $T(a_0) = a_1$. We can construct such a T as follows: we let $\eta = \eta(\epsilon, t)$ be a family of time dependent sections of A which determines the equivalence $a_0 \sim a_1$ (see Proposition 3.15), so that $\eta(\epsilon, 0) = \eta(\epsilon, 1) = 0$ (we may assume η has compact support, so that all the flows involved are everywhere defined). Given an A -path b_0 , we consider a time dependent section ξ_0 so that $\xi_0(t, \gamma_0(t)) = b_0(t)$ and denote by ξ the solution of equation (3.19) with initial condition ξ_0 . If we set $\gamma_\epsilon(t) = \Phi_{\rho(\eta_t)}^{\epsilon, 0}(\gamma_0(t))$ and $b_\epsilon(t) = \xi_\epsilon(t, \gamma_\epsilon(t))$, then $T_\eta(b_0) \equiv b_1$ is homotopic to b_0 via b_ϵ , and maps a_0 into a_1 . \square

The following exercises provides further evidence of the importance of the notion of A -homotopy and the naturality of the construction of the groupoid $\mathcal{G}(A)$:

Exercise 62. Let E be a representation of the Lie algebroid A . Show that if a_0 and a_1 are A -homotopic paths from x to y , then the parallel transports $\tau_{a_0}, \tau_{a_1} : E_x \rightarrow E_y$ along a_0 and a_1 coincide. Conclude that every representation $E \in \text{Rep}(A)$ determines a representation of $\mathcal{G}(A)$, which in the integrable case is the induced smooth representation.

Exercise 63. Show that every algebroid homomorphism $\phi : A \rightarrow B$ determines a continuous groupoid homomorphism $\Phi : \mathcal{G}(A) \rightarrow \mathcal{G}(B)$. If A and B are integrable, show that Φ is smooth and $\Phi_* = \phi$. Finally, use this to prove Lie II (Theorem 2.19).

Exercise 64. Show that, for any Lie algebroid A , there exists an exponential map $\exp : \Gamma_c(A) \rightarrow \Gamma(\mathcal{G}(A))$, which generalizes the exponential map in the integrable case, and for any $\alpha, \beta \in \Gamma_c(A)$ satisfies:

$$\exp(t\alpha) \exp(\beta) \exp(-t\alpha) = \exp(\phi_\alpha^t \beta),$$

where ϕ_α^t denotes the flow of α .

3.4. Monodromy

In this section we will introduce the *monodromy groups* that control the integrability of a Lie algebroid A .

Let us assume first that A is an integrable Lie algebroid and that \mathcal{G} is a source 1-connected Lie groupoid integrating A . If $x \in M$, there are two Lie groups that integrate the isotropy Lie algebra \mathfrak{g}_x :

- The isotropy Lie groups $\mathcal{G}_x := \mathfrak{s}^{-1}(x) \cap \mathfrak{t}^{-1}(x)$.
- The 1-connected Lie group $G_x := \mathcal{G}(\mathfrak{g}_x)$ with Lie algebra \mathfrak{g}_x .

Let us denote by \mathcal{G}_x^0 the connected component of \mathcal{G}_x containing the identity element. By simple ordinary Lie theory arguments, there exists a subgroup $\tilde{\mathcal{N}}_x \subset Z(G_x)$ of the center of G_x such that:

$$\mathcal{G}_x^0 \simeq G_x / \tilde{\mathcal{N}}_x.$$

Note that $\tilde{\mathcal{N}}_x$ can be identified with $\pi_1(\mathcal{G}_x^0)$ and that it is a discrete subgroup of G_x . The group $\tilde{\mathcal{N}}_x$ will be called the *monodromy group* of A at x . We will show now that one can define these monodromy groups even when A is non-integrable, but then they may fail to be discrete. This lack of discreteness is the clue to understand the non-integrability of A .

Let $\pi : A \rightarrow M$ be *any* Lie algebroid. Notice that the isotropy group of the Weinstein groupoid $\mathcal{G}(A)$ at x is formed by the equivalence classes of **A -loops** based at x :

$$\mathcal{G}_x(A) = \{[a] \in \mathcal{G} : \pi(a(0)) = \pi(a(1)) = x\}.$$

We emphasize that *only* the base path of a is a loop and, in general, we will have $a(0) \neq a(1)$. Also, the base loop must lie inside the orbit \mathcal{O}_x through the base point.

It will be also convenient to consider the **restricted isotropy group** formed by those A -loops whose base loop is contractible in the orbit \mathcal{O}_x :

$$\mathcal{G}_x(A)^0 = \{[a] \in \mathcal{G}_x(A) : \gamma \sim * \text{ in } \mathcal{O}_x\}.$$

It is clear that $\mathcal{G}_x(A)^0$ is the connected component of the identity of the isotropy group $\mathcal{G}_x(A)$. Moreover, the map $[a] \mapsto [\gamma]$, associating to an A -homotopy class of A -paths the homotopy class of its base path gives a short exact sequence:

$$1 \longrightarrow \mathcal{G}_x(A)^0 \longrightarrow \mathcal{G}_x(A) \longrightarrow \pi_1(\mathcal{O}_x) \longrightarrow 1.$$

We know that when A is integrable, $\mathcal{G}_x(A)^0$ is a connected Lie group integrating the isotropy Lie algebra $\mathfrak{g}_x = \mathfrak{g}_x(A)$. We would like to understand this restricted isotropy Lie group for a general, possibly non-integrable, Lie algebroid. Recall from Lecture 2, that over an orbit \mathcal{O}_x of any Lie algebroid, the anchor gives a short exact sequence of Lie algebroids:

$$0 \longrightarrow \mathfrak{g}_{\mathcal{O}_x} \longrightarrow A|_{\mathcal{O}_x} \xrightarrow{\rho} T\mathcal{O}_x \longrightarrow 0.$$

If we think of this sequence as a fibration, the next proposition shows that there exists (the first terms of) an associated homotopy long exact sequence:

Proposition 3.21. *There exists a homomorphism $\partial : \pi_2(\mathcal{O}_x) \rightarrow \mathcal{G}(\mathfrak{g}_x)$ which makes the following sequence exact:*

$$\cdots \longrightarrow \pi_2(\mathcal{O}_x) \xrightarrow{\partial} \mathcal{G}(\mathfrak{g}_x) \longrightarrow \mathcal{G}_x(A) \longrightarrow \pi_1(\mathcal{O}_x).$$

Proof. To define ∂ let $[\gamma] \in \pi_2(\mathcal{O}_x)$ be represented by some smooth path $\gamma : I \times I \rightarrow \mathcal{O}_x$ which maps the boundary into x . We choose a morphism of algebroids

$$adt + bde : TI \times TI \rightarrow A_{\mathcal{O}_x}$$

(i.e., (a, b) satisfies equation (3.16)) which lifts $d\gamma : TI \times TI \rightarrow T\mathcal{O}_x$ via the anchor, and such that $a(0, t)$, $b(\epsilon, 0)$, and $b(\epsilon, 1)$ vanish. This is always possible: we can take $b(\epsilon, t) = \sigma(\frac{d}{d\epsilon}\gamma(\epsilon, t))$ where $\sigma : T\mathcal{O}_x \rightarrow A_{\mathcal{O}_x}$ is any splitting of the anchor map, and take a to be the unique solution of the differential equation (3.16) with the initial conditions $a(0, t) = 0$. Since γ is constant on the boundary, $a_1 = a(1, -)$ stays inside the Lie algebra \mathfrak{g}_x , i.e., defines a \mathfrak{g}_x -path $a_1 : I \rightarrow \mathfrak{g}_x$. Its integration (see Proposition 3.12 applied to the Lie algebra \mathfrak{g}_x) defines a path in $\mathcal{G}(\mathfrak{g}_x)$, and its end point is denoted by $\partial(\gamma)$.

We need to check that ∂ is well defined. For that we assume that

$$\gamma^i = \gamma^i(\epsilon, t) : I \times I \rightarrow \mathcal{O}_x, \quad i \in \{0, 1\}$$

are two homotopic paths relative to the boundary, and that

$$a^i dt + b^i d\epsilon : TI \times TI \rightarrow A_{\mathcal{O}_x}, \quad i \in \{0, 1\},$$

are lifts of $d\gamma^i$ as above. We prove that the paths $a^i(1, t)$ ($i \in \{0, 1\}$) are homotopic as \mathfrak{g}_x -paths.

By hypothesis, there is a homotopy $\gamma^u = \gamma^u(\epsilon, t)$ ($u \in I$) between γ^0 and γ^1 . We choose a family $b^u(\epsilon, t)$ joining b^0 and b^1 , such that $\rho(b^u(\epsilon, t)) = \frac{d\gamma^u}{d\epsilon}$ and $b^u(\epsilon, 0) = b^u(\epsilon, 1) = 0$. We also choose a family of sections η depending on u , ϵ and t , such that

$$\eta^u(\epsilon, t, \gamma^u(\epsilon, t)) = b^u(\epsilon, t), \quad \text{with } \eta = 0 \text{ when } t = 0, 1.$$

As in the proof of Proposition 3.15, let ξ and θ be the solutions of

$$\begin{cases} \frac{d\xi}{d\epsilon} - \frac{d\eta}{dt} = [\xi, \eta], & \text{with } \xi = 0 \text{ when } \epsilon = 0, 1, \\ \frac{d\theta}{d\epsilon} - \frac{d\eta}{du} = [\theta, \eta], & \text{with } \theta = 0 \text{ when } \epsilon = 0, 1. \end{cases}$$

Setting $u = 0, 1$ we get

$$a^i(\epsilon, t) = \xi^i(\epsilon, t, \gamma^i(\epsilon, t)), \quad i = 0, 1.$$

On the other hand, setting $t = 0, 1$ we get $\theta = 0$ when $t = 0, 1$. A brief computation shows that $\phi \equiv \frac{d\xi}{du} - \frac{d\theta}{dt} - [\xi, \theta]$ satisfies

$$\frac{d\phi}{d\epsilon} = [\phi, \eta],$$

and since $\phi = 0$ when $\epsilon = 0$, it follows that

$$\frac{d\xi}{du} - \frac{d\theta}{dt} = [\xi, \theta].$$

If in this relation we choose $\epsilon = 1$, and use $\theta^u(1, t) = 0$ when $t = 0, 1$, we conclude that $a^i(1, t) = \xi^i(1, t, \gamma^i(1, t))$, $i = 0, 1$, are equivalent as \mathfrak{g}_x -paths.

Finally, to check that the sequence is exact we only need to check exactness at the level $\mathcal{G}(\mathfrak{g}_x)$. However, it is clear from the definition of ∂ that its image is exactly the subgroup of $\mathcal{G}(\mathfrak{g}_x)$ which consists of the equivalence classes $[a] \in \mathcal{G}(\mathfrak{g}_x)$ of \mathfrak{g}_x -paths with the property that, as an A -path, a is equivalent to the trivial A -path. \square

The previous proposition motivates our next definition:

Definition 3.22. The homomorphism $\partial : \pi_2(\mathcal{O}_x) \rightarrow \mathcal{G}(\mathfrak{g}_x)$ of Proposition 3.21 is called the **monodromy homomorphism** of A at x . Its image:

$$\tilde{\mathcal{N}}_x(A) = \{[a] \in \mathcal{G}(\mathfrak{g}_x) : a \sim 0_x \text{ as an } A\text{-path}\},$$

is called the **monodromy group** of A at x .

The reason for using the tilde in the notation for the monodromy group is explained in the next exercise.

Exercise 65. Show that $\tilde{\mathcal{N}}_x(A)$ is a subgroup of $\mathcal{G}(\mathfrak{g}_x)$ contained in the center $Z(\mathcal{G}(\mathfrak{g}_x))$, and its intersection with the connected component $Z(\mathcal{G}(\mathfrak{g}_x))^0$ of the center is isomorphic to

$$(3.23) \quad \mathcal{N}_x(A) = \{v \in Z(\mathfrak{g}_x) : v \sim 0_x \text{ as } A\text{-paths}\} \subset \mathfrak{g}_x(A).$$

(Hint: Use the fact that for any $g \in \tilde{\mathcal{N}}_x(A) \subset \mathcal{G}(\mathfrak{g}_x)$ which can be represented by a \mathfrak{g}_x -path a , parallel transport $T_a = Ad_g : \mathfrak{g}_x \rightarrow \mathfrak{g}_x$ along a is the identity. Then apply the exponential map $\exp : Z(\mathfrak{g}_x) \rightarrow Z(\mathcal{G}(\mathfrak{g}_x))^0$).

Exercise 66. Check that if x and y lie in the same orbit of A then there exists a canonical isomorphism $\tilde{\mathcal{N}}_x(A) \simeq \tilde{\mathcal{N}}_y(A)$.

It follows from Proposition 3.21 that the restricted isotropy group is given by:

$$(3.24) \quad \mathcal{G}_x(A)^0 = \mathcal{G}(\mathfrak{g}_x) / \tilde{\mathcal{N}}_x(A).$$

This leads immediately to the following result:

Proposition 3.25. *For any Lie algebroid A , and any $x \in M$, the following are equivalent:*

- (i) $\mathcal{G}_x(A)^0$ is a Lie group with Lie algebra \mathfrak{g}_x .
- (ii) $\tilde{\mathcal{N}}_x(A)$ is closed;
- (iii) $\tilde{\mathcal{N}}_x(A)$ is discrete;
- (iv) $\mathcal{N}_x(A)$ is closed;
- (v) $\mathcal{N}_x(A)$ is discrete.

Proof. We just need to observe that the group $\tilde{\mathcal{N}}_x(A)$ (and hence $\mathcal{N}_x(A)$) is countable since it is the image under ∂ of $\pi_1(\mathcal{O})$, which is always a countable group. \square

At this point, we notice that we have an obvious *necessary* condition for a Lie algebroid to be integrable: if A is integrable, then each $\mathcal{G}_x(A)^0$ is a Lie group, and hence the monodromy groups must be discrete. This is enough to explain the non-integrability in the examples at the beginning of this lecture. But before we can do that, we need to discuss briefly how the monodromy groups can be explicitly computed in many examples.

Again, let us consider the short exact sequence of an orbit

$$0 \longrightarrow \mathfrak{g}_{\mathcal{O}} \longrightarrow A|_{\mathcal{O}} \xrightarrow{\rho} T\mathcal{O} \longrightarrow 0.$$

and any linear splitting $\sigma : T\mathcal{O} \rightarrow A|_{\mathcal{O}}$ of ρ . The *curvature* of σ is the element $\Omega_{\sigma} \in \Omega^2(\mathcal{O}; \mathfrak{g}_{\mathcal{O}})$ defined by:

$$\Omega_{\sigma}(X, Y) := \sigma([X, Y]) - [\sigma(X), \sigma(Y)].$$

In favorable cases, the computation of the monodromy can be reduced to the following

Lemma 3.26. *If there is a splitting σ with the property that its curvature Ω_{σ} is $Z(\mathfrak{g}_{\mathcal{O}})$ -valued, then*

$$\tilde{\mathcal{N}}_x(A) \simeq \mathcal{N}_x(A) = \left\{ \int_{\gamma} \Omega_{\sigma} : [\gamma] \in \pi_2(\mathcal{O}, x) \right\} \subset Z(\mathfrak{g}_x)$$

for all $x \in \mathcal{O}$.

Remark 3.27. Note that $Z(\mathfrak{g}_{\mathcal{O}})$ is canonically a flat vector bundle over \mathcal{O} . The corresponding flat connection can be expressed with the help of the splitting σ as

$$\nabla_X^{\sigma} \alpha = [\sigma(X), \alpha],$$

and it is easy to see that the definition does not depend on σ . In this way Ω_{σ} appears as a 2-cohomology class with coefficients in the local system defined by $Z(\mathfrak{g}_{\mathcal{O}})$ over \mathcal{O} , and then the integration is just the usual pairing between cohomology and homotopy. In practice one can always avoid working with local coefficients: if $Z(\mathfrak{g}_{\mathcal{O}})$ is not already trivial as a vector bundle, one can achieve this by pulling back to the universal cover of \mathcal{O} (where parallel transport with respect to the flat connection gives the desired trivialization).

We should specify what we mean by integrating forms with coefficients in a local system. Assume $\omega \in \Omega^2(M; E)$ is a 2-form with coefficients in some flat vector bundle E . Integrating ω over a 2-cycle $\gamma : \mathbb{S}^2 \rightarrow M$ means (i) taking the pull-back $\gamma^*\omega \in \Omega^2(\mathbb{S}^2; \gamma^*E)$, and (ii) integrate $\gamma^*\omega$ over \mathbb{S}^2 . Here γ^*E should be viewed as a flat vector bundle over \mathbb{S}^2 for the pull-back connection. Notice that the connection enters the integration part, and this matters for the integration to be invariant under homotopy.

Proof of Lemma 3.26. We may assume that $\mathcal{O} = M$, i.e., A is transitive. In agreement with the Remark above, we also assume for simplicity that $Z(\mathfrak{g})$ is trivial as a vector bundle (here and below $\mathfrak{g} := \mathfrak{g}_{\mathcal{O}}$).

The formula above defines a connection ∇^σ on the entire \mathfrak{g} . We use σ to identify A with $TM \oplus \mathfrak{g}$ so the bracket becomes

$$[(X, v), (Y, w)] = ([X, Y], [v, w] + \nabla_X^\sigma(w) - \nabla_Y^\sigma(v) - \Omega_\sigma(X, Y)).$$

Now choose some connection ∇^M on M and consider the connection $\nabla = (\nabla^M, \nabla^\sigma)$ on $A = TM \oplus \mathfrak{g}$. Note that

$$T_\nabla((X, v), (Y, w)) = (T_{\nabla^M}(X, Y), \Omega_\sigma(X, Y) - [v, w])$$

for all $X, Y \in TM$, $v, w \in \mathfrak{g}$. This shows that the two A -paths a and b in Proposition 3.15, will take the form $a = (\frac{d\gamma}{dt}, \phi)$, $b = (\frac{d\gamma}{d\epsilon}, \psi)$, where ϕ and ψ are paths in \mathfrak{g} satisfying

$$\partial_t \psi - \partial_\epsilon \phi = \Omega_\sigma\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}\right) - [\phi, \psi].$$

Now we only have to apply the definition of ∂ : Given $[\gamma] \in \pi_2(M, x)$, we choose the lift $adt + bde$ of $d\gamma$ with $\psi = 0$ and

$$\phi = - \int_0^\epsilon \Omega_\sigma\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}\right).$$

Then ϕ takes values in $Z(\mathfrak{g}_x)$ and we obtain $\partial[\gamma] = [\int_\gamma \Omega_\sigma]$. □

We can now justify why the examples of Lie algebroids given at the beginning of this lecture are non-integrable.

Example 3.28. Let $\omega \in \Omega^2(M)$ be a closed two-form, and consider the Lie algebroid A_ω (see Examples 2.26 and 3.1). This is a transitive Lie algebroid and its anchor fits into the short exact sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow A_\omega \xrightarrow{\rho} TM \longrightarrow 0.$$

Using the obvious splitting of this sequence, Lemma 3.26 tells us that the monodromy group

$$\mathcal{N}_x(A_\omega) = \left\{ \int_\gamma \omega : [\gamma] \in \pi_2(M, x) \right\} \subset \mathbb{R}$$

is just the group Γ_ω of spherical periods of ω .

Assume that A_ω is integrable. Then the restricted isotropy group $\mathcal{G}_x(A_\omega)^0$ is a Lie group, and so Γ_ω must be discrete. Hence, A_ω is non-integrable if the group of spherical periods Γ_ω is non-discrete.

Example 3.29. Let us come back to the example of $\mathfrak{su}(2)$ with the modified Poisson structure $\{, \}_a$. (Example 3.3). Under the identification of $\mathfrak{su}(2)^* \simeq \mathbb{R}^3$, with coordinates (x, y, z) , the Lie bracket of the cotangent Lie algebroid of $\mathfrak{su}(2)^*$ is:

$$[dx, dy] = dz, \quad [dy, dz] = dx, \quad [dz, dx] = dy,$$

while the anchor $\rho : T^*\mathbb{R}^3 \rightarrow T\mathbb{R}^3$ is given by

$$\rho(dx) = X, \quad \rho(dy) = Y, \quad \rho(dz) = Z,$$

where X , Y , and Z are the infinitesimal generators of rotations around the coordinate axis:

$$X = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}, \quad Y = x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x}, \quad Z = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}.$$

After we rescale the Poisson bracket by the factor $a(r)$, the new anchor $\rho_a : T^*\mathbb{R}^3 \rightarrow T\mathbb{R}^3$ is related to the old one by:

$$\rho_a = a\rho.$$

Exercise 67. Determine the Lie bracket on 1-forms for the rescaled Poisson structure.

We now compute the monodromy of this new Lie algebroid. We restrict to a leaf \mathbb{S}_r^2 , with $r > 0$, and we pick the splitting of ρ_a defined by

$$\begin{aligned} \sigma(X) &= \frac{1}{a} \left(dx - \frac{x}{r^2} \bar{n} \right), & \sigma(Y) &= \frac{1}{a} \left(dy - \frac{y}{r^2} \bar{n} \right), \\ \sigma(Z) &= \frac{1}{a} \left(dz - \frac{z}{r^2} \bar{n} \right), \end{aligned}$$

where $\bar{n} = \frac{1}{r}(x dx + y dy + z dz)$. Then we obtain the center-valued 2-form:

$$\Omega_\sigma = \frac{a'(r)r - a(r)}{a(r)^2 r^4} \omega \bar{n}$$

where $\omega = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$. Since $\int_{\mathbb{S}_r^2} \omega = 4\pi r^3$ it follows that the monodromy groups are given by:

$$\mathcal{N}_{(x,y,z)} \simeq 4\pi \frac{A'(r)}{r} \mathbb{Z} \bar{n} \subset \mathbb{R} \bar{n},$$

where $A'(r) = 4\pi \frac{a'(r)r - a(r)}{a(r)^2}$ is the derivative of the symplectic areas. So the monodromy might vary in a non-trivial fashion, even nearby regular leaves.

Exercise 68. Verify that

$$\mathcal{G}_{(x,y,z)} = \begin{cases} SU(2) & \text{if } r = 0 \text{ and } a(r) \neq 0, \\ \mathbb{R}^3 & \text{if } r = 0 \text{ and } a(r) = 0, \\ \mathbb{S}^1 & \text{if } r \neq 0 \text{ and } A'(r) \neq 0, \\ \mathbb{R} & \text{if } r \neq 0 \text{ and } A'(r) = 0. \end{cases}$$

Also, explain why we obtain a non-integrable Lie algebroid whenever the symplectic area undergoes a critical point (i.e, $A'(r) = 0$).

Exercise 69. Determine the isotropy groups $\mathcal{G}_x(A)$ for the Lie algebroid of Example 3.2 and explain why A is not integrable.

Exercise 70. Give an example where a splitting as in Lemma 3.26 does not exist.

(Hint: Verify that the groups \mathcal{N}_x and $\tilde{\mathcal{N}}_x$ are distinct for the cotangent Lie algebroid of the Poisson manifold $M = \mathfrak{su}^*(3)$, at points lying in a orbit of dimension 4.)

3.5. Notes

Special instances of the integrability problem, for special classes of Lie algebroids, are well-known and have a positive solution. For example, we have:

- (i) For algebroids over a point (i.e., Lie algebras), the integrability problem is solved by Lie's third theorem on the integrability of (finite dimensional) Lie algebras by Lie groups;
- (ii) For algebroids with zero anchor map (i.e., bundles of Lie algebras), it is Douady-Lazard [19] extension of Lie's third theorem which ensures that the Lie groups integrating each Lie algebra fiber fit into a smooth bundle of Lie groups;
- (iii) For algebroids with injective anchor map (i.e., involutive distributions $\mathcal{F} \subset TM$), the integrability problem is solved by Frobenius' integrability theorem.

The integrability problem for general Lie algebroids goes back to Pradines original works [51, 52, 53, 54]. These notes contain little proofs, and he made the erroneous statement that all Lie algebroids are integrable. Almeida and Molino in [2] gave the first example of a non-integrable Lie algebroid in connection with developability of foliations. Other fundamental examples were discovered over many years by different authors, such as Élie Cartan's infinite continuous groups (Singer and Sternberg, [56]), the integrability of infinitesimal actions on manifolds (Palais, [50], Moerdijk and Mrčun [47]), abstract Atiyah sequences and transitive Lie algebroids (Almeida and Molino [2]; Mackenzie [37]), of Poisson manifolds (Weinstein, [62]) and of algebras of vector fields (Nistor, [48]).

The strategy to integrate Lie algebras to Lie groups that we have explained above appears in the book of Duistermaat and Kolk [21] (with minor changes). Alan Weinstein told one of us (RLF) about the possibility of applying this strategy to integrate Lie algebroids during a short visit to Berkeley, in late 2000. In [8], Cattaneo and Felder describe an approach to symplectic groupoids via an infinite dimensional symplectic reduction (the Poisson Sigma model), and which can be seen as a special instance of this strategy (however they fail to recognize the monodromy groups). Finally, Severa describes in the preprint [55] a similar strategy and makes some comments that are reminiscent of monodromy, without giving details.

The monodromy groups were introduced by us in [15]. Of course these groups have appeared before (in disguised form) in connection with some special classes of Lie algebroids. One example is the class of transitive groupoids, where the monodromy is equivalent to Mackenzie's cohomological class [37] that represents the obstruction to integrability. Another example, is provided by regular Poisson manifolds, where the monodromy groups appeared in the work of Alcade Cuesta and Hector [1].

LECTURE 4

Integrability: Smooth Theory

In the previous lecture, for *any* Lie algebroid A , we have constructed a topological groupoid $\mathcal{G}(A)$, by taking the equivalence classes of A -paths modulo A -homotopy. Moreover, we noticed that an obvious *necessary* condition for a Lie algebroid to be integrable is that the monodromy groups be discrete. In this lecture, we use the monodromy groups to give a complete answer to the problem we raised at the beginning of that lecture:

- What are the precise obstructions to integrate a Lie algebroid?

In the next lecture, we will give applications of the integrability criteria.

4.1. The Main Theorem

Here is one of the possible statements of our main integrability criteria:

Theorem 4.1. *For a Lie algebroid A , the following statements are equivalent:*

- (a) A is integrable.
- (b) $\mathcal{G}(A)$ is smooth.
- (c) the monodromy groups $\mathcal{N}_x(A)$ are locally uniformly discrete.

Moreover, in this case, $\mathcal{G}(A)$ is the unique s -simply connected Lie groupoid integrating A .

The precise meaning of (c) is the following: for any $x \in M$, there is an open set $U \subset A$ containing 0_x , such that $\mathcal{N}_y(A) \cap U = \{0\}$, for all y 's close enough to x . One can have a better understanding of the nature of this condition if one splits it into two conditions: a longitudinal one and a transversal one, relative to the orbits of A . To do this, we fix a norm on A , i.e., a norm on each fiber A_x which varies continuously with respect to x ¹, and we use it to measure the discreteness of the monodromy group $\mathcal{N}_x(A)$:

$$r_{\mathcal{N}}(x) = d(0, \mathcal{N}_x(A) - \{0\}).$$

Here we adopt the convention $d(0, \emptyset) = +\infty$. Note that the group \mathcal{N}_x is discrete iff $r_{\mathcal{N}}(x) > 0$. Now we can restate our theorem as follows:

¹For example, one can use a simple partition of unit argument to construct even a smooth field of norms.

Theorem 4.2. *A Lie algebroid A over M is integrable if and only if, for all $x \in M$, the following conditions hold:*

- (i) $r_{\mathcal{N}}(x) > 0$;
- (ii) $\liminf_y r_{\mathcal{N}}(y) > 0$, where the limit is over $y \rightarrow x$ with y outside the orbit through x .

Notice that, since $\mathcal{N}_x \simeq \mathcal{N}_y$ whenever x and y are in the same orbit, condition (i) can be seen as a longitudinal obstruction, while condition (ii) can be seen as a transversal obstruction. We should emphasize, also, that the transversal obstruction is *not* a stable condition: it can hold in a deleted neighborhood of a point without being true at the point. This comes from the fact that, in general, the map $r_{\mathcal{N}}$ is neither upper nor lower semicontinuous. Also, the two integrability obstructions are independent.

Here are a few examples to illustrate how wild $r_{\mathcal{N}}$ can behave.

Example 4.3. Consider the Lie algebroid A_{ω} associated to a 2-form ω on M (cf. Examples 2.26, 3.1 and 3.28). We deduce that, as promised, A_{ω} is integrable if and only if Γ_{ω} is discrete. Note that, in this case, the second obstruction is void, since our Lie algebroid is transitive.

Example 4.4. Let us look one more time at the example of $\mathfrak{su}^*(2)$ with the modified Poisson structure $\{, \}_a$ (cf. Examples 3.3 and 3.29). If we choose the standard euclidian norm on the fibers $A = T^*(\mathfrak{su}^*(2))$, the discussion in Example 3.29 shows that:

$$r_{\mathcal{N}}(x, y, z) = \begin{cases} 4\pi \frac{A'(r)}{r} \mathbb{Z}\bar{n}, & \text{if } A'(r) \neq 0, \\ \infty, & \text{otherwise.} \end{cases}$$

In this case, $r_{\mathcal{N}}$ is not upper semicontinuous at the critical points of the symplectic area $A(r)$. Also, the theorem above implies that the integrability of the underlying cotangent Lie algebroid is equivalent to the fact that the symplectic area has no critical points. In this example, only the second obstruction is violated.

Example 4.5. Consider the manifold $M = \mathbb{S}^2 \times \mathbb{H}$, where \mathbb{H} denotes the quaternions. We take the Lie algebroid A over M which, as a vector bundle, is trivial of rank 3, with a fixed basis $\{e_1, e_2, e_3\}$. The Lie bracket is defined by:

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.$$

To define the anchor, we let v_1, v_2 and v_3 be the vector fields on \mathbb{S}^2 obtained by restriction of the infinitesimal generators of rotations around the coordinate axis (the X, Y and Z in Example 3.3). We also consider the vector fields on \mathbb{H} corresponding to multiplication by i, j and k , denoted w_1, w_2 and w_3 . The anchor of A is then defined by

$$\rho(e_i) = (v_i, \frac{1}{2}w_i).$$

A simple computation shows that this Lie algebroid has only two leaves. More precisely,

- (a) the anchor is injective on $\mathbb{S}^2 \times (\mathbb{H} - \{0\})$.
- (b) $\mathbb{S}^2 \times \{0\}$ is an orbit.

In particular, the monodromy groups at all points outside $\mathbb{S}^2 \times \{0\}$ are zero. On the other hand, when restricted to $\mathbb{S}^2 \times \{0\}$, A becomes the algebroid on \mathbb{S}^2 associated to the area form. Hence we obtain

$$r_{\mathcal{N}}(x) = \begin{cases} r_0 & \text{if } x \in \mathbb{S}^2 \times \{0\}, \\ \infty & \text{otherwise,} \end{cases}$$

where $r_0 > 0$. In particular, $r_{\mathcal{N}}$ is not lower semicontinuous. However, in this case, there are no obstructions and A is integrable.

4.2. Smooth structure

In order to describe the smooth structure on $\mathcal{G}(A)$, we first describe the smooth structure on $P(A)$. We consider the larger space $\tilde{P}(A)$ of all C^1 -curves $a : I \rightarrow A$ with base path $\gamma = \pi \circ a$ of class C^2 . It has an obvious structure of Banach manifold.

Exercise 71. Show that the tangent space $T_a(\tilde{P}(A))$ consists of all C^0 -curves $U : I \rightarrow TA$ such that $U(t) \in T_{a(t)}A$. Using a TM -connection ∇ on A , check that such curves can be viewed as pairs (u, ϕ) formed by a curve $u : I \rightarrow A$ over γ and a curve $\phi : I \rightarrow TM$ over γ (the *vertical* and *horizontal* component of U).

From now on, we fix a TM -connection ∇ on A and we use the description of $T_a(\tilde{P}(A))$ given in this exercise. Also, we will use the associated A -connections $\bar{\nabla}$ that were discussed in Exercise 46.

Lemma 4.6. $P(A)$ is a (Banach) submanifold of $\tilde{P}(A)$. Moreover, its tangent space $T_a P(A)$ consists of those paths $U = (u, \phi)$ with the property that

$$\rho(u) = \bar{\nabla}_a \phi.$$

Proof. We consider the smooth map $F : \tilde{P}(A) \rightarrow \tilde{P}(TM)$ given by

$$F(a) = \rho(a) - \frac{d}{dt} \pi \circ a.$$

Clearly $P(A) = F^{-1}(Q)$, where Q is the submanifold of $\tilde{P}(TM)$ consisting of zero paths. Fix $a \in P(A)$, with base path $\gamma = \pi \circ a$, and let us compute the image of $U = (u, \phi) \in T_a \tilde{P}(A)$ by the differential

$$(dF)_a : T_a \tilde{P}(A) \rightarrow T_{0_\gamma} \tilde{P}(TM).$$

The result will be some path $t \mapsto (dF)_a \cdot U(t) \in T_{0_{\gamma(t)}} TM$, hence, using the canonical splitting $T_{0_x} TM \cong T_x M \oplus T_x M$, it will have a horizontal and vertical component. We claim that for any connection ∇ , if (u, ϕ) are the components of U , then

$$((dF)_a \cdot U)^{\text{hor}} = \phi, \quad ((dF)_a \cdot U)^{\text{ver}} = \rho(u) - \bar{\nabla}_a \phi.$$

Note that this immediately implies that F is transverse to Q , so the assertion of the proposition follows. Since this decomposition is independent of the connection ∇ and it is local (we can look at restrictions of a to smaller intervals), we may assume that we are in local coordinates, and that ∇ is the standard flat connection. We let $x = (x^1, \dots, x^n)$ denote local coordinates on M , and we denote by $\frac{\partial}{\partial x^i}$ the horizontal basis of $T_{0_x} TM$, and by $\frac{\delta}{\delta x^i}$ the

vertical basis. Also, we denote by $\{e_1, \dots, e_k\}$ a (local) basis of A over this chart. The anchor and the bracket of A decompose as⁽²⁾

$$\rho(e_p) = b_p^i \frac{\partial}{\partial x^i}, \quad [e_p, e_q] = c_{pq}^r e_r,$$

and an A -path a can be written $a(t) = a^p(t)e_p$. A simple computation shows that the horizontal component of $(dF)_a(u, \phi)$ is $\phi^i \frac{\partial}{\partial x^i}$, while its vertical component is

$$\left(-\dot{\phi}^j(t) + u^p(t)b_p^j(\gamma(t)) + a^p(t)\phi^i(t)\frac{\partial b_p^j}{\partial x^i}(\gamma(t)) \right) \frac{\delta}{\delta x^j}.$$

That this is precisely $\rho(u) - \bar{\nabla}_a \phi$ immediately follows by computing

$$\begin{aligned} \bar{\nabla}_{e_p} \frac{\partial}{\partial x^i} &= \rho(\nabla_{\frac{\partial}{\partial x^i}} e_p) - \left[\frac{\partial}{\partial x^i}, \rho(e_p) \right] \\ &= -\frac{\partial b_p^j}{\partial x^i} \frac{\partial}{\partial x^j} \end{aligned}$$

□

Finally, we can be more precise about the smoothness of $\mathcal{G}(A)$.

Definition 4.7. We say that $\mathcal{G}(A)$ is smooth if it admits a smooth structure with the property that the projection $P(A) \rightarrow \mathcal{G}(A)$ is a submersion.

Note that, by the local form of submersions, it follows that the smooth structure on $\mathcal{G}(A)$ will be unique if it exists. Invoking the local form of submersions requires some care, since we are in the infinite dimensional setting. However, there are no problems in our case since, as we shall see in the next section, the fibers of the projection map are submanifolds of $P(A)$ of finite codimension.

Exercise 72. Check that if $\mathcal{G}(A)$ is smooth, then $\mathcal{G}(A)$ is a Lie groupoid. (Hint: At the level of $P(A)$, the structure maps (i.e., concatenation, inverses, etc.) are smooth).

4.3. A -homotopy revisited

Before we can proceed with the proof of our main result, we need a better control on the A -homotopy equivalence relation defining the groupoid $\mathcal{G}(A)$. In this section we will show that we can describe the equivalence classes of the A -homotopy relation \sim as the leaves of a foliation $\mathcal{F}(A)$ on $P(A)$, of finite codimension. Hence, $\mathcal{G}(A)$ is the leaf space of a foliation $\mathcal{F}(A)$ on $P(A)$ of finite codimension.

Using that the basic connection $\bar{\nabla}$ associated to ∇ commutes with the anchor (see Exercise 46), there is a simple way of constructing vector fields on $P(A)$. To make this more explicit, for a path γ in M , we consider the space

$$\tilde{P}_\gamma(A) = \{b \in \tilde{P}(A) : b(0) = 0, b(t) \in A_{\gamma(t)}\}.$$

²Recall that we use the Einstein sum convention.

Definition 4.8. Let ∇ be a connection on A . Given an A -path a_0 with base path γ_0 , and $b_0 \in \tilde{P}_{\gamma_0}(A)$, we define

$$X_{b_0, a_0} \in T_{a_0} \tilde{P}(A)$$

as the tangent vector with components (u, ϕ) relative to the connection ∇ given by

$$u = \bar{\nabla}_{a_0} b_0, \quad \phi = \rho(b_0).$$

The following lemma gives a geometric interpretation of these vector fields, showing that they are “tangent” to A -homotopies:

Lemma 4.9. *The vectors X_{b_0, a_0} are tangent to $P(A)$ and they do not depend on the choice of the connection. More precisely, choosing a and b satisfying the homotopy condition (3.16):*

$$(4.10) \quad \partial_t b - \partial_\epsilon a = T_{\nabla}(a, b), \quad b(\epsilon, 0) = 0,$$

and such that $a(0, t) = a_0(t)$, $b(0, t) = b_0(t)$, then

$$(4.11) \quad X_{b_0, a_0} = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} a_\epsilon \in T_{a_0} P(A).$$

Proof. The components of X_{b_0, a_0} clearly satisfy the condition from Lemma 4.6, hence we obtain a vector tangent to $P(A)$. Choosing a and b as in the statement, we compute the vertical and horizontal components (u, ϕ) with respect to ∇ of the tangent vector

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} a_\epsilon(t) \in T_{a_0(t)} A.$$

We find:

$$u = \partial_\epsilon a|_{\epsilon=0} = \partial_t b_0 - T_{\nabla}(a_0, b_0) = \bar{\nabla}_{a_0} b_0,$$

$$\phi = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \gamma_\epsilon(t) = \rho(b_0(t)),$$

which are precisely the components of X_{b_0, a_0} . \square

Remark 4.12. One should point out that, in the previous lemma, given a_0 and b_0 , one can always extend them to families a_ϵ and b_ϵ satisfying the required conditions. For instance, write $b_0(t) = \eta(t, \gamma_0(t))$ for some time dependent section $\eta = \eta(t, x)$ with $\eta_0 = 0$ (one may also assume that η has compact support) and put

$$\gamma(\epsilon, t) = \phi_{\rho(\eta_\epsilon)}^{\epsilon, 0}(\gamma_0(t)), \quad b(\epsilon, t) = \eta(t, \gamma(\epsilon, t)).$$

Next, one solves the equation in (4.10), with the unknown $a = a(\epsilon, t)$, satisfying the initial condition $a(0, t) = a_0(t)$.

Of course, the solution a can be described without any reference to the connection, and this is what we have seen in Proposition 3.15: write $a_0(t) = \xi_0(t, \gamma_0(t))$ for some time dependent ξ_0 , then consider the (ϵ, t) -dependent solution ξ of

$$(4.13) \quad \frac{d\xi}{d\epsilon} - \frac{d\eta}{dt} = [\xi, \eta], \quad \xi(0, t, x) = \xi_0(t, x),$$

and, finally,

$$a(\epsilon, t) = \xi(\epsilon, t, \gamma(\epsilon, t)).$$

Next, we move towards our aim of realizing homotopy classes of A -paths as leaves of a foliation on $P(A)$. Due to the definition of the homotopy, we have to restrict to paths b_0 vanishing at the end points. So, for a path γ in M , consider

$$\tilde{P}_{0,\gamma}(A) = \{b \in \tilde{P}_\gamma(A) : b(1) = 0\}.$$

Definition 4.14. We denote by \mathcal{D} the distribution on $P(A)$ whose fiber at $a \in P(A)$ is given by

$$\mathcal{D}_a \equiv \left\{ X_{b,a} \in T_a P(A) : b \in \tilde{P}_{0,\gamma}(A) \right\}.$$

Now we have:

Lemma 4.15. *A family $\{a_\epsilon\}$ of A -paths is a homotopy if and only if the path $I \ni \epsilon \mapsto a_\epsilon \in P(A)$ is tangent to \mathcal{D} . In particular, two A -paths are homotopic if and only if they are connected by a path tangent to \mathcal{D} .*

Proof. We will continue to use the notations from Proposition 3.15 (in particular, we assume that a connection ∇ on A has been fixed). Exactly as in the proof of Lemma 4.9, we compute the components of the derivatives (this time at arbitrary ϵ 's)

$$\frac{d}{d\epsilon} a_\epsilon \in T_{a_\epsilon} P(A)$$

and we find that this derivative coincides with $X_{b_\epsilon, a_\epsilon}$. Hence, if $\{a_\epsilon\}$ is a homotopy, then $b_\epsilon(1) = 0$ and we deduce that $\epsilon \mapsto a_\epsilon$ is tangent to \mathcal{D} .

For the converse, we fix ϵ and we want to prove that $b_\epsilon(1) = 0$. But the assumption implies that there exists $c \in \tilde{P}_{0,\gamma_\epsilon}(A)$ (remember that we have fixed ϵ) such that $X_{c, a_\epsilon} = X_{b_\epsilon, a_\epsilon}$. Writing the components with respect to a connection, we find that

$$\bar{\nabla}_a(b_\epsilon - c) = 0.$$

Since $b_\epsilon(0) - c(0) = 0$, we see that $b_\epsilon = c$. In particular, $b_\epsilon(1) = c(1) = 0$. \square

Next we show that the distribution \mathcal{D} is integrable. Set:

$$P_0\Gamma(A) := \{I \ni t \mapsto \eta_t \in \Gamma(A) : \eta_0 = \eta_1 = 0, \eta \text{ is of class } C^2 \text{ in } t\}.$$

Then $P_0\Gamma(A)$ is a Lie algebra with Lie bracket the pointwise bracket induced from $\Gamma(A)$. For each $\eta \in P_0\Gamma(A)$ define a vector field X_η on $P(A)$ as follows. If $a \in P(A)$, let $b = \eta(t, \gamma(t))$, where γ is the base path of a , and set:

$$X_\eta|_a := X_{b,a} \in T_a P(A).$$

Exercise 73. Show that, X_η can be written in terms of flows as:

$$X_\eta|_a = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \phi_{\eta_t}^{\epsilon,0}(a(t)) + \frac{d\eta_t}{dt}(\gamma(t)).$$

(Hint: Show that once ξ_0 has been chosen, then ξ is given by:

$$\xi(\epsilon, t) = \int_0^\epsilon (\phi_{\eta_t}^{\epsilon',\epsilon})^* \frac{d\eta}{dt}(t) d\epsilon' + (\phi_{\eta_t}^{0,\epsilon})^* \xi_0$$

(cf. Remark 4.12). Deduce the explicit integral formula:

$$a_\epsilon(t) = \int_0^\epsilon \phi_{\eta_t}^{\epsilon,\epsilon'} \left(\frac{d\eta_t}{dt}(\phi_{\rho(\eta_t)}^{\epsilon',0}(\gamma_0(t))) \right) d\epsilon' + \phi_{\eta_t}^{\epsilon,0}(a_0(t))$$

(cf. Proposition 3.15). Now differentiate at $\epsilon = 0$ to find $X_\eta|_a$.)

Now we can show that \mathcal{D} is an involutive distribution so that there exists a foliation $\mathcal{F}(A)$ integrating it:

Lemma 4.16. *The vector fields X_η span the distribution \mathcal{D} . Moreover, the map*

$$P_0\Gamma(A) \rightarrow \mathfrak{X}(P(A)), \quad \eta \mapsto X_\eta$$

defines an action of the Lie algebra $P_0\Gamma(A)$ on $P(A)$. In particular, \mathcal{D} is involutive.

Proof. The first assertion is clear since for any $b \in \tilde{P}_{0,\gamma}(A)$ we can find $\eta \in P_0\Gamma(A)$ such that $b(t) = \eta(t, \gamma(t))$. The proof of the second assertion follows by a computing in local coordinates. \square

Finally, we can put together the main properties of the foliation $\mathcal{F}(A)$ integrating \mathcal{D} :

Theorem 4.17. *For a Lie algebroid A , there exists a foliation $\mathcal{F}(A)$ on $P(A)$ such that:*

- (i) $\mathcal{F}(A)$ is a foliation of finite codimension equal to $n + k$, where $n = \dim M$ and $k = \text{rank } A$.
- (ii) Two A -paths are equivalent (homotopic) if and only if they are in the same leaf of $\mathcal{F}(A)$.

Proof. We already know that $\mathcal{F}(A)$ is a foliation whose leaves are precisely the homotopy classes of A -paths.

To determine the codimension of $\mathcal{D}_a = T_a\mathcal{F}(A)$, we use a connection ∇ to construct a surjective map

$$\nu_a : T_aP(A) \longrightarrow A_{\gamma(1)} \oplus T_{\gamma(0)}M$$

whose kernel is precisely \mathcal{D}_a . Explicitly, given a vector tangent to $P(A)$ with components (u, ϕ) , we put

$$\nu_a(u, \phi) = (b(1), \phi(0)),$$

where b is the solution of the equation $\bar{\nabla}_a(b) = u$ with initial condition $b(0) = 0$ (which can be expressed in terms of the parallel transport along a with respect to $\bar{\nabla}$). Assume now that $\nu_a(u, \phi) = 0$, i.e., $b(1) = 0$ and $\phi(0) = 0$. Since (u, ϕ) is tangent to $P(A)$, we must have $\rho(u) = \bar{\nabla}_a(\phi)$. Replacing u by $\bar{\nabla}_a(b)$ we deduce that

$$\bar{\nabla}_a(\rho(b) - \phi) = 0.$$

Since $\rho(b) - \phi$ vanishes at the initial point, we deduce that $\phi = \rho(b)$. Hence (u, ϕ) gives precisely the vector $X_{b,a}$ tangent to $\mathcal{F}(A)$. We deduce that the kernel of ν_a is \mathcal{D}_a . The surjectivity of ν_a is easily checked: given $b_1 \in A_{\gamma(1)}$ and $\phi_0 \in T_{\gamma(0)}M$, we denote by ϕ the solution of the equation $\bar{\nabla}_a(\phi) = 0$ with initial condition ϕ_0 and we choose any path $b : I \longrightarrow A$ above γ joining 0 and b_1 . Then $(\bar{\nabla}_a(b), \phi + \rho(b))$ is tangent to $P(A)$ and is mapped by ν_a into (b_1, ϕ_0) . This proves (i). \square

4.4. The exponential map

Recall that for a Lie group G with Lie algebra \mathfrak{g} , the exponential map $\exp : \mathfrak{g} \rightarrow G$ is a local diffeomorphism around the origin. Hence the smooth structure on G around the identity element can be obtained from \mathfrak{g} and the exponential map; using then right translations, the same is true around each point in G . With this in mind, our plan is to construct a similar exponential map for algebroids, and then use it to obtain a smooth structure on $\mathcal{G}(A)$.

Let \mathcal{G} be a Lie groupoid with Lie algebroid A . The exponential map that we will use will require the choice of a connection ∇ on the vector bundle A . Let ∇ be such a connection, and take the pull-back of ∇ along the target map

$$\mathbf{t} : \mathcal{G}(x, -) \rightarrow M.$$

This will define a connection on the vector bundle \mathbf{t}^*A restricted to $\mathcal{G}(x, -)$, which is isomorphic to $T\mathcal{G}(x, -)$. Hence we obtain a connection, denoted by ∇^x , on the s -fiber $\mathcal{G}(x, -)$. We consider the associated exponential, defined on a neighborhood of the origin in $T_x\mathcal{G}(x, -) = A_x$, which by abuse of notation, we write it as if it was defined on the entire A_x :

$$\text{Exp}_{\nabla^x} : A_x \longrightarrow \mathcal{G}(x, -).$$

Putting together these exponential maps we obtain a global exponential map

$$\overline{\text{Exp}}_{\nabla} : A \rightarrow \mathcal{G}$$

Again, in spite of the notation, this is only defined on an open neighborhood of the zero section. This exponential map has the desired property:

Lemma 4.18. *$\overline{\text{Exp}}_{\nabla}$ is a diffeomorphism from an open neighborhood of the zero section in A to an open neighborhood of M in \mathcal{G} .*

Proof. Abbreviate $E = \overline{\text{Exp}}_{\nabla}$ and $E^x = \overline{\text{Exp}}_{\nabla^x}$. It is enough to prove that, for an arbitrary $x \in M$, the differential of E at $0_x \in A$ is a linear isomorphism from $T_{0_x}A$ to $T_{1_x}\mathcal{G}$. To see this, we first remark that (dE_x) is a linear isomorphism from A_x into $T_{1_x}\mathcal{G}(x, -)$. Secondly, we remark that we have a commutative diagram whose horizontal lines are short exact sequences:

$$\begin{array}{ccccc} A_x & \longrightarrow & T_{0_x}A & \xrightarrow{(d\pi)} & T_xM \\ (dE^x) \downarrow & & (dE) \downarrow & & \downarrow \text{Id} \\ T_x\mathcal{G}(x, -) & \longrightarrow & T_x\mathcal{G} & \xrightarrow{(ds)} & T_xM \end{array}$$

The desired conclusion now follows from a simple diagram chasing. \square

Next, we define a version of $\overline{\text{Exp}}_{\nabla}$ which makes sense on the (possibly non-smooth) $\mathcal{G}(A)$, and which coincides with the construction above when $\mathcal{G} = \mathcal{G}(A)$ is smooth.

Let A be any Lie algebroid and let ∇ be a connection on A . We denote by the same letter the induced A -connection on A : $\nabla_{\alpha}\beta = \nabla_{\rho(\alpha)}\beta$. Similar to the classical case, we consider A -geodesics with respect to ∇ , i.e., A -paths a with the property that $\nabla_a a = 0$. Still as in the classical case (and exactly by the same arguments), for each $v \in A_x$ there is a unique geodesic a_v with

$a_v(0) = v$. Also, for v close enough to zero, a_v is defined on the entire interval I . Hence we obtain a map $v \mapsto a_v$, denoted

$$\text{Exp}_\nabla : A \rightarrow P(A)$$

and defined on an open neighborhood of M in A . Taking homotopy classes of A -paths, we obtain a map $v \mapsto [a_v]$ denoted

$$\overline{\text{Exp}}_\nabla : A \longrightarrow \mathcal{G}(A).$$

Next, let us check that the two versions of $\overline{\text{Exp}}_\nabla$ are compatible.

Lemma 4.19. *If $\mathcal{G}(A)$ is smooth, then the two constructions above for $\overline{\text{Exp}}_\nabla : A \rightarrow \mathcal{G}(A)$ coincide.*

Proof. To emphasize that the first construction of $\overline{\text{Exp}}_\nabla$ uses the smooth structure of $\mathcal{G}(A)$, let us assume that \mathcal{G} is an arbitrary s -connected Lie groupoid integrating A ; at the end we will set $\mathcal{G} = \mathcal{G}(A)$. Again, we abbreviate the notations for the exponential maps associated to ∇ : $E : A \rightarrow \mathcal{G}(A)$ and $E' : A \rightarrow \mathcal{G}$. The key remark now is: if $g : I \rightarrow \mathcal{G}(x, -)$ is a geodesic with respect to ∇^x , then the A -path $a = D^R(g)$ is an A -geodesic with respect to ∇ . This shows that the two exponential maps coincide modulo the isomorphism D^R between $\tilde{\mathcal{G}} = P(\mathcal{G})/\sim$ and $\mathcal{G}(A)$. \square

As a consequence we obtain

Corollary 4.20. *If a Lie algebroid A is integrable, then*

$$\overline{\text{Exp}}_\nabla : A \rightarrow \mathcal{G}(A)$$

is injective around the zero section.

We will see later in the lecture, in Theorem 4.23, that the converse is also true. At this point, we can detect the monodromy groups $\mathcal{N}_x(A)$ as the obvious obstructions to the integrability of A . Indeed, since $\overline{\text{Exp}}_\nabla$ restricted to \mathfrak{g}_x is the composition of the exponential map of \mathfrak{g}_x with the obvious map $i : \mathcal{G}(\mathfrak{g}_x) \rightarrow \mathcal{G}(A)_x$, we will have $\overline{\text{Exp}}_\nabla(v_x) = 1_x$ for all $v_x \in \mathcal{N}_x(A)$ in the domain of the exponential map. We conclude:

Corollary 4.21. *If $\mathcal{G}(A)$ is smooth (i.e., if A is integrable), then the monodromy groups $\mathcal{N}_x(A)$ are locally uniformly discrete.*

Next, to understand how the exponential map can be used to obtain a smooth structure on $\mathcal{G}(A)$ (when it is injective), we have to understand its behaviour at the level of the paths space.

Proposition 4.22. *For any (local) connection ∇ on A , the exponential map $\text{Exp}_\nabla : A \rightarrow P(A)$ is transverse to $\mathcal{F}(A)$.*

Proof. We assume, for simplicity, that we are in local coordinates and that ∇ is the trivial flat connection (this is actually all we will use for the proof of the main theorem, and this in turn will imply the full statement of (iii)). Also, we only need to show that $\text{Exp}_\nabla(A)$ is transverse to $\mathcal{F}(A)$ at any trivial A -path $a = O_x$ over $x \in M$. Now, the equations for the geodesics show that if (u, ϕ) is a tangent vector to $\text{Exp}_\nabla(A)$ at a then we must have:

$$\dot{\phi}^i = b_p^i(x)u^p, \quad \dot{u}^p = 0.$$

Therefore, we see that:

$$T_a \text{Exp}_{\nabla}(A) = \{(u, \phi) \in T_a P(A) : u(t) = u_0, \phi(t) = \phi_0 + t\rho(u_0)\}.$$

Suppose now that a tangent vector (u, ϕ) belongs to this $n + k$ dimensional space and is also tangent to $\mathcal{F}(A)$. Since (u, ϕ) is tangent to $\mathcal{F}(A)$ we must have $\phi_0 = 0$ and $b(1) = 0$ (see the proof of Theorem 4.17). Therefore, we must have $\phi_0 = 0$ and $u_0 = 0$, so (u, ϕ) is the null tangent vector. This shows that $\text{Exp}_{\nabla}(A)$ is transverse to $\mathcal{F}(A)$ at 0_x , for any x . \square

Finally, we have:

Theorem 4.23. *$\mathcal{G}(A)$ is smooth if and only if the exponential map is injective around the zero section. Moreover, in this case $\mathcal{G}(A)$ is a Lie groupoid integrating A .*

Proof. Assuming injectivity, we will prove that $\mathcal{G}(A)$ is smooth. We have to prove that the foliation $\mathcal{F}(A)$ on $P(A)$ is simple. For this it suffices to show that, for each $a \in P(A)$, there exists $S_a \subset P(A)$ which is transverse to $\mathcal{F}(A)$, and which intersects each leaf of $\mathcal{F}(A)$ in at most one point. We will call such an S_a , a simple transversal through a .

Let a be an arbitrary A -path. We denote by γ its base path and by $x \in M$ its initial point, and we let $a(t) = \xi(t, \gamma(t))$ for some compactly supported, time dependent, section ξ of A . Then we define

$$\sigma_{\xi} : M \rightarrow P(A), \quad \sigma_{\xi}(y)(t) = \xi(t, \phi_{\rho\xi}^{t,0}(y)).$$

Composing on the left with σ_{ξ} , we obtain a smooth injective map

$$T_{\xi} : P(A) \rightarrow P(A), \quad T_{\xi}(b) = \sigma_{\xi}(\mathbf{t}(b))b.$$

By the hypothesis and the previous proposition, there exists some open set $U \subset A$ containing 0_x such that

$$\text{Exp}_U = \text{Exp}_{\nabla} : U \rightarrow P(A)$$

induces a simple transversal $S_x \subset P(A)$ through 0_x . Applying T_{ξ} , we obtain a simple transversal $T_{\xi}(S_x)$ through $T_{\xi}(0_x)$. But $T_{\xi}(0_x) = 0_{\xi}a$ is homotopic to (hence in the same leaf) as a . Using the holonomy of the foliation $\mathcal{F}(A)$ along any path from $T(0_x)$ to a , we obtain a simple transversal through a .

Next, we have to prove that the Lie algebroid of $\mathcal{G}(A)$ is isomorphic to A . It is clear from the definitions that A can be identified with $T_M^s \mathcal{G}(A)$, as vector bundles, and that under this identification ρ coincides with the differential of the target \mathbf{t} . So we need only to check that the bracket of right-invariant vector fields on $\mathcal{G}(A)$ is identified with the bracket of sections of A . For this we note that, on one hand, the bracket is completely determined by the infinitesimal flow of sections through the basic formula (3.14). On the other hand, we also know that the exponential $\exp : \Gamma(A) \rightarrow \Gamma(\mathcal{G}(A))$ is injective in a neighborhood of the zero section, and so Exercise 64 shows that the infinitesimal flow of a section α is the infinitesimal flow of the right-invariant vector field on $\mathcal{G}(A)$ determined by α . Hence, we must have $A(\mathcal{G}(A)) = A$. \square

4.5. End of the proof: injectivity of the exponential

In this section we complete the proof of the main theorem. According to Theorem 4.23 in the last section, it suffices to show:

Proposition 4.24. *Any $x \in M$ admits a neighborhood V such that the exponential $\overline{\text{Exp}}_{\nabla} : V \rightarrow \mathcal{G}(A)$ is injective.*

In other words, we look for V such that $\text{Exp}_{\nabla} : V \rightarrow P(A)$ intersects each leaf of $\mathcal{F}(A)$ in at most one point. This will be proven in several steps by a sequence of reductions and careful choices.

Fix $x \in M$, choose local coordinates around x , and let ∇ be the canonical flat connection on the coordinate neighborhood. We also choose a small neighborhood U of 0_x in A so that the exponential map $\text{Exp}_{\nabla} : U \rightarrow P(A)$ is defined and is transverse to $\mathcal{F}(A)$.

Claim 1. We may choose U such that, for any $v \in U \cap \mathfrak{g}_y$ ($y \in M$) with the property that $\text{Exp}_{\nabla}(v)$ is homotopic to 0_y , we must have $v \in Z(\mathfrak{g}_y)$.

Given a norm $|\cdot|$ on A , the set $\{|[v, w]| : v, w \in \mathfrak{g}_y \text{ with } |v| = |w| = 1\}$, where $y \in M$ varies in a neighborhood of x , is bounded. Rescaling $|\cdot|$ if necessary, we find a neighborhood D of x in M , and a norm $|\cdot|$ on $A_D = \{v : \pi(v) \in D\}$, such that $|[v, w]| \leq |v||w|$ for all $v, w \in \mathfrak{g}_y$ with $y \in D$. We now choose U so that $U \subset A_D$ and $|v| < 2\pi$ for all $v \in U$. If v is as in the claim, it follows (see Exercise 62) that parallel transport $T_v : \mathfrak{g}_y \rightarrow \mathfrak{g}_y$ along the constant A -path v is the identity. But T_v is precisely the exponential of the linear map $\text{ad}_v : \mathfrak{g}_y \rightarrow \mathfrak{g}_y$. Since $\|\text{ad}_v\| \leq |v| < 2\pi$, it follows that all the eigenvalues of ad_v are of norm less than 2π , and then we deduce that all these eigenvalues must be zero. In conclusion $\text{ad}_v = 0$, and the claim follows.

Claim 2. We may choose U such that, if $v \in U \cap \mathfrak{g}_y$ ($y \in M$) has the property that $\text{Exp}_{\nabla}(v)$ is homotopic to 0_y then $v = 0_y$.

Obviously this is just a restatement of the obstruction assumptions, combined with the previous claim.

Claim 3. We may choose U such that, if $v \in U$ has the property that the base path of $\text{Exp}_{\nabla}(v)$ is closed, then $v \in \mathfrak{g}_y$.

To see this, we write the equations for the geodesics:

$$\dot{x}^i = b_p^i(x(t))a^p, \quad \dot{a}^p = 0,$$

and we apply the following Period Bounding Lemma:

Lemma 4.25. *Let $\Omega \subset \mathbb{R}^n$ be an open set, let $F : \Omega \rightarrow \mathbb{R}^n$ be a smooth map, and assume that*

$$\sup_{x \in \Omega} \left| \frac{\partial F^j}{\partial x_k}(x) \right| \leq L, \quad (1 \leq j, k \leq n).$$

Then, any periodic solution of the equation $\dot{x}(t) = F(x(t))$ must have period T such that

$$T \geq \frac{2\pi}{L}.$$

Hence, it suffices to choose $U \subset A_D$, where D is chosen small enough so that

$$\sup_{\substack{1 \leq j, k \leq m \\ x \in \bar{D}}} \left\| \frac{\partial b_p^j}{\partial x^k}(x) a^p \right\| < 2\pi,$$

and the claim follows.

Now, for any open set $O \subset P(A)$, we consider the plaques in O of $\mathcal{F}(A)$, or, equivalently, the leaves of $\mathcal{F}(A)|_O$. For $a, b \in O$, we write $a \sim_O b$ if a and b lie in the same plaque. From now on, we fix U satisfying all the conditions above, and we choose an open set O such that $\text{Exp}_\nabla : U \rightarrow P(A)$ intersects each plaque inside O exactly in one point. This is possible since Exp_∇ is transversal to $\mathcal{F}(A)$. Apart from the pair (O, U) , we also choose similar pairs (O_i, U_i) , $i = 1, 2$, such that $O_1 O_1 \subset O$, $O_2 O_2 \subset O_1$ and $O_i^{-1} = O_i$.

Claim 4. It is possible to choose a neighborhood V of x in U_2 so that, for all $v \in V$,

$$0_y \cdot \text{Exp}_\nabla(v) \sim_O \text{Exp}_\nabla(v).$$

We know that for any v there is a natural homotopy between the two elements above. This homotopy can be viewed as a smooth map $h : I \times U \rightarrow P(A)$ with $h(0, v) = 0 \cdot \text{Exp}_\nabla(v)$, $h(1, v) = \text{Exp}_\nabla(v)$, $h(t, 0_y) = 0_y$. Since I is compact and O is open, we can find V around x such that $h(I \times V) \subset O$. Obviously, V has the desired property.

Claim 5. It is possible to choose V so that, for all $v, w \in V$,

$$(\text{Exp}_\nabla(v) \cdot \overline{\text{Exp}_\nabla(w)}) \cdot \text{Exp}_\nabla(w) \sim_O \text{Exp}_\nabla(v)$$

This is proved exactly as the previous claim. Our final claim is:

Claim 6. $\text{Exp}_\nabla : V \rightarrow P(A)$ intersects each leaf of $\mathcal{F}(A)$ in at most one point.

To see this, let us assume that $v, w \in V$ have $\text{Exp}_\nabla(v) \sim \text{Exp}_\nabla(w)$. Then $a_1 := \text{Exp}_\nabla(v) \cdot \overline{\text{Exp}_\nabla(w)} \in O_1$ will be homotopic to the trivial A -path 0_y . On the other hand, by the choice of the pair (O_1, U_1) , $a_1 \sim_{O_1} \text{Exp}_\nabla(u)$ for a unique $u \in U_1$. Since $\text{Exp}_\nabla(u)$ is equivalent to 0_y , its base path must be closed, hence, by claim 4 above, $u \in \mathfrak{g}_y$. Using Claim 3, it follows that $u = 0$, hence $a_1 \sim_{O_1} 0_y$. Since $O_1 O_1 \subset O$, this obviously implies that

$$a_1 \cdot \text{Exp}_\nabla(w) \sim_O 0_y \cdot \text{Exp}_\nabla(w).$$

Since V satisfies Claim 5 and Claim 6, we get $\text{Exp}_\nabla(v) \sim_O \text{Exp}_\nabla(w)$. Hence, by the construction of O , $v = w$. This concludes the proof of the main theorem. \square

Exercise 74. How can one modify this proof, to show that, in the main theorem, it suffices to require that for each leaf L , there exists $x \in L$ satisfying the two obstructions.

Exercise 75. Define the notion of a *local Lie groupoid* (so that the structure maps are defined only on appropriate small neighborhoods of the identity section) and adapt this proof to show that every Lie algebroid integrates to a local Lie groupoid.

4.6. Notes

Theorem 4.2 is the main theorem of these lectures and appears first in [15]. The proof is the same that we describe here, and makes use of the key Period Bounding Lemma which is due to Yorke [60].

LECTURE 5

An example: integrability and Poisson geometry

In this last lecture we will illustrate some of the previous results on integrability of Lie bracket with an application to Poisson geometry.

5.1. Integrability of Poisson brackets

As we saw in Example 2.28, to a Poisson manifold $(M, \{ , \})$ there is associated a cotangent Lie algebroid $(T^*M, \rho, [,])$ where the anchor $\rho : \Omega^1(M) \rightarrow \mathfrak{X}(M)$ and the bracket $[,] : \Omega^1(M) \times \Omega^1(M) \rightarrow \Omega^1(M)$, on exact 1-forms is given by:

$$(5.1) \quad \rho(df) = \pi^\#df = X_f, \quad [df_1, df_2] = d\{f_1, f_2\},$$

for all $f, g \in C^\infty(M)$. Moreover, these formulas determine uniquely the Lie algebroid structure. Exercise 40 shows that the Koszul bracket defined by

$$(5.2) \quad [\alpha, \beta] = \mathcal{L}_{\pi^\#(\alpha)}(\beta) - \mathcal{L}_{\pi^\#(\beta)}(\alpha) - d(\pi(\alpha, \beta)),$$

is the unique Lie algebroid bracket on $\Omega^1(M)$ that satisfies (5.1).

Exercise 76. Show that a Lie algebroid structure on T^*M is induced from a Poisson bracket on M if and only if (i) the anchor $\rho : T^*M \rightarrow TM$ is skew-symmetric and (ii) the bracket of closed 1-forms is a closed 1-form.

Therefore, Poisson manifolds form a nice class of Lie algebroid structures. Moreover, all the symplectic and foliated geometry that combines into a Poisson manifold, often leads to beautiful geometric interpretations of many of the results and constructions in abstract Lie algebroid theory. Then, a basic questions is:

- What are the Lie groupoids integrating Poisson manifolds?

Before we initiate our study of the integrability of Poisson brackets, it is worth pointing out that there exists yet another connection between Poisson geometry and Lie groupoid/algebroid theory. In fact, one can think of Lie algebroids as a special class of Poisson manifolds, a generalization of the well-known equivalence between finite dimensional Lie algebras and linear Poisson brackets.

To explain this, let $p : A \rightarrow M$ be a Lie algebroid over a manifold M , with anchor $\rho : A \rightarrow TM$, and Lie bracket $[,] : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$. We define a Poisson bracket on the total space of the dual bundle A^* as

follows. The algebra $C^\infty(A^*)$ of smooth functions on A^* is generated by the following two types of functions:

- the *basic functions* $f \circ p$, where $f \in C^\infty(M)$, and
- the *evaluation by a section* $\alpha \in \Gamma(A)$, denoted F_α , and defined by

$$F_\alpha(\xi) := \langle \alpha, \xi \rangle \quad (\xi \in A^*).$$

In order to define the Poisson bracket on A^* it is enough to define it on these two kinds of functions, in a way that is compatible with the Jacobi identity and the Leibniz identity. We do this as follows:

- (i) The bracket of basic functions is zero:

$$\{f \circ \pi, g \circ \pi\}_{A^*} = 0, \quad (f, g \in C^\infty(M));$$

- (ii) The bracket of two evaluation functions is the evaluation function of their Lie brackets:

$$\{F_\alpha, F_\beta\}_{A^*} = F_{[\alpha, \beta]} \quad (\alpha, \beta \in \Gamma(A));$$

- (iii) The bracket of a basic function and an evaluation function is the basic function given by applying the anchor:

$$\{F_\alpha, f \circ \pi\}_{A^*} = \rho(\alpha)(f) \circ \pi \quad (\alpha \in \Gamma(A), f \in C^\infty(M)).$$

Exercise 77. Show that these definitions are compatible with the Jacobi and Leibniz identities, so they define a Poisson bracket on A^* .

The following exercise gives some basic properties of this Poisson bracket.

Exercise 78. Show that the Poisson bracket just defined satisfies the following properties:

- (a) it is fiberwise linear, i.e., the bracket of functions linear on the fibers is a function linear on the fibers;
- (b) the hamiltonian vector fields X_{F_α} project to the vector fields $\rho(\alpha)$.
- (c) the flow of a hamiltonian vector field X_{F_α} is the fiberwise transpose of the flow of the section $\alpha \in \Gamma(A)$:

$$\langle \phi_\alpha^t(a), \xi \rangle = \langle a, \phi_{X_{F_\alpha}}^t(\xi) \rangle, \quad \forall a \in A, \xi \in A^*.$$

Conversely, show that a fiberwise linear Poisson structure on a vector bundle $A^* \rightarrow M$ induces a Lie algebroid structure on the dual bundle A , whose associated Poisson bracket is the original one.

Therefore, we see that Lie algebroids and Poisson geometry come together hand on hand. This lecture is an elaboration on this theme.

5.2. Contravariant geometry and topology

For a Poisson manifold $(M, \{ , \})$ there is a contravariant version of geometry, dual in certain sense to the usual covariant geometry.

Let us start by recalling a few basic notions of Poisson geometry. First of all, we recall that a *Poisson map* is a map $\phi : M \rightarrow N$ between Poisson manifolds that preserves the Poisson brackets:

$$\{f \circ \phi, g \circ \phi\}_M = \{f, g\}_N \circ \phi,$$

for all $f, g \in C^\infty(N)$. We recall also that a *Poisson vector field* is an infinitesimal automorphism of π , i.e., a vector field $X \in \mathfrak{X}(M)$ such that

$\mathcal{L}_X\pi = 0$. We denote by $\mathfrak{X}_\pi(M)$ the space of Poisson vector fields and by $\mathfrak{X}_{\text{Ham}}(M) \subset \mathfrak{X}_\pi(M)$ the subspace of Hamiltonian vector fields.

Some of the concepts we have introduced before for a general Lie algebroid A , are known in Poisson geometry (when $A = T^*M$) under different names. For example, T^*M -paths, T^*M -loops and T^*M -homotopies are known as **cotangent paths**, **cotangent loops** and **cotangent homotopies**, respectively. We will denote by $P_\pi(M)$ the space of cotangent paths (warning: the notation $P(T^*M)$ will have a different meaning in this lecture). On the other hand, a T^*M -connection on a vector bundle E over a Poisson manifold is known as a **contravariant connection** on E . Of special interest are the contravariant connections on $E = T^*M$. By the usual procedure such a connection induces contravariant connections on any associated bundle (TM , $\wedge TM$, $\wedge T^*M$, $\text{End}(TM)$, etc.)

Exercise 79. Let (M, π) be a Poisson manifold. Show that there exists a contravariant connection ∇ on T^*M such that $\nabla\pi = 0$. On the other hand, show that there exists an ordinary connection $\bar{\nabla}$ on T^*M such that $\bar{\nabla}\pi = 0$ iff π is a regular Poisson structure.

The cohomology of the cotangent Lie algebroid of a Poisson manifold (M, π) is known as the **Poisson cohomology** of the Poisson manifold (M, π) and will be denoted $H_\pi^\bullet(M)$ (instead of $H^\bullet(T^*M)$).

Exercise 80. Show that Poisson cohomology is just the cohomology of the complex of multivector fields:

$$\mathfrak{X}^\bullet(M) := \Gamma(\wedge^\bullet TM),$$

with differential $d_\pi : \mathfrak{X}^\bullet(M) \rightarrow \mathfrak{X}^{\bullet+1}(M)$ given by taking the Schouten bracket with π :

$$d_\pi\theta = [\pi, \theta].$$

Moreover, verify the following interpretations of the first few cohomology groups:

- $H_\pi^0(M)$ is formed by the functions in the center of the Poisson algebra $(C^\infty(M), \{, \})$ (often called *Casimirs*).
- $H_\pi^1(M)$ is the quotient of the Poisson vector fields by the hamiltonian vector fields.
- $H_\pi^2(M)$ is the space of classes of non-trivial infinitesimal deformations of the Poisson structure π .

Given a vector field $X \in \mathfrak{X}(M)$ and a cotangent path $a : I \rightarrow T^*M$ we define the *integral* of X along a by:

$$\int_a X = \int_0^1 \langle a(t), X|_{\gamma(t)} \rangle dt,$$

where $\gamma : I \rightarrow M$ is the base path of a . Notice that if $X_h \in \mathfrak{X}_{\text{Ham}}(M)$ is a hamiltonian vector field, then:

$$\int_a X_h = h(\gamma(1)) - h(\gamma(0)),$$

so the integral in this case depends only on the end-points.

Though the definition of the integral along cotangent paths makes sense for *any* vector field, we will only be interested in the integral along *Poisson vector fields*. The reason is that homotopy invariance holds only for Poisson vector fields, so that one should think of Poisson vector fields as the analogue of closed 1-forms in this contravariant calculus:

Proposition 5.3. *Let $a_\epsilon \in P_\pi(M)$ be a family of cotangent paths, whose base paths have fixed end-points. Then a_ϵ is a cotangent homotopy iff for all $X \in \mathfrak{X}_\pi(M)$*

$$\frac{d}{d\epsilon} \int_{a_\epsilon} X = 0.$$

Proof. Let $a = a(\epsilon, t)$ be a variation of A -paths, fix a connection ∇ on T^*M and let $b = b(\epsilon, t)$ be a solution of equation (3.16). If we set $I = \langle a, X \rangle$ and $J = \langle b, X \rangle$, then a straightforward computation using the defining equation (3.16) and the expression (5.2) for the Lie bracket, shows that

$$\frac{dI}{d\epsilon} - \frac{dJ}{dt} = \mathcal{L}_X \pi(a, b).$$

Integrating first with respect to t , using $b(\epsilon, 0) = 0$, and then integrating with respect to ϵ , we find that

$$\int_{a_1} X - \int_{a_0} X = \int_{b(\cdot, 1)} X + \int_0^1 \int_0^1 \mathcal{L}_X \pi(a, b) dt d\epsilon,$$

for any vector field X . When $X \in \mathfrak{X}_\pi(M)$, the second term on the right-hand side drops out.

Now if a_ϵ is a cotangent homotopy, then $b(\epsilon, 1) = 0$ and we find that $\int_{a_0} X = \int_{a_1} X$. A scaling argument shows that $\int_{a_0} X = \int_{a_\epsilon} X$ for any epsilon, so that:

$$\frac{d}{d\epsilon} \int_{a_\epsilon} X = 0.$$

Similarly, if this holds for any $X \in \mathfrak{X}_\pi(M)$, we conclude that $b(\epsilon, 1) = 0$ so that a_ϵ is a cotangent homotopy. \square

We will denote by $\Sigma(M, x)$ the isotropy group at x , which is the topological group formed by all cotangent homotopy classes of loops based at $x \in M$ (in the notation of the previous lectures, this is just $\mathcal{G}_x(T^*M)$). Also, we will denote by $\Sigma(M, x)^0$ the restricted isotropy group, formed by classes of cotangent loops covering loops that are contractible in the symplectic leaf. Then we have the short exact sequence:

$$1 \longrightarrow \Sigma(M, x)^0 \longrightarrow \Sigma(M, x) \longrightarrow \pi_1(S, x) \longrightarrow 1,$$

where S is the symplectic leaf through x .

Exercise 81. Show that integration gives a homomorphism:

$$\int : \Sigma(M, x) \rightarrow H_\pi^1(M)^*.$$

All these suggest to think of $\Sigma(M, x)$ as the Poisson fundamental group of M at x . We remind that these groups will be, in general, non-isomorphic as x varies (but, of course, $\Sigma(M, x)$ is isomorphic to $\Sigma(M, y)$ if x and y belong

to the same symplectic leaf). We should also keep in mind that these groups, in general, are not discrete. In fact, we know that we have an isomorphism:

$$\Sigma(M, x)^0 \simeq \mathcal{G}(\mathfrak{g}_x)/\tilde{\mathcal{N}}_x,$$

where $\tilde{\mathcal{N}}_x$ is the monodromy group of our Poisson manifold at x and \mathfrak{g}_x is the isotropy Lie algebra at x .

As one could expect, for a Poisson manifold M the isotropy Lie algebra and the monodromy groups have nice geometric interpretations.

First, recall that if the Poisson structure vanishes at a point x_0 (i.e., $\pi|_{x_0} = 0$), then there is a well-defined linear approximation to π at x_0 , which is a certain linear Poisson structure on the tangent space $T_{x_0}M$. Equivalently, we have the Lie algebra structure on $T_{x_0}^*M$ defined by:

$$[d_{x_0}f, d_{x_0}g] = d_{x_0}\{f, g\}.$$

At a point $x \in M$ of higher rank, the Weinstein splitting theorem shows that any small transverse N to the symplectic leaf S through x , inherits a Poisson structure which vanishes at x . This gives a linear Poisson structure on T_xN , or equivalently, a Lie algebra structure on T_x^*N . A different choice of transversal produces isomorphic linear approximations. It is easy to check that on the normal space $\nu_x(S)$ we obtain an intrinsically defined Poisson structure, and so also an intrinsically defined Lie algebra structure on the dual $\nu_x^*(S)$. This just the isotropy Lie algebra at x :

$$\mathfrak{g}_x = \nu_x^*(S).$$

Now, let us turn to a geometric interpretation of the monodromy. We will assume that our Poisson manifold is *regular*⁽¹⁾ so both $T\mathcal{F} = \text{Im } \pi^\#$ and $\nu^*(\mathcal{F}) = \ker \#$ are vector bundles over M , and the isotropy Lie algebras $\nu^*(\mathcal{F})_x$ are all abelian. This amounts to several simplifications. For instance, the long exact sequence of the monodromy reduces to

$$\cdots \longrightarrow \pi_2(S, x) \xrightarrow{\partial} \nu^*(S)_x \longrightarrow \Sigma(M, x) \longrightarrow \pi_1(S, x) .$$

The monodromy groups can be described (or defined) as the image of ∂ , which, in turn, is given by integration of a canonical cohomology 2-class $[\Omega_\sigma] \in H^2(S, \nu_x^*(S))$. This class can be computed explicitly by using a section σ of $\pi^\# : T_S^*M \rightarrow TS$.

Fix a point x in a Poisson manifold M , let S be the symplectic leaf through x , and consider a 2-sphere $\gamma : S^2 \rightarrow S$, which maps the north pole p_N to x . The symplectic area of γ is given, as usual, by

$$A_\omega(\gamma) = \int_{S^2} \gamma^*\omega,$$

where ω is the symplectic 2-form on the leaf S . By a **deformation** of γ we mean a family $\gamma_t : S^2 \rightarrow M$ of 2-spheres parameterized by $t \in (-\varepsilon, \varepsilon)$, starting at $\gamma_0 = \gamma$, and such that for each fixed t the sphere γ_t has image

¹By regular, we mean that the rank of the Poisson structure π is constant, so the symplectic leaves form a regular foliation \mathcal{F} . If you prefer, you can assume that we are restricting to a neighborhood of a regular leaf.

lying entirely in a symplectic leaf. The **transversal variation** of γ_t (at $t = 0$) is the class of the tangent vector

$$\text{var}_\nu(\gamma_t) \equiv \left[\frac{d}{dt} \gamma_t(p_N) \Big|_{t=0} \right] \in \nu_x(\mathcal{F}).$$

We shall see below that the quantity

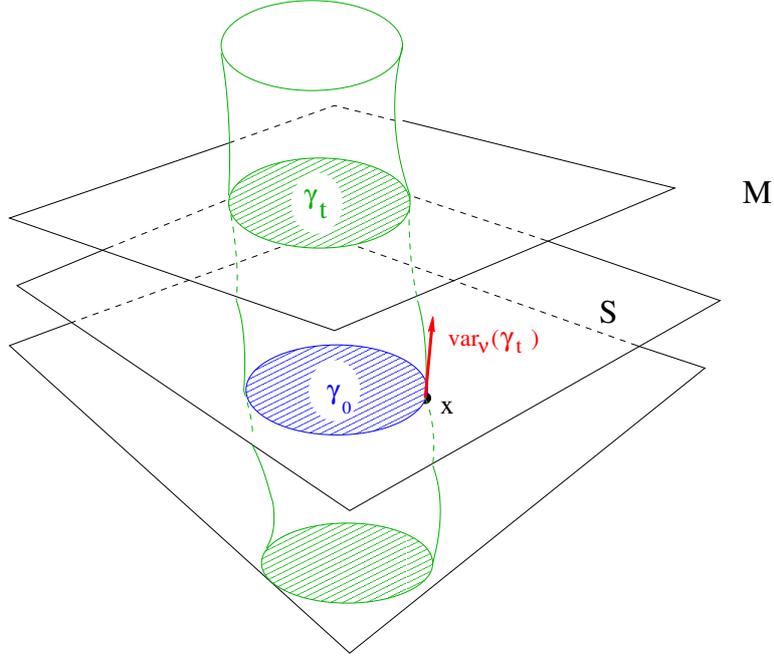
$$\frac{d}{dt} A_\omega(\gamma_t) \Big|_{t=0}$$

only depends on the homotopy class of γ and on $\text{var}_\nu(\gamma_t)$. Finally, the formula

$$\langle A'_\omega(\gamma), \text{var}_\nu(\gamma_t) \rangle = \frac{d}{dt} A_\omega(\gamma_t) \Big|_{t=0},$$

applied to different deformations of γ , gives a well defined element

$$A'_\omega(\gamma) \in \nu_x^*(S).$$



Now, we have:

Proposition 5.4. *For any regular manifold M ,*

$$\mathcal{N}_x = \{A'_\omega(\sigma) : \sigma \in \pi_2(S, x)\},$$

where S is the symplectic leaf through x .

Proof. Recall first that the normal bundle ν (hence also any associated tensor bundle) has a natural flat \mathcal{F} -connection $\nabla : \Gamma(T\mathcal{F}) \times \Gamma(\nu) \rightarrow \Gamma(\nu)$, given by

$$\nabla_X \bar{Y} = \overline{[X, Y]}.$$

In terms of the Lie algebroid $T\mathcal{F}$, ∇ is a flat Lie algebroid connection giving a Lie algebroid representation of $T\mathcal{F}$ on the vector bundle ν (and also, on any associated tensor bundle). Therefore, we have the foliated cohomology

with coefficients in ν , denoted $H^*(\mathcal{F}; \nu)$ (see Exercise 50). Similarly one can talk about cohomology with coefficients in any tensorial bundle associated to ν . In the special case of trivial coefficients, we recover $H^*(\mathcal{F})$.

Now, the Poisson tensor in M determines a foliated 2-form $\omega \in \Omega^2(\mathcal{F})$, which is just another way of looking at the symplectic forms on the leaves. Therefore, we have a foliated cohomology class in the second foliated cohomology group:

$$[\omega] \in H^2(\mathcal{F}).$$

On the other hand, as we saw in Lecture 3, a splitting σ determines a foliated 2-form $\Omega_\sigma \in \Omega^2(\mathcal{F}; \nu^*)$, with coefficients in the co-normal bundle, and the corresponding foliated cohomology class

$$[\Omega] \in H^2(\mathcal{F}; \nu^*)$$

does not depend on the choice of splitting.

These two classes are related in a very simple way: there is a map

$$d_\nu : H^2(\mathcal{F}) \longrightarrow H^2(\mathcal{F}; \nu^*),$$

which can be described as follows. We start with a class $[\theta] \in H^2(\mathcal{F})$, represented by a foliated 2-form θ . As with any foliated form, we have $\theta = \tilde{\theta}|_{T\mathcal{F}}$ for some 2-form $\tilde{\theta} \in \Omega^2(M)$. Since $d\tilde{\theta}|_{\mathcal{F}} = 0$, it follows that the map $\Gamma(\wedge^2 \mathcal{F}) \rightarrow \Gamma(\nu^*)$ defined by

$$(X, Y) \mapsto d\tilde{\theta}(X, Y, -),$$

gives a closed foliated 2-form with coefficients in ν^* . It is easily seen that its cohomology class does not depend on the choice of $\tilde{\theta}$, and this defines d_ν . Now the formula $\omega(X_f, X) = X(f)$, immediately implies that

$$d_\nu([\omega]) = [\Omega].$$

Notice that this construction of d_ν is functorial with respect to foliated maps (i.e., maps between foliated spaces which map leaves into leaves).

From this perspective, a deformation γ_t of 2-spheres is a foliated map $S^2 \times I \rightarrow M$, where in $S^2 \times I$ we consider the foliation \mathcal{F}_0 whose leaves are the spheres $S^2 \times \{t\}$. From $H^2(S^2) \simeq \mathbb{R}$, we get

$$\begin{aligned} H^2(\mathcal{F}_0) &\simeq C^\infty(I) = \Omega^0(I), \\ H^2(\mathcal{F}_0; \nu) &\simeq C^\infty(I)dt = \Omega^1(I), \end{aligned}$$

where the isomorphisms are obtained by integrating over S^2 . Hence, d_ν for \mathcal{F}_0 becomes the de Rham differential $d : \Omega^0(I) \rightarrow \Omega^1(I)$. Now, the functoriality of d_ν with respect to γ_t when applied to ω gives

$$\frac{d}{dt} \int_{S^2} \gamma_t^* \omega = \left\langle \int_{\gamma_t} \Omega, \frac{d}{dt} \gamma_t(p_N) \right\rangle.$$

This proves the last part of the proposition (and also the properties of the variation of the symplectic area stated before). \square

5.3. Symplectic groupoids

Let us now turn to the study of the groupoid

$$\Sigma(M) = P_\pi(M) / \sim .$$

It is probably a good idea to recall now a few basic examples.

Example 5.5 (symplectic manifolds). Let $M = S$ be a symplectic manifold with symplectic form $\omega = \pi^{-1}$. The symplectic form gives an inverse $\omega^\sharp : TS \rightarrow T^*S$ to the anchor $\pi^\sharp : T^*S \rightarrow TS$. Hence, the cotangent Lie algebroid T^*S is isomorphic to the tangent Lie algebroid TS and a cotangent path is determined by its base path.

It follows that, if we assume S to be simply connected, then $\Sigma(S)$ is just the pair groupoid $S \times S$. Note that $\Sigma(S)$ becomes a symplectic manifold with symplectic form $\omega \oplus (-\omega)$ (the reason for choosing a minus sign in the second factor will be apparent later).

Exercise 82. Extend this example to a non-simply connected symplectic manifold, showing that $\Sigma(S)$ still is a symplectic manifold in this case.

Example 5.6 (trivial Poisson manifold). At the other extreme, let M be any manifold with the trivial bracket $\{ , \} \equiv 0$. Then T^*M becomes a bundle of abelian Lie algebras. Cotangent paths are just paths in the fibers of T^*M and any such path is cotangent homotopic to its average. It follows that $\Sigma(M) = T^*M$, with multiplication being addition on the fibers. Note that $\Sigma(M)$ is a symplectic manifold with the usual canonical symplectic form on the cotangent bundle.

Example 5.7 (linear Poisson structures). Let \mathfrak{g} be any finite dimensional Lie algebra, and take $M = \mathfrak{g}^*$ with the Kostant-Kirillov-Souriau bracket:

$$\{f_1, f_2\}(\xi) = \langle \xi, [d_\xi f_1, d_\xi f_2] \rangle, \quad f_1, f_2 \in C^\infty(M), \quad \xi \in \mathfrak{g}^*.$$

If G is the 1-connected Lie group integrating \mathfrak{g} , we have seen before that the cotangent Lie algebroid of \mathfrak{g}^* integrates to the 1-connected Lie groupoid $\mathcal{G} = T^*G$. By uniqueness, we conclude that $\Sigma(\mathfrak{g}^*) = T^*G$. Note again that $\Sigma(\mathfrak{g}^*)$ carries a symplectic structure.

In all these examples, the groupoid $\Sigma(M)$ that integrates the Poisson manifold M carries a symplectic form. It turns out that the symplectic form is compatible with the groupoid structure in the following sense:

Definition 5.8. Let \mathcal{G} be a Lie groupoid and $\omega \in \Omega^\bullet(\mathcal{G})$. We call ω a **multiplicative form** if

$$m^*\omega = p_1^*\omega + p_2^*\omega,$$

where we denote by $m : \mathcal{G}_2 \rightarrow \mathcal{G}$ the multiplication in \mathcal{G} defined on the submanifold $\mathcal{G}_2 \subset \mathcal{G} \times \mathcal{G}$ of composable arrows, and by $p_1, p_2 : \mathcal{G}_2 \rightarrow \mathcal{G}$ the projections to the first and second factors. A **symplectic groupoid** is a pair (\mathcal{G}, ω) where \mathcal{G} is a Lie groupoid and $\omega \in \Omega^2(\mathcal{G})$ is a multiplicative symplectic form.

Exercise 83. Check that in all examples above the symplectic forms are multiplicative, so all those Poisson manifolds integrate to s -simply connected symplectic groupoids.

In the sequel, if (S, ω) is a symplectic manifold, we will denote by \bar{S} the manifold S furnished with the symplectic form $-\omega$.

Exercise 84. Recall that a map $\phi : (S_1, \omega_1) \rightarrow (S_2, \omega_2)$ is symplectic if $\phi^*\omega_2 = \omega_1$. Show that:

- (i) A map ϕ is symplectic iff its graph is a Lagrangian submanifold of $S \times \bar{S}$.
- (ii) A symplectic form ω in a groupoid \mathcal{G} is multiplicative iff the graph of the multiplication

$$\gamma_m = \{(g, h, gh) \in \mathcal{G} \times \mathcal{G} \times \mathcal{G} : (g, h) \in \mathcal{G}^{(2)}\}$$

is a Lagrangian submanifold of $\mathcal{G} \times \mathcal{G} \times \bar{\mathcal{G}}$.

Here are some basic properties of a symplectic groupoid, whose proof is a very instructive exercise.

Exercise 85. Let (\mathcal{G}, ω) be a symplectic groupoid. Show that:

- (i) the \mathbf{s} -fibers and the \mathbf{t} -fibers are symplectic orthogonal;
- (ii) the inverse map $i : \mathcal{G} \rightarrow \mathcal{G}$ is an anti-symplectic map;
- (iii) M , viewed as the unit section, is a Lagrangian submanifold.

Finally, we can state and prove the main property of a symplectic groupoid: the base manifold of a symplectic groupoid has a canonical Poisson bracket.

Proposition 5.9. *Let (\mathcal{G}, ω) be a symplectic groupoid over M . Then, there exists a unique Poisson structure on M such that:*

- (i) \mathbf{s} is Poisson and \mathbf{t} is anti-Poisson;
- (ii) the Lie algebroid of \mathcal{G} is canonically isomorphic to T^*M .

Proof. Let us call a function on \mathcal{G} of the form $f \circ \mathbf{s}$ a *basic function*. We can assume that \mathcal{G} is s -connected, so a function $f : \mathcal{G} \rightarrow \mathbb{R}$ is basic iff it is constant on s -fibers. Then, using the exercise above, we have the chain of equivalences:

$$\begin{aligned} f \text{ is basic} &\iff df \in (\text{Ker } ds)^0 \\ &\iff X_f \in (\text{Ker } ds)^\perp \iff X_f \in \text{Ker } dt. \end{aligned}$$

From this last characterization and the fact that

$$X_{\{f_1, f_2\}_{\mathcal{G}}} = [X_{f_1}, X_{f_2}],$$

we see that the Poisson bracket of any two basic functions is a basic function.

We conclude that M carries a unique Poisson bracket such that \mathbf{s} is a Poisson map. Since $\mathbf{t} = \mathbf{s} \circ i$, and the inversion is anti-symplectic, we see that \mathbf{t} is anti-Poisson. Now we use the following:

Exercise 86. Show that if $f \in C^\infty(M)$, then the Hamiltonian vector field $-X_{f \circ \mathbf{t}} \in \mathfrak{X}_{\text{Ham}}(\mathcal{G})$ is the right invariant vector field in \mathcal{G} that corresponds to the section $df \in \Omega^1(M)$.

Hence, to compute the anchor in $A(\mathcal{G})$ we start with an exact 1-form $df \in \Omega^1(M)$, we extend it to a right invariant vector field $-X_{f \circ \mathbf{t}}$, and apply the differential of the target map, obtaining:

$$\rho(df) = dt(-X_{f \circ \mathbf{t}}) = X_f.$$

This shows that the anchor for $A(\mathcal{G})$ coincides with the anchor for T^*M .

On the other hand, to compute the $A(\mathcal{G})$ -bracket of two exact 1-forms $df_i \in \Omega^1(M)$, we extend them to the right invariant vector fields $-X_{f_i \circ t}$ and we compute their Lie bracket:

$$\begin{aligned} [-X_{f_1 \circ t}, -X_{f_2 \circ t}] &= X_{\{f_1 \circ t, f_2 \circ t\}_{\mathcal{G}}} \\ &= -X_{\{f_1, f_2\}_{M \circ t}}. \end{aligned}$$

Therefore, the $A(\mathcal{G})$ -bracket is given by $d\{f_1, f_2\}_M$, which coincides with the Koszul bracket $[df_1, df_2]$. Since we already know that the anchors coincide, the Leibniz identity implies that the Lie brackets on $A(\mathcal{G})$ and T^*M coincide on *any* 1-forms. \square

In the examples of Poisson manifolds that we saw before, $\Sigma(M)$ was a symplectic groupoid. This situation is by no means exceptional: we will now show that whenever $\Sigma(M)$ is smooth, it is a symplectic groupoid!

In order to prove this, we will need to look closer at the description of cotangent homotopies through a Lie algebra action (we saw this in the previous lecture for general A -homotopies). We denote now by $P(T^*M)$ the space of C^1 -paths $a : I \rightarrow T^*M$ with base path $\gamma : I \rightarrow M$ of class C^2 . This is a Banach manifold in the obvious way, and the space of cotangent paths $P_\pi(M) \subset P(T^*M)$ is a Banach submanifold. Also, we let $P(M)$ denote the space of C^2 -paths $\gamma : I \rightarrow M$. A basic fact, which in fact explains the existence of a symplectic structure on $\Sigma(M)$, is that

$$P(T^*M) = T^*P(M),$$

so that $P(T^*M)$ carries a natural (weak) symplectic structure ω_{can} .

Exercise 87. Identify the tangent space $T_a P(T^*M)$ with the space of vector fields along a :

$$T_a P(T^*M) = \{U : I \rightarrow TT^*M : U(t) \in T_{a(t)} T^*M\}.$$

and denote by ω_0 the canonical symplectic form on T^*M . Show that the 2-form ω_{can} on $P(T^*M) = T^*P(M)$ is given by:

$$(5.10) \quad (\omega_{\text{can}})_a(U_1, U_2) = \int_0^1 \omega_0(U_1(t), U_2(t)) dt,$$

for all $U_1, U_2 \in T_a P(T^*M)$. Moreover, check that $d\omega_{\text{can}} = 0$ and that $\omega_{\text{can}}^\sharp : T_a P(T^*M) \rightarrow T_a^* P(T^*M)$ is injective, so that ω_{can} is a weak symplectic form.

On the other hand, the Lie algebra

$$P_0\Omega^1(M) := \{\eta_t \in \Omega^1(M), t \in I : \eta_0 = \eta_1 = 0, \eta_t \text{ of class } C^1 \text{ in } t\}$$

with the pointwise Lie bracket, acts on $P(T^*M)$ in such a way that:

- (a) the action is tangent to $P_\pi(M)$;
- (b) two cotangent paths are homotopic if and only if they belong to the same orbit.

Now we have the following remarkable fact:

Theorem 5.11. *The Lie algebra action of $P_0\Omega^1(M)$ on $(P(T^*M), \omega_{can})$ is Hamiltonian, with equivariant moment map $J : P(T^*M) \rightarrow P_0\Omega^1(M)^*$ given by*

$$(5.12) \quad \langle J(a), \eta \rangle = \int_0^1 \left\langle \frac{d}{dt} \pi(a(t)) - \#a(t), \eta(t, \gamma(t)) \right\rangle dt.$$

Proof. We need to check that if $\eta_t \in P_0\Omega^1(M)$ is a time dependent 1-form vanishing at $t = 0, 1$, the the infinitesimal generator X_{η} coincides with the Hamiltonian vector field with Hamiltonian function $a \mapsto \langle J(a), \eta \rangle$. We leave this for the reader to check. (Hint: It is enough to prove this at A -paths a fitting in a coordinate domain, so one can use local coordinates.) \square

Note that set of cotangent paths $P_{\pi}(M)$ is just the level set $J^{-1}(0)$. Hence, our groupoid $\Sigma(M)$ can be described alternatively as a symplectic quotient:

$$(5.13) \quad \Sigma(M) = \tilde{P}(T^*M) // P_0\Omega(M).$$

and we have:

Corollary 5.14. *Whenever $\Sigma(M)$ is smooth, it admits a canonical multiplicative symplectic form ω , so that $(\Sigma(M), \omega)$ is a symplectic groupoid. Moreover, the induced Poisson bracket on the base is the original one.*

Proof of Theorem 5.11. We only need to check that the reduced symplectic form ω is multiplicative, i.e., satisfies

$$m^* \omega = \pi_1^* \omega + \pi_2^* \omega,$$

where $m : \Sigma(M)_2 \rightarrow \Sigma(M)$ is the multiplication in $\Sigma(M)$ and $\pi_1, \pi_2 : \Sigma(M)_2 \rightarrow \Sigma(M)$ are the projections to the first and second factors. But the additivity of the integral and expression (5.10) shows that that this condition holds already at the level of $P(T^*M)$, hence it must hold also on the reduced space $\Sigma(M)$. \square

Putting all these together, and usings the general integrability results of Lecture 4, we arrive at the following esult:

Theorem 5.15. *For a Poisson manifold M , the following are equivalent:*

- (i) *M is integrable by a symplectic groupoid.*
- (ii) *The algebroid T^*M is integrable.*
- (iii) *The groupoid $\Sigma(M)$ is a smooth manifold.*
- (iv) *The monodromy groups \mathcal{N}_x , with $x \in M$, are locally uniformly discrete.*

In this case, $\Sigma(M)$ is the unique s -simply connected, symplectic groupoid which integrates M .

Proof. If Σ is a symplectic groupoid integrating M , then its associated algebroid is isomorphic to T^*M , and this shows that (i) implies (ii). We have seen in Lecture 2 that (ii) implies (iii), by taking any Lie groupoid \mathcal{G} integrating T^*M , and then taking the groupoid formed by the universal covers of the s -fibers of \mathcal{G} together. Obviously (iii) implies (i), since we just saw that $\Sigma(M)$ is a symplectic groupois. Finaly, the equivalence of (iii) and (iv) is just a special case of our main integrability theorem for Lie algebroids. \square

Note that not every Lie groupoid integrating the cotangent bundle T^*M of a Poisson manifold is a symplectic groupoid, as the following example shows:

Example 5.16. Take $M = \mathbb{S}^3$ with the trivial Poisson bracket, so that the s -simply connected groupoid integrating M is $\Sigma(\mathbb{S}^3) = T^*\mathbb{S}^3$, multiplication being addition on the fibers. Another integrating groupoid is, for example, $\mathcal{G} = \mathbb{S}^3 \times \mathbb{T}^3$ where $\mathbf{s} = \mathbf{t}$ is projection in the first factor and multiplication is multiplication on the torus \mathbb{T}^3 . This groupoid cannot be a symplectic groupoid because of the following exercise:

Exercise 88. Show that the manifold $\mathbb{S}^3 \times \mathbb{T}^3$ does not admit any symplectic structure.

At this point it is natural to ask, for a general Lie algebroid A , what is the relationship between the integrability of A and the integrability of the Poisson manifold A^* . The following exercise shows that these two problems are actually equivalent.

Exercise 89. Let A be a Lie algebroid and consider the natural fiberwise linear Poisson structure on A^* . Check that the natural injection $A \hookrightarrow T^*A^*$ is a Lie algebroid morphism, so that:

- (a) If A is integrable, then the Poisson manifold A^* is also integrable and $\Sigma(A^*) = T^*\mathcal{G}(A)$.
- (b) If A^* is integrable, then A is also integrable and $\mathcal{G}(A) \hookrightarrow \Sigma(A^*)$ is a subgroupoid.

This exercise is one instance of the intrinsic connection between Poisson and symplectic geometry on one hand, and Lie algebroid and groupoid theory on the other hand. In the next section, we will go deeper into this connection which is sometimes not so obvious.

It is now time to look at some examples.

Example 5.17 (Poisson manifolds of dimension 2). The lowest dimension one can have non-trivial Poisson manifolds is 2. However, such a Poisson manifold will have it follows immediately from Theorem 5.15 and the description of the monodromy groups that in dimension 2 all Poisson manifolds are integrable:

Corollary 5.18. *Any 2-dimensional Poisson manifold is integrable.*

Corollary 5.18 can be partially generalized to higher dimensions in the following sense: any $2n$ -dimensional Poisson manifold whose Poisson tensor has rank $2n$ on a dense, open set, is integrable. The proof of this fact is more involved (see the notes at the end of this chapter).

Example 5.19 (A non-integrable Poisson manifold). Already in dimension 3 there are examples of non-integrable Poisson manifolds. Let us consider $M = \mathbb{R}^3$ with the Poisson bracket:

$$(5.20) \quad \{f, g\} = \det \begin{pmatrix} x & y & z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix}.$$

Notice that this is just the linear Poisson structure on $\mathfrak{su}(2)^*$ when we identify $\mathfrak{su}(2)$ and \mathbb{R}^3 with the exterior product.

Let us choose any smooth function $a = a(R)$ on M , which depends only on the radius R , and which is strictly positive for $R > 0$. We multiply the previous brackets by a , and we denote by M_a the resulting Poisson manifold. The bracket on $\Omega^1(M_a)$ is computed using the Leibniz identity and we get

$$\begin{aligned} [dx^2, dx^3] &= a dx^1 + b x^1 R \bar{n}, \\ [dx^3, dx^1] &= a dx^2 + b x^2 R \bar{n}, \\ [dx^1, dx^2] &= a dx^3 + b x^3 R \bar{n}, \end{aligned}$$

where $\bar{n} = \frac{1}{R} \sum_i x^i dx^i$ and $b(R) = a'(R)/R$. The bundle map $\# : T^*M_a \rightarrow TM_a$ is just

$$\#(dx^i) = a \bar{v}^i, \quad i = 1, 2, 3$$

where \bar{v}^i is the infinitesimal generator of a rotation about the i -axis:

$$\bar{v}^1 = x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3}, \quad \bar{v}^2 = x^1 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^1}, \quad \bar{v}^3 = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}.$$

The leaves of the symplectic foliation of M_a are the spheres $S_R^2 \subset \mathbb{R}^3$ centered at the origin, and the origin is the only singular point.

We will compute the monodromy function $r_{\mathcal{N}}$ in two distinct fashions. First, using the obvious metric on T^*M_a , we restrict to a leaf S_R^2 with $R > 0$, and we choose as a splitting of $\#$ the map defined by

$$\sigma(\bar{v}^i) = \frac{1}{a} (dx^i - \frac{x^i}{R} \bar{n}),$$

with curvature the center-valued 2-form

$$\Omega_{\sigma} = \frac{Ra' - a}{a^2 R^3} \omega \bar{n},$$

where $\omega = x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2$. Since $\int_{S_R^2} \omega = 4\pi R^3$ it follows that

$$\mathcal{N}_{(x,y,z)} \simeq 4\pi \frac{Ra' - a}{a^2} \mathbb{Z} \bar{n} \subset \mathbb{R} \bar{n}.$$

On the other hand, the canonical generator of $\pi_2(S_R^2)$ defines the symplectic area of S_R^2 , which is easily computable:

$$A_a(R) = 4\pi \frac{R}{a(R)}.$$

We recover in this way the relationship between the monodromy and the variation of the symplectic area (Proposition 5.4).

Also, observe that

$$r_{\mathcal{N}}(x, y, z) = \begin{cases} +\infty & \text{if } R = 0 \text{ or } A'_a(R) = 0, \\ A'_a(R) & \text{otherwise,} \end{cases}$$

so the monodromy might vary in a non-trivial fashion, even for nearby regular leaves. Our computation also gives the isotropy groups

$$\Sigma(M_a, (x, y, z)) \cong \begin{cases} \mathbb{R}^3 & \text{if } R = 0, a(0) = 0, \\ SU(2) & \text{if } R = 0, a(0) \neq 0, \\ \mathbb{R} & \text{if } R \neq 0, A'_a(R) = 0, \\ S^1 & \text{if } R \neq 0, A'_a(R) \neq 0. \end{cases}$$

Example 5.21 (Heisenberg-Poisson manifolds). The **Heisenberg-Poisson manifold** $M(S)$ associated to a symplectic manifold S , is the manifold $S \times \mathbb{R}$ with the Poisson structure given by $\{f, g\} = t\{f_t, g_t\}_S$, where t stands for the real parameter, and f_t denotes the function on S obtained from f by fixing the value of t . We have the following result:

Corollary 5.22. *For a symplectic manifold S , the following are equivalent:*

- (i) *The Poisson-Heisenberg manifold $M(S)$ is integrable;*
- (ii) *\tilde{S} is pre-quantizable.*

We recall that condition (ii) is usually stated as follows: when we pull back the symplectic form ω on S to a 2-form $\tilde{\omega}$ on the covering space \tilde{S} , the group of periods

$$\left\{ \int_{\gamma} \tilde{\omega} : \gamma \in \mathcal{H}_2(\tilde{S}, \mathbb{Z}) \right\} \subset \mathbb{R}$$

is a multiple of \mathbb{Z} . Note that this group coincides with the **group of spherical periods** of ω

$$\mathcal{P}(\omega) = \left\{ \int_{\gamma} \omega : \gamma \in \pi_2(S) \right\},$$

so that (ii) says that $\mathcal{P}(\omega) \subset \mathbb{R}$ is a multiple of \mathbb{Z} .

Proof of Corollary 5.22. We have to compute the monodromy groups. The singular symplectic leaves are the points in $S \times \{0\}$ and they clearly have vanishing monodromy groups. The regular symplectic leaves are the submanifolds $S \times \{t\}$, where $t \neq 0$, with symplectic form ω/t . To compute their monodromy groups we invoke Proposition 5.4, so that we immediately get

$$\mathcal{N}_{(x,t)} = \frac{1}{t} \mathcal{P}(\omega) \subset \mathbb{R},$$

and the result follows. □

Exercise 90. Show that the arrows in $\Sigma(M(S))$ are of two types:

- (a) arrows which start and end at $(x, 0)$, which form a group isomorphic to the additive subgroup of $T_{x,0}^*M(S)$;
- (b) arrows inside the symplectic leaves $S \times \{t\}$, $t \neq 0$, which consist of equivalence classes of pairs (γ, v) , where γ is a path in S and $v \in \mathbb{R}$. Two such pairs (γ_i, v_i) are equivalent if and only if there is a homotopy $\gamma(\varepsilon, s)$ (with fixed end points) between the γ_i 's, such that $v_1 - v_0 = \frac{1}{t} \int \gamma^* \omega$.

Remark 5.23. The two necessary and sufficient conditions that must hold in order for $\Sigma(M)$ to be smooth, namely:

- (i) $r_{\mathcal{N}}(x) > 0$, and
- (ii) $\liminf_{y \rightarrow x} r_{\mathcal{N}}(y) > 0$,

are independent. In fact, if we choose a symplectic manifold which does not satisfy the pre-quantization condition, the Poisson-Heisenberg manifold of Example 5.21 gives a non-integrable Poisson manifold violating condition (i). On the other hand, Example 5.19 gives examples of non-integrable Poisson manifolds in which condition (i) is satisfied, but condition (ii) is not.

5.4. The symplectization functor

So far, to every Poisson manifold (M, π) we have associated a groupoid $\Sigma(M)$. In the integrable case, this is a symplectic groupoid integrating the cotangent algebroid T^*M . If $\phi : (M_1, \pi_1) \rightarrow (M_2, \pi_2)$ is a Poisson map, in general, it *does not* induce a Lie algebroid morphism of the associated cotangent Lie algebroids, and hence it will not yield a Lie groupoid homomorphism.

We will see now that we can still define Σ on Poisson morphisms such that we obtain a functor from the *Poisson category* to a certain *symplectic groupoid “category”*. This functor should not be confused with *integration functor* \mathcal{G} which we have studied before and which goes from the category of Lie algebroids to the category of Lie groupoids. We will call Σ the **symplectization functor** and will see that it entails a rich geometry, which is not present in the integration functor.

Henceforth, we will denote by **Poiss** the Poisson category, in which the objects are the Poisson manifolds and the morphisms are the Poisson maps. We already know what the effect of Σ on objects is, so let us look at its effect on a Poisson morphism $\phi : (M_1, \pi_1) \rightarrow (M_2, \pi_2)$. In order to find out what the answer should be, let us recall different ways of expressing the condition for a map to be Poisson:

Exercise 91. Let $\phi : (M_1, \pi_1) \rightarrow (M_2, \pi_2)$ be a smooth map between two Poisson manifolds. Show that the following conditions are equivalent:

- (a) The map ϕ preserves Poisson brackets: $\{f \circ \phi, g \circ \phi\}_1 = \{f, g\}_2 \circ \phi$.
- (b) The Poisson bivectors are ϕ -related: $\phi_*\pi_1 = \pi_2$.
- (c) $\text{Graph}(\phi) \subset M_1 \times \overline{M}_2$ is a coisotropic submanifold ⁽²⁾.

(Hint: A submanifold N of a Poisson manifold (M, π) is called *coisotropic* if $\pi^\sharp(TN)^0 \subset TN$, where $(TN)^0 \subset T^*M$ denotes the annihilator of TN .)

Hence, to integrate a Poisson morphism, we just need to know what objects integrate coisotropic submanifolds of a Poisson manifold. This is solved as follows:

Theorem 5.24. *If $\mathcal{G} \rightrightarrows C$ is a Lagrangian subgroupoid of a symplectic groupoid $\Sigma \rightrightarrows M$, then $C \subset M$ is a coisotropic submanifold. Conversely, if C is a coisotropic submanifold of an integrable Poisson manifold (M, π) , there exists a Lagrangian subgroupoid $\mathcal{G} \rightrightarrows C$ of $\Sigma(M)$ that integrates C .*

Proof. Let us explain why coisotropic submanifolds integrate to Lagrangian subgroupoids. Note that C is a coisotropic submanifold of (M, π) iff its conormal bundle $\nu^*(C) := (TC)^0 \subset T^*M$ is a Lie subalgebroid of the cotangent Lie algebroid T^*M . Therefore, if (M, π) is integrable, then there exists

²For a Poisson manifold M , the notation \overline{M} means the same manifold with the symmetric Poisson structure.

a source connected Lie subgroupoid $\mathcal{G} \rightrightarrows C$ of the groupoid $\Sigma(M) \rightrightarrows M$ that integrates $\nu^*(C)$. Now we claim that the restriction of the symplectic form ω to \mathcal{G} vanishes which, combined with $\dim \mathcal{G} = 1/2 \dim \Sigma(M)$, implies that \mathcal{G} is Lagrangian.

To prove our claim, we observe that for the canonical symplectic form ω_0 on T^*M the submanifold $\nu^*(C) \subset T^*M$ is Lagrangian. Using the explicit expression (5.10) for the symplectic form ω_{can} on $P(T^*M)$, we see immediately that space of paths $P(\nu^*(C)) \subset P(T^*M)$ is isotropic. It follows that the symplectic form ω on the symplectic quotient $\Sigma(M)$ restricts to zero on the submanifold $\mathcal{G} = P(\nu^*(C))/P_0\Omega^1(M)$, as we claimed. \square

We have now found what Poisson morphisms integrate to:

Corollary 5.25. *Let $\phi : (M_1, \pi_1) \rightarrow (M_2, \pi_2)$ be a Poisson map between two integrable Poisson manifolds. Then ϕ integrates to a Lagrangian subgroupoid $\Sigma(\phi) \subset \Sigma(M_1) \times \overline{\Sigma(M_2)}$.*

Let us introduce now the symplectic “category” of Alan Weinstein, which will be denoted by **Symp**. In this “category” the objects are the symplectic manifolds and the morphisms are the *canonical relations*, which are defined as follows.

Definition 5.26. If (S_1, ω_1) and (S_2, ω_2) are two symplectic manifolds, then a **canonical relation** from S_1 to S_2 is a Lagrangian submanifold $L \subset S_1 \times \overline{S_2}$.

If $L_1 \in \text{Mor}(S_1, S_2)$ and $L_2 \in \text{Mor}(S_2, S_3)$ are canonical relations, their composition is the usual composition of relations:

$$L_1 \circ L_2 := \{(x, z) \in S_1 \times S_3 \mid \exists y \in S_2, \text{ with } (x, y) \in L_1 \text{ and } (y, z) \in L_2\}.$$

Exercise 92. Let $\phi : S_1 \rightarrow S_2$ be a diffeomorphism between two symplectic manifolds. Show that ϕ is a symplectomorphism iff its graph is a canonical relation.

Hence, the canonical relations enlarge the space of symplectomorphisms. There is, however, a problem: $L_1 \circ L_2$ may not be a smooth submanifold of $S_1 \times \overline{S_3}$. One needs the two canonical relations to intersect cleanly, as explained in the following exercise.

Exercise 93. Let $L_1 \in \text{Mor}(S_1, S_2)$ and $L_2 \in \text{Mor}(S_2, S_3)$ be canonical relations, and denote by $L_1 \star L_2$ their fiber product over S_2 :

$$\begin{array}{ccc} L_1 \star L_2 & \longrightarrow & L_2 \\ \downarrow & & \downarrow \\ L_1 & \longrightarrow & S_2 \end{array}$$

Finally, let $p : S_1 \times S_2 \times S_3 \rightarrow S_1 \times S_3$ be the projection into the first and last factor.

- (a) Verify that $L_1 \circ L_2 = p(L_1 \star L_2)$;

Assume the **clean intersection property**: $L_1 \star L_2$ is a manifold and its tangent spaces are fiber products of the tangent spaces of L_1 and L_2 :

$$\begin{array}{ccc} T_{(s_1, s_2, s_3)}(L_1 \star L_2) & \longrightarrow & T_{(s_2, s_3)}L_2 \\ \downarrow & & \downarrow \\ T_{(s_1, s_2)}L_1 & \longrightarrow & T_{s_2}S_2 \end{array}$$

- (b) Show that $L_1 \circ L_2$ is an (immersed) Lagrangian submanifold of $S_1 \times S_3$.

Therefore, in **Symp** composition is not always defined and Weinstein proposed to name it a “category”, with quotation marks. One can make an analogy between this “category” and the category of hermitian vector spaces, as in the following table:

Symplectic “category”	hermitian category
Symplectic manifold (S, ω)	hermitian vector space H
\overline{S}	dual vector space H^*
$S_1 \times S_2$	tensor product $H_1 \otimes H_2$
$S_1 \times \overline{S_2}$	homomorphisms $\text{Hom}(S_1, S_2)$
a point 0	\mathbb{C}

Just as an element in the hermitian vector space H can be thought of as morphism from $\mathbb{C} \rightarrow H$, we can think of an element in the symplectic manifold S as a morphism in $\text{Mor}(0, S)$, i.e., a Lagrangian submanifold of S . As a special case, an element in the Hom-object $S_1 \times \overline{S_2}$ is a Lagrangian submanifold, so we recover the canonical relations.

Exercise 94. In the symplectic “category” there exists an involution which takes an object S to itself $S^\dagger := S$ and a morphism $L \in \text{Mor}(S_1, S_2)$ to the *dual morphism*

$$L^\dagger := \{(y, x) \in S_2 \times \overline{S_1} \mid (x, y) \in L\} \in \text{Mor}(S_2, S_1).$$

Call a morphism *unitary* if its dual equals its inverse: $L^\dagger = L^{-1}$. Check that the unitary morphisms are just the symplectomorphisms.

The symplectic groupoids form a “subcategory” **SympGrp** of the symplectic “category”, where the morphisms $\text{Mor}(\mathcal{G}_1, \mathcal{G}_2)$ are the Lagrangian subgroupoids of $\mathcal{G}_1 \times \overline{\mathcal{G}_2}$. We can summarize the previous discussion by saying that Σ is a covariant functor from **Poiss** to **SympGrp**. The remainder of this section is dedicated to the study of some basic properties of the functor Σ , namely, we look at its effect on various geometric constructions in Poisson geometry.

5.4.1. Sub-objects

A sub-object in the Poisson category is just a Poisson submanifold N of a Poisson manifold (M, π) . As the following example shows, a Poisson submanifold of an integrable Poisson manifold may fail to be integrable.

Example 5.27. Let $M = \mathfrak{so}(4)^*$ with its linear Poisson structure. Note that we have an isomorphism

$$\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3) = \mathfrak{su}(2) \oplus \mathfrak{su}(2),$$

so that we can think of M as $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ with the product Poisson structure, where each factor has the Poisson structure (5.20). For the usual euclidean structure on \mathbb{R}^6 , the function $f(x) = \|x\|^2$ is a Casimir, so that the unit sphere

$$\mathbb{S}^5 = \{x \in \mathbb{R}^6 : \|x\| = 1\},$$

is a Poisson submanifold of M .

Exercise 95. Determine which points of \mathbb{S}^5 are regular points. Using Proposition 5.4, show that if $x = (x_1, x_2) \in \mathbb{S}^5$ is a regular point then:

$$\mathcal{N}_x \simeq \{n_1\|x_1\| + n_2\|x_2\| : n_1, n_2 \in \mathbb{Z}\} \subset \mathbb{R}.$$

Therefore, \mathbb{S}^5 is not an integrable Poisson manifold.

Henceforth, we will assume that both (M, π) and the Poisson submanifold N are integrable. Then the inclusion $i : N \hookrightarrow M$ is a Poisson morphism which, according to Corollary 5.25, integrates to a Lagrangian subgroupoid $\Sigma(i) \subset \Sigma(N) \times \overline{\Sigma(M)}$. However, this is not the end of the story.

For a Poisson submanifold $N \subset M$, let us consider the set of equivalence classes of cotangent paths that take their values in the restricted subbundle T_N^*M :

$$\Sigma_N(M) := \{[a] \in \Sigma(M) \mid a : I \rightarrow T_N^*M\}.$$

This is a Lie subgroupoid of $\Sigma(M)$: it is the Lie subgroupoid that integrates the Lie subalgebroid $T_N^*M \subset T^*M$. Moreover, this subalgebroid is a coisotropic submanifold of the symplectic manifold T^*M , and it follows that $\Sigma_N(M) \subset \Sigma(M)$ is a coisotropic Lie subgroupoid. The fact that the closed 2-form ω on $\Sigma_N(M)$ is multiplicative implies that if we factor by its kernel foliation, we still obtain a symplectic groupoid, and in fact (see [15]):

$$\Sigma(N) \simeq \Sigma_N(M) / \text{Ker } \omega.$$

These two constructions are related as follows:

Theorem 5.28. *Let $i : N \hookrightarrow M$ be an integrable Poisson submanifold of an integrable Poisson manifold, and $\Sigma(i) \subset \Sigma(N) \times \overline{\Sigma(M)}$ the corresponding Lagrangian subgroupoid. For the restriction of the projections on each factor:*

$$\begin{array}{ccc} & \Sigma(i) & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \Sigma(N) & & \overline{\Sigma(M)} \end{array}$$

π_2 is a diffeomorphism onto the coisotropic subgroupoid $\Sigma_N(M) \subset \Sigma(M)$ above, and the groupoid morphism:

$$(\pi_1) \circ (\pi_2)^{-1} : \Sigma_N(M) \rightarrow \Sigma(N),$$

corresponds to the quotient map $\Sigma_N(M) \rightarrow \Sigma_N(M) / \text{Ker } \omega \simeq \Sigma(N)$.

A very special situation happens when the exact sequence of Lie algebroids

$$(5.29) \quad 0 \longrightarrow \nu^*(N) \longrightarrow T_N^*M \longrightarrow T^*N \longrightarrow 0$$

splits: in this case, the splitting $\phi : T^*M \rightarrow T_N^*M \subset T^*M$ integrates to a symplectic Lie groupoid homomorphism $\Phi : \Sigma(N) \rightarrow \Sigma(M)$, which realizes $\Sigma(N)$ as a symplectic subgroupoid of $\Sigma(M)$. Poisson submanifolds of this sort maybe called **Lie-Poisson submanifolds**, and there are topological obstructions on a Poisson submanifold for this to happen. The following exercise discusses the case of a regular symplectic leaf, so that $\nu^*(N)$ is bundle of abelian Lie algebras.

Exercise 96.³ Let N be an embedded Poisson submanifold of a Poisson manifold (M, π) , and choose any splitting ϕ of the short exact sequence (5.29). Denote by ∇ the connection defined by this splitting:

$$\nabla_\alpha \gamma = [\phi(\alpha), \gamma], \quad \alpha \in \Omega^1(N), \gamma \in \Gamma(\nu^*(N)),$$

and by Ω the curvature of this splitting:

$$\Omega(\alpha, \beta) := [\phi(\alpha), \phi(\beta)] - \phi([\alpha, \beta]).$$

Show that:

- (a) The connection ∇ is independent of the choice of the splitting so that $\nu^*(N)$ is canonically a flat Poisson vector bundle over N .
- (b) The curvature defines a Poisson cohomology 2-class $[\Omega] \in H_{\Pi}^2(N, \nu^*(N))$ which is the obstruction for N to be a Lie-Poisson submanifold.
- (c) If N is a Lie-Poisson submanifold, then N is integrable.
- (d) If N is simply connected, then N is a Lie-Poisson submanifold if and only if its monodromy group vanishes.

5.4.2. Quotients

What we have just seen for sub-objects is typical: though the functor Σ gives us some indication of what the integration of a certain geometric construction is, there is often extra geometry hidden in the symplectization. Another instance of this happens when one looks at quotients.

Let G be a Lie group that acts smoothly by Poisson diffeomorphisms on a Poisson manifold (M, π) . We will denote the action by $\Psi : G \times M \rightarrow M$ and we will also write $\Psi(g, x) = g \cdot x$. For each $g \in G$, we set:

$$\Psi_g : M \rightarrow M, \quad x \mapsto g \cdot x,$$

so that each Ψ_g is a Poisson diffeomorphism.

Now we apply the functor Σ . For each $g \in G$, we obtain a Lagrangian subgroupoid $\Sigma(\Psi_g) \subset \Sigma(M) \times \overline{\Sigma(M)}$. This Lagrangian subgroupoid is, in fact, the graph of a symplectic Lie groupoid automorphism, which we denote by the same symbol $\Sigma(\Psi_g) : \Sigma(M) \rightarrow \Sigma(M)$. Also, it is not hard to check that

$$\Sigma(\Psi) : G \times \Sigma(M) \rightarrow \Sigma(M), \quad (g, [a]) \mapsto g \cdot [a] := \Sigma(\Psi_g)([a]),$$

³to clarify

defines a symplectic smooth action of G on $\Sigma(M)$. Briefly, Σ lifts a Poisson action $\Psi : G \times M \rightarrow M$ to a symplectic action $\Sigma(\Psi) : G \times \Sigma(M) \rightarrow \Sigma(M)$ by groupoid automorphisms. However, this is not the end of the story.

Let us look closer at how one lifts the action from M to $\Sigma(M)$. First of all, recall that any smooth action $G \times M \rightarrow M$ has a lifted cotangent action $G \times T^*M \rightarrow T^*M$. This yields, by composition, an action of G on cotangent paths: if $a : I \rightarrow T^*M$ is a cotangent path we just move it around

$$(g \cdot a)(t) := g \cdot a(t).$$

The fact that the original action $G \times M \rightarrow M$ is Poisson yields that (i) $g \cdot a$ is a cotangent path whenever a is a cotangent path, and (ii) if a_0 and a_1 are cotangent homotopic then so are the translated paths $g \cdot a_0$ and $g \cdot a_1$. Therefore, we have a well-defined action of G on cotangent homotopy classes and this is just the lifted action: $\Sigma(\Psi_g)([a]) = [g \cdot a]$.

Now we invoke a simple (but important) fact from symplectic geometry: for any action $G \times M \rightarrow M$ the lifted cotangent action $G \times T^*M \rightarrow T^*M$ is a hamiltonian action with equivariant momentum map $j : T^*M \rightarrow \mathfrak{g}^*$ given by:

$$j : T^*M \rightarrow \mathfrak{g}^*, \langle j(\alpha_x), \xi \rangle := \langle \alpha_x, X_\xi(x) \rangle,$$

where $X_\xi \in \mathfrak{X}(M)$ is the infinitesimal generator associated with $\xi \in \mathfrak{g}$. This yields immediately the fact that the lifted action $\Sigma(\Psi) : G \times \Sigma(M) \rightarrow \Sigma(M)$ is also hamiltonian ⁽⁴⁾. The equivariant momentum map $J : \Sigma(M) \rightarrow \mathfrak{g}^*$ for the lifted action is given by:

$$(5.30) \quad \langle J([a]), \xi \rangle := \int_0^1 j(a(t)) dt = \int_a X_\xi.$$

Since each X_ξ is a Poisson vector field, the last expression shows that only the cotangent homotopy class of a matters, and J is indeed well-defined. Expression 5.30 means that we can see the momentum map of the $\Sigma(\Psi)$ -action in two ways:

- It is the integration of the momentum map of the lifted cotangent action;
- It is the integration of the infinitesimal generators along cotangent paths.

In any case, expression (5.30) for the momentum map shows that it satisfies the following additive property:

$$J([a_0] \cdot [a_1]) = J([a_0]) + J([a_1]).$$

Hence J is a groupoid homomorphism from $\Sigma(M)$ to the additive group $(\mathfrak{g}^*, +)$ or, which is the same, J is differentiable *groupoid 1-cocycle*. Moreover, this cocycle is *exact* iff there exists a map $\mu : M \rightarrow \mathfrak{g}^*$ such that

$$J = \mu \circ \mathbf{t} - \mu \circ \mathbf{s},$$

and this happens precisely iff the original Poisson action $\Psi : G \times M \rightarrow M$ is hamiltonian with equivariant momentum map $\mu : M \rightarrow \mathfrak{g}^*$. We summarize all this in the following theorem:

⁴Recall again that, after all, the symplectic structure on $\Sigma(M)$ comes from canonical symplectic structures on cotangent bundles.

Theorem 5.31. *Let $\Psi : G \times M \rightarrow M$ be a smooth action of a Lie group G on a Poisson manifold M by Poisson diffeomorphisms. There exists a lifted action $\Sigma(\Psi) : G(M) \rightarrow \Sigma(M)$ by symplectic groupoid automorphisms. This lifted G -action is Hamiltonian and admits the momentum map $J : \Sigma(M) \rightarrow \mathfrak{g}^*$ given by (5.30). Furthermore:*

- (i) *The momentum map J is G -equivariant and is a groupoid 1-cocycle.*
- (ii) *The G -action on M is hamiltonian with momentum map $\mu : M \rightarrow \mathfrak{g}^*$ if and only if J is an exact cocycle.*

Let us now assume that the Poisson action $\Psi : G \times M \rightarrow M$ is proper and free. These assumptions guarantee that M/G is a smooth manifold. The space $C^\infty(M/G)$ of smooth functions on the quotient is naturally identified with the space $C^\infty(M)^G$ of G -invariant functions on M . Since the Poisson bracket of G -invariant functions is a G -invariant function, we have a quotient Poisson structure on M/G such that the natural projection $M \rightarrow M/G$ is a Poisson map.

Exercise 97. Show that if (M, π) is an integrable Poisson structure and $\Psi : G \times M \rightarrow M$ is proper and free Poisson action, then M/G is also an integrable Poisson manifold.

So the question arises: what is the relationship between the symplectic groupoids $\Sigma(M)$ and $\Sigma(M/G)$?

First notice that if the original Poisson action $\Psi : G \times M \rightarrow M$ is proper and free, so is the lifted action $\Sigma(\Psi) : G \times \Sigma(M) \rightarrow \Sigma(M)$. Therefore $0 \in \mathfrak{g}^*$ is a regular value of the momentum map $J : \Sigma(M) \rightarrow \mathfrak{g}^*$. Let us look at the symplectic quotient:

$$\Sigma(M)//G := J^{-1}(0)/G.$$

Since J is a groupoid homomorphism, its kernel $J^{-1}(0) \subset \Sigma(M)$ is a Lie subgroupoid. Since J is G -equivariant, the action leaves $J^{-1}(0)$ invariant and the restricted action is a free action by groupoid automorphisms. Hence, the groupoid structure descends to a groupoid structure $\Sigma(M)//G \rightrightarrows M/G$.

Exercise 98. Show that $\Sigma(M)//G$ is a symplectic groupoid that integrates the quotient Poisson manifold M/G .

In general, however, it is not true that:

$$\Sigma(M/G) = \Sigma(M)//G,$$

so, in general, symplectization *does not* commute with reduction. First of all, $J^{-1}(0)$ may not be connected, so that $\Sigma(M)//G$ may not have source connected fibers. Even if we restrict to $J^{-1}(0)^c$, the connected component of the identity section (so that the source fibers of $\Sigma(M)//G$ are connected) these fibers may have a non-trivial fundamental group, as shown by the following exercise:

Exercise 99. Consider the proper and free action of \mathbb{S}^1 on $M = \mathbb{C}^2 - \{0\}$ given by $\theta \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{-i\theta} z_2)$. For the canonical symplectic structure $\omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$, this \mathbb{S}^1 -action on M is Poisson. Determine both groupoids $\Sigma(M)//G$ and $\Sigma(M/G)$ and show that they are not isomorphic.

We will not get here into the subtleties of this problem.

5.5. Notes

The notion of a symplectic groupoid and its relation to Poisson geometry appears first in Weinstein [62] and is discussed in greater detail in [12]. The integration of Poisson manifolds presented has its roots in the work of Cattaneo and Felder [8] on the Poisson sigma model and was presented first by us in [16].

The main positive integrability result of [8] states that any Poisson structure on \mathbb{R}^2 is integrable. This, in turn, is a consequence of a general result due to Debord [18] (see also [15, Corollary 5.9]) that states that a Lie algebroid with almost injective anchor is integrable. The results on the integrability of regular Poisson manifolds and its relation to the variation of symplectic areas are due to Alcade Cuesta and Hector [1]. The two examples we presented (the modified $\mathfrak{su}^*(2)$ -bracket and the Heisenberg-Poisson manifolds) are both due to Weinstein [62, 64].

The “symplectic category” appears early in the modern history of symplectic geometry and is due to Weinstein [61]. Also, in his first note on symplectic groupoids he observes that a Poisson morphism integrates to a canonical relation. Coisotropic submanifolds are first studied in a systematic way by Weinstein in [63], in connection with Poisson groupoids. The problem of integrating coisotropic submanifolds was solved by Cattaneo and Felder in [9]. The integration of Poisson submanifolds (and more general submanifolds) was discussed first by Xu in [67], and then completed in [16]. Actions of groups on groupoids, and some of the properties on lifted actions can be found, in one form or another, in [10, 12, 42, 66, 68]. This problem, as well as other issues such as non-free actions, convexity, etc, is the subject of ongoing research (see [26]).

The symplectization functor can be (and should be!) applied to many other constructions in Poisson geometry. For example, one can look at the symplectization of Poisson fibrations (in the sense of [59]) and this yields groupoids which are symplectic fibrations (in the sense of [29, 41]). Another example, is provided by the connected sum construction in Poisson geometry (see [33]) which out of two Poisson manifolds M_1 and M_2 yields, under some conditions, a new Poisson manifold $M_1 \# M_2$. A result of the sort:

$$\Sigma(M_1 \# M_2) = \Sigma(M_1) \# \Sigma(M_2),$$

should be true (here, on the right-hand side one has a symplectic connected sum). A third example, is in the theory of Poisson-Nijenhuis manifolds where the application of the Σ functor leads to a symplectic-Nijenhuis groupoid ([13]), and this is relevant both in the study of the generalized complex structures and of integrable hierarchies associated with a PN-manifold. The symplectization functor Σ , which we have just started understanding, should play an important role in many other problems in Poisson geometry.

BIBLIOGRAPHY

1. F. Alcade Cuesta and G. Hector, Intégration symplectique des variétés de Poisson régulières, *Israel J. Math.* **90** (1995), 125–165.
2. R. Almeida and P. Molino, Suites d’Atiyah et feuilletages transversalement complets, *C. R. Acad. Sci. Paris Sér. I Math.* **300** (1985), 13–15.
3. K. Behrend, L. Fantechi, W. Fulton, L. Goettsche and A. Kresch, An Introduction to Stacks, book in preparation.
4. K. Behrend and P. Xu, Differentiable Stacks and Gerbes, preprint *math.DG/0605694* (2006).
5. W. Brandt, Über eine Verallgemeinerung des Gruppenbegriffes, *Math. Ann.* **96** (1926), 360–366.
6. R. Bryant, Bochner-Kähler metrics, *J. Amer. Math. Soc.* **14** (2001), no. 3, 623–715.
7. A. Cannas da Silva and A. Weinstein, *Geometric Models for Noncommutative Algebras*, Berkeley Mathematics Lectures, vol. **10**, American Math. Soc. , Providence, 1999.
8. A.S. Cattaneo and G. Felder, Poisson sigma models and symplectic groupoids, in *Quantization of Singular Symplectic Quotients*, (ed. N. P. Landsman, M. Pflaum, M. Schlichenmeier), Progress in Mathematics **198** (2001), 41–73.
9. A.S. Cattaneo and G. Felder, Coisotropic submanifolds in Poisson geometry and branes in the Poisson sigma model, *Lett. Math. Phys.* **69** (2004), 157–175.
10. M. Condevaux, P. Dazord, and P. Molino, Géométrie du moment, Travaux du Séminaire Sud-Rhodanien de Géométrie, I, *Publ. Dép. Math. Nouvelle Sér. B*, **88**, Univ. Claude-Bernard, Lyon, 1988, pp. 131–160.
11. A. Connes, *Noncommutative Geometry*, Academic Press, 1984.
12. A. Coste, P. Dazord, and A. Weinstein, Groupoïdes symplectiques, Publications du Département de Mathématiques. Nouvelle Série. A, Vol. 2, *Publ. Dép. Math. Nouvelle Sér. A*, **87**, Univ. Claude-Bernard, Lyon, 1987, pp. i–ii, 1–62.
13. M. Crainic, Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes, *Comment. Math. Helv.* **78** (2003), no. 4, 681–721.

14. M. Crainic, Generalized complex structures and Lie brackets, preprint math.DG/0412097.
15. M. Crainic and R. L. Fernandes, Integrability of Lie brackets, *Ann. of Math. (2)* **157** (2003), 575–620.
16. M. Crainic and R. L. Fernandes, Integrability of Poisson brackets, *J. Differential Geom.* **66** (2004), 71–137.
17. P. Dazord and G. Hector, Intégration symplectique des variétés de Poisson totalement asphériques, in *Symplectic Geometry, Groupoids and Integrable Systems*, MSRI Publ., **20** (1991), 37–72.
18. C. Debord, *Feuilletages singuliers et groupoïdes d'holonomie*, Ph. D. Thesis, Université Paul Sabatier, Toulouse, 2000.
19. A. Douady and M. Lazard, Espaces fibrés en algèbres de Lie et en groupes, *Invent. Math.* **1** (1966), 133–151.
20. J.-P. Dufour, N.T. Zung, *Poisson structures and their normal forms*, Progress in Mathematics, **242**, Birkhäuser Verlag, Basel, 2005.
21. J. Duistermaat and J. Kolk, *Lie Groups*, Springer-Verlag Berlin Heidelberg, 2000.
22. C. Ehresmann, Œuvres complètes et commentées. I-1,2. Topologie algébrique et géométrie différentielle, *Cahiers Topologie Géom. Différentielle* **24** (1983) suppl. 1.
23. S. Evens, J.-H. Lu and A. Weinstein, Transverse measures, the modular class and a cohomology pairing for Lie algebroids, *Quart. J. Math. Oxford (2)* **50** (1999), 417–436.
24. R.L. Fernandes, Lie algebroids, holonomy and characteristic classe, *Adv. in Math.* **170** (2002), 119–179.
25. R.L. Fernandes, Connections in Poisson Geometry I: Holonomy and Invariants, *J. Differential Geometry* **54** (2000), 303–366.
26. R.L. Fernandes, J.P. Ortega and T. Ratiu, Momentum maps in Poisson geometry, in preparation.
27. V. Ginzburg, Grothendieck groups of Poisson vector bundles, *J. Symplectic Geometry* **1** (2001), 121–169.
28. A. Grothendieck, Séminaire Henri Cartan, 13ième année: 1960/61. Fasc. 1, Exp. 7, 9–13; Fasc. 2, Exp. 14–17, deuxième édition, Secrétariat mathématique, Paris, 1962.
29. V. Guillemin, E. Lerman and S. Sternberg, *Symplectic fibrations and multiplicity diagrams*, Cambridge University Press, Cambridge, 1996.
30. A. Haefliger, Homotopy and integrability, Lecture Notes in Mathematics, Vol. 197, Springer, Berlin, 1971, 133–163.
31. P.J. Higgins and K. Mackenzie, Algebraic constructions in the category of Lie algebroids, *J. Algebra* **129** (1990), 194–230.
32. J. Huebschmann, *Duality for Lie-Rinehart algebras and the modular class*, J. reine angew. Math. **510** (1999), 103–159.
33. A. Ibort and D. Martínez Torres, A new construction of Poisson manifolds, *J. Symplectic Geometry* **2** (2003), 83–107.
34. M. Karasëv, Analogues of objects of the theory of Lie groups for nonlinear Poisson brackets, *Izv. Akad. Nauk SSSR Ser. Mat.* **50** (1986), no.3, 508–538, 638.

35. A. Kumpera, D. Spencer, *Lie equations. Vol. I: General theory*, Annals of Mathematics Studies, no.73, Princeton University Press, Princeton, N.J., 1972.
36. S. Lang, *Differential Manifolds*, Springer-Verlag, New-York, 1985.
37. K. Mackenzie, *Lie Groupoids and Lie Algebroids in Differential Geometry*, London Math. Soc. Lecture Notes Series **124**, Cambridge Univ. Press, Cambridge, 1987.
38. K. Mackenzie, Lie algebroids and Lie pseudoalgebras, *Bull. London Math. Soc.* **27** (1995), 97–147.
39. K. Mackenzie, *General theory of Lie groupoids and Lie algebroids*, London Mathematical Society Lecture Note Series **213**, Cambridge Univ. Press, Cambridge, 2005.
40. K. Mackenzie and P. Xu, Integration of Lie bialgebroids, *Topology* **39** (2000), 445–467.
41. D. McDuff and D. Salamon, *Introduction to symplectic topology*, 2nd edition, Oxford Mathematical Monographs, Oxford University Press, New York, 1998.
42. K. Mikami and A. Weinstein, Moments and reduction for symplectic groupoids, *Publ. Res. Inst. Math. Sci.* **24** (1988), no. 1, 121–140.
43. P. Molino, Étude des feuilletages transversalement complets et applications, *Ann. Scient. Éc. Norm. Sup.* **10** (1977), 289–307
44. I. Moerdijk and J. Mrčun, *Introduction to foliations and Lie groupoids*. Cambridge University Press, Cambridge, 2003.
45. I. Moerdijk, Orbifolds as groupoids: an introduction, in *Orbifolds in mathematics and physics*, 205–222, Contemp. Math., **310**, Amer. Math. Soc., Providence, RI, 2002.
46. I. Moerdijk, Introduction to the language of stacks and gerbes, University of Utrecht, 2002, preprint *math.AT/0212266*.
47. I. Moerdijk and J. Mrčun, On integrability of infinitesimal actions, preprint *math.DG/0406558*.
48. V. Nistor, Groupoids and the integration of Lie algebroids, *J. Math. Soc. Japan* **52** (2000), 847–868.
49. V. Nistor, A. Weinstein and P. Xu, Pseudodifferential operators on differential groupoids, *Pacific J. Math.* **189** (1999), 117–152.
50. R. Palais, A global formulation of the Lie theory of transformations, *Mem. Amer. Math. Soc.* **22**, 1957.
51. J. Pradines, Théorie de Lie pour les groupoïdes différentiables. Relations entre propriétés locales et globales, *C. R. Acad. Sci. Paris, Série A* **263** (1966) 907–910.
52. J. Pradines, Théorie de Lie pour les groupoïdes différentiables. Calcul différentiel dans la catégorie des groupoïdes infinitésimaux, *C. R. Acad. Sci. Paris, Série A* **264** (1967) 245–248.
53. J. Pradines, Géométrie différentielle au-dessus d'un groupoïde, *C. R. Acad. Sci. Paris, Série A* **266** (1968) 1194–1196.
54. J. Pradines, Troisième théorème de Lie pour les groupoïdes différentiables, *C. R. Acad. Sci. Paris, Série A* **267** (1968) 21–23.
55. P. Severa, Some title containing the words "homotopy" and "symplectic", e.g. this one, preprint *math.SG/0105080*.

56. I. M. Singer and S. Sternberg, The infinite groups of Lie and Cartan, *J. Analyse Math.* **15** (1965), 1–114.
57. H. Sussmann, Orbits of families of vector fields and integrability of distributions, *Trans. Amer. Math. Soc.* **180** (1973), 171–188.
58. A. Văintrob, Lie algebroids and homological vector fields, *Uspekhi Mat. Nauk* **52** (1997), no. 2(314), 161–162.
59. Y. Vorobjev, Coupling tensors and Poisson geometry near a single symplectic leaf, *Banach Center Publ.* **54** (2001), 249–274.
60. J. Yorke, Periods of periodic solutions and the Lipschitz constant, *Proc. Amer. Math. Soc.* **22** (1969), 509–512.
61. A. Weinstein, Symplectic geometry, *Bull. Amer. Math. Soc. (N.S.)* **5** (1981), no. 1, 1–13.
62. A. Weinstein, Symplectic groupoids and Poisson manifolds, *Bull. (New Series) Amer. Math. Soc.* **16** (1987), 101–104.
63. A. Weinstein, Coisotropic calculus and Poisson groupoids, *J. Math. Soc. Japan* **40** (1988), 705–727.
64. A. Weinstein, Blowing up realizations of Heisenberg-Poisson manifolds, *Bull. Sci. Math.* **113** (1989), 381–406.
65. A. Weinstein, Linearization of regular proper groupoids, *J. Inst. Math. Jussieu* **1** (2002), no.3, 493–511.
66. A. Weinstein and P. Xu, Extensions of symplectic groupoids and quantization, *J. Reine Angew. Math.* **417** (1991), 159–189.
67. P. Xu, Dirac submanifolds and Poisson involutions, *Ann. Sci. École Norm. Sup. (4)* **36** (2003), 403–430.
68. P. Xu, *Symplectic groupoids of reduced Poisson spaces*, *C. R. Acad. Sci. Paris, Série I Math.* **314** (1992) 457–461.
69. P. Xu, *Gerstenhaber algebras and BV-algebras in Poisson geometry*, *Comm. Math. Phys.* **200** (1999), 545–560.
70. P. Xu, Morita equivalence of Poisson manifolds, *Comm. Math. Phys.* **142** (1991), 493–509.
71. P. Xu, Morita equivalent symplectic groupoids, in *Symplectic geometry, groupoids, and integrable systems* (Berkeley, CA, 1989), 291–311, *Math. Sci. Res. Inst. Publ.*, **20**, Springer, New York, 1991.
72. S. Zakrzewski, Quantum and classical pseudogroups I and II, *Comm. Math. Phys.* **134** (1990), 347–395.
73. N.T. Zung, Proper Groupoids and Momentum Maps: Linearization, Affinity and Convexity, preprint *math.SG/0407208* (2004).