Symmetry beyond groups

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Main Reference:

http://www.math.ist.utl.pt/~rfern/
1. Introduction

Why groupoids?
Usual Credo:

Symmetry = Group Theory
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In this talk:

Symmetry ≠ Group Theory
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Symmetry = Group Theory

In this talk:

Symmetry ≠ Group Theory

Symmetry = Groupoid Theory
Basic Remark:

Many objects which we recognize as symmetric admit few or no non-trivial symmetries.
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Many objects which we recognize as symmetric admit few or no non-trivial symmetries.

*Groupoids* allow one to fix this.
2. Usual credo...

symmetries = groups
A **group** is a set $G$ together with a **multiplication**

$$G \times G \to G$$

$$(g_1, g_2) \mapsto g_1g_2$$

satisfying:
A group is a set $G$ together with a multiplication

$$G \times G \rightarrow G$$

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- **Associativity.** For all $g_1, g_2, g_3 \in G$:

  $$(g_1g_2)g_3 = g_1(g_2g_3).$$
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- **Identity.** There exists an element \( e \in G \):

\[
ge e = e g = e.
\]

- **Inverse.** For all \( g \in G \) there exists \( g^{-1} \in G \):

\[
g g^{-1} = g^{-1} g = e.
\]
Main example: group of isometries of $\mathbb{R}^n$
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If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$:

$$d(x, y) \equiv ||x - y|| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}.$$
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The **Euclidean group** is:

$$E(n) = \{ \phi : \mathbb{R}^n \rightarrow \mathbb{R}^n : d(\phi(x), \phi(y)) = d(x, y), \forall x, y \in \mathbb{R}^n \}$$
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with multiplication composition of isometries:

\[E(n) \times E(n) \to E(n)\]

\[(\phi_1, \phi_2) \mapsto \phi_1 \circ \phi_2.\]
Group of isometries of $\mathbb{R}^n$ (cont.)

Every isometry $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is of the form:

$$\phi(x) = Ax + b,$$

where $b \in \mathbb{R}^n$ and $A$ is an **orthogonal matrix**:

$$AA^T = A^TA = I.$$
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**ISOMETRY = ORTHOGONAL TRANSFORMATION + TRANSLATION**
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**ISOMETRY = ORTHOGONAL TRANSFORMATION + TRANSLATION**

**Remark:**

A **proper isometry** is an isometry which preserves orientation $\iff \phi(x) = Ax + b$ with $\det A = 1$. 

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  \[ \mathbb{R}^n = \{ \phi \in E(n) : \phi \text{ is a translation} \} , \]
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- **The special orthogonal group** ("rotations"):\[SO(n) = \{ \phi \in O(n) : \phi \text{ is proper} \}\]
  \[\simeq \{ A : AA^T = A^T A = I, \ \det A = 1 \} .\]
Symmetries
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Philosophic principle:
An object is symmetric if it has many symmetries.
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$$G_\Omega = O(n)$$

$$\tilde{G}_\Omega = SO(n)$$
Example: Tiling by rectangles of $\mathbb{R}^2$
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Take $\Omega \subset \mathbb{R}^2$ the tiling of $\mathbb{R}^2$ by 2 : 1 rectangles:

What is the group of symmetries $G_\Omega$?
Example: Tiling by rectangles of $\mathbb{R}^2$ (cont.)

The group $G_\Omega$ consists of:
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The group $G_\Omega$ consists of:

- Translations by elements of the lattice $\Lambda = 2\mathbb{Z} \times \mathbb{Z}$:

  $$(x, y) \mapsto (x, y) + (2n, m), \quad n, m \in \mathbb{Z}.$$
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- Reflections through horizontal and vertical lines:
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The tiling has a lot of symmetry!
This gives a very successful theory:

• symmetry groups of tilings;
• symmetry groups of crystals;
• symmetry groups of differential equations;
• symmetry groups of geometric structures;

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...But ...
3. Need for a new credo
Instead of tiling, take $B$ a **real** bathroom floor:

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Instead of tiling, take $B$ a **real** bathroom floor:

The group of symmetries shrinks drastically:

$$G_B = \mathbb{Z}_2 \times \mathbb{Z}_2.$$ 

It contains only 4 elements!
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\[
\begin{array}{cccc}
\emptyset & \emptyset & \emptyset & \emptyset \\
\emptyset & \emptyset & \emptyset & \emptyset \\
\emptyset & \emptyset & \emptyset & \emptyset \\
\end{array}
\]

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It contains only 4 elements!

However, we can still recognize a repetitive pattern...
Not surprising! There are *very few* symmetry groups:
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**Theorem 3.1.** The possible finite proper symmetry groups of a bounded region $\Omega \subset \mathbb{R}^3$ are:
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**Theorem 3.1.** The possible finite proper symmetry groups of a bounded region $\Omega \subset \mathbb{R}^3$ are:

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![Diagram of a regular polygon with a star]

- The group $D_n$ of symmetries of a regular $n$-side polyhedron:

![Diagram of a regular octahedron]

- The 3 groups of symmetries of the platonic solids.
For example, the molecule of the fullerene $C_{60}$:

has the same symmetry group as the icosahedron:
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has the same symmetry group as the icosahedron:

(just truncate the vertexes of the icosahedron).
4. Symmetry groupoids

To distinguish the soccer ball from the icosahedron, to describe the symmetry of a bathroom floor, and in many other problems, we need groupoids.
Look again at the tiling $\Omega$. 
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\[ G_\Omega = \{(x, \phi, y) : x, y \in \mathbb{R}^2, \phi \in G_\Omega \text{ and } x = \phi(y)\} \]
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with the partially defined multiplication:

$$(x, \phi, y)(y, \psi, z) = (x, \phi \circ \psi, z).$$
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with the *partially defined multiplication*:

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We can view each $g = (x, \phi, y) \in G$ as an arrow:

$$
\begin{array}{c}
\bullet \\
\downarrow \\
x \\
\bullet \\
y
\end{array}
\begin{array}{c}
\bullet \\
\downarrow \\
g \\
\bullet \\
\end{array}
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Now, we have:

- **source** and **target maps** $s, t : \mathcal{G} \to \mathbb{R}^2$:

  $$s(x, \phi, y) = y, \quad t(x, \phi, y) = x.$$
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\[
\begin{array}{cc}
\bullet & \bullet \\
g & \\
\bullet & \bullet \\
x & y
\end{array}
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- \textit{identity arrows} $1_x = (x, I, x)$:
  
  $$\begin{array}{c}
  1_x \\
  \circ \\
  x
  \end{array}$$
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- **inverse arrows** $g^{-1} = (y, \phi^{-1}, x)$:
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4. **Inverse:** \(gg^{-1} = 1_x\) and \(g^{-1}g = 1_y\).

**Definition 4.1.** A **groupoid** with base \(B\) is a set \(\mathcal{G}\) with maps \(s, t : \mathcal{G} \to B\) and operation satisfying 1–4.
We can restrict the symmetry groupoid $G_\Omega$ of the tiling, to the real bathroom floor $B \subset \mathbb{R}^2$:

$$G_B = \{(x, \phi, y) : x, y \in B, \phi \in G_\Omega \text{ and } x = \phi(y)\}.$$
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The groupoid $G_B$ captures the symmetry of the real bathroom floor.

We need two elementary concepts from groupoid theory:
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- Two elements $x, y \in B$ belong to the same orbit of $\mathcal{G}$ if they can be connected by an arrow:

$$\xrightarrow{g}$$

where $x \xrightarrow{g} y$. 

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  $\begin{array}{c}
  x \\
  \downarrow \\
  \downarrow \\
  y
  \end{array} \xrightarrow{g} \begin{array}{c}
  \bullet \\
  \bullet
  \end{array}$

- The isotropy group of $x \in B$ is the set of arrows $g \in \mathcal{G}$ from $x$ to $x$:

  $\begin{array}{c}
  x \\
  \circlearrowleft
  \end{array}$
For the symmetry groupoid $g_B$ of the real bathroom floor:
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●          ●
●          ●
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- The only points with non-trivial isotropy are those in $(\mathbb{Z} \times \frac{1}{2}\mathbb{Z}) \cap B$. For these, the isotropy group is:

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2.$$
5. Other groupoids

Groupoids play an important role in many other contexts, not related with symmetry.
Fundamental Groupoid of a space
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$X$ any topological space
Look at continuous curves $\gamma : [0, 1] \rightarrow X$
Fundamental Groupoid of a space

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Fundamental Groupoid of a space

$X$ any \textit{topological} space

Look at \textit{continuous} curves $\gamma : [0, 1] \rightarrow X$
Fundamental Groupoid of a space

\( X \) any topological space

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Fundamental Groupoid of a space

$X$ any *topological* space
Look at *continuous* curves $\gamma : [0, 1] \to X$
Fundamental Groupoid of a space

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Look at *continuous* curves $\gamma : [0, 1] \rightarrow X$
Fundamental Groupoid of a space

$X$ any topological space

Look at continuous curves $\gamma : [0, 1] \to X$

$$[\gamma] \equiv \text{homotopy class of } \gamma$$
Fundamental Groupoid of a space

$X$ any topological space
Look at continuous curves $\gamma : [0, 1] \rightarrow X$

$[\gamma] \equiv \text{homotopy class of } \gamma$ \hspace{1em} (e.g. $[\gamma_0] = [\gamma_1]$ but $[\gamma_0] \neq [\eta]$).
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The fundamental groupoid of $X$ is:

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- **source** and **target** give initial and final points:

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- **inverse** is the opposite curve:
  
  \[ [\gamma]^{-1} = [\overline{\gamma}], \quad \text{where } \overline{\gamma}(t) = \gamma(1 - t). \]
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**Examples:**

- If \( X = SO(2) \) one has \( \pi(X, x) = \mathbb{Z} \).
- If \( X = SO(n) \) one has \( \pi(X, x) = \mathbb{Z}_2 = \{+1, -1\} \).
Groupoids and control theory

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- Orbits are the leaves of $\mathcal{F}$;
- Isotropy groups are the fundamental groups of the leaves.
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This is where the *real math* starts and where this talk stops...