Notations:
- $X$: a smooth manifold, the space in which our particle moves.
- $M$: spacetime (usually taken to be Minkovski space)
- $\mathcal{M}$: phase space, the space of states of classical system
- $O$: observable, function on $\mathcal{M}$
- $O(t, f)$: a family of real-valued observable parameterized by pairs $(t, f)$

1 Classical Mechanics

1.1 particle motion:
We take the space $X$ in which our particle moves to be a smooth manifold. To determine the particle motion on $X$, we need the potential energy function:

$$V : X \rightarrow \mathbb{R}$$

We model the time difference as a principal $\mathbb{R}$ bundle $M^1$ over a point (i.e. $M_1$ is $\mathbb{R}^1$).

There is one more piece of structure to include, as we have not yet specified the choice of units seconds, hours, etc. in which to measure time. For that we ask that $M^1$ carry a translation-invariant Riemannian metric, or equivalently that $V$ be an inner product space.

We can describe all potential particle motions in $X$ as the infinite dimensional function space $F = Map(M^1, X)$ of smooth maps from time to space. Actual particle trajectories are those which satisfy the differential equation

$$m\ddot{x}(t) = -V'(x(t))$$ (1)

This is Newton’s second law expressed in terms of energy.

1.2 phase space, states and observables
Let $\mathcal{M}$ denote the space of all solutions to (1). It’s a submanifold of the function space $F = Map(M^1, X)$ cut out by (1). It is the space of states of our classical system, often called the phase space. $\mathcal{M}$ is typically infinite-dimensional.
In face $\mathcal{M}$ a symplectic manifold, and that motion is described by a particular one-parameter group of symplectic diffeomorphisms called time translations. 

$$(T_s x)(t) = x(t - s) \text{ for } x \in \mathcal{M} \text{ and } s \in \mathbb{R}.$$ 

We can describe the symplectic structure by considering

$$\mathcal{M} \rightarrow TX$$

$$x \mapsto (x(t_0), \dot{x}(t_0))$$

(2)

If $X$ is a complete Riemann manifold, then this map is a diffeomorphism. The Riemannian structure gives an isomorphism of vector bundles $TX \cong T^*X$, so composing with (2) gives us a diffeomorphism $\mathcal{M} \rightarrow T^*X$. The symplectic structure on $\mathcal{M}$ is the pullback of the natural symplectic structure on $T^*X$.

The observables of our classical system are simply functions on $\mathcal{M}$. The pairing of observables and states is then the evaluation of functions.

A typical family of real-valued observables $O_{(t,f)}$ is parametrized by pairs $(t,f)$ consisting of a time $t \in \mathcal{M}$ and a function $f : X \rightarrow \mathbb{R}$. Then we have evaluation:

$$O_{(t,f)}(x) = f(x(t)), x \in \mathcal{M}$$

2 Hamiltonian mechanics

If $\mathcal{M}$ is a symplectic manifold with symplectic form $\Omega$, then there is an exact sequence

$$0 \rightarrow H^0(\mathcal{M}; \mathbb{R}) \rightarrow \Omega^0(\mathcal{M}) \xrightarrow{d} \mathfrak{X}_\Omega(M) \rightarrow H^1(\mathcal{M}; \mathbb{R}) \rightarrow 0$$

where $\mathfrak{X}_\Omega(M)$ is the space of vector fields $\xi$ on $X$ which (infinitesimally) preserve $\Omega$, i.e. $L_\xi \Omega = 0$. The symplectic gradient of a function $O$ is the unique vector field $\xi_O$ such that

$$dO = -\iota(\xi_O)\Omega$$

Thus any observable $O$ determines an infinitesimal group of symplectic automorphisms, so in good cases a one-parameter group of symplectic diffeomorphism. Conversely, an infinitesimal group of symplectic automorphisms which satisfies a certain cohomological constraint determines a set of observables any two elements of which differ by a locally constant function.

So the state space $\mathcal{M}$ carries a one-parameter group of time translations, $\xi_t \text{ s.t. } dO_t = -\iota(\xi_O)\Omega$. So if $\xi$ is the infinitesimal generator, we assume that $\iota(\xi)\Omega$ is exact. A choice of corresponding observable is the negative of a quantity we call the energy or Hamiltonian of the system.

A classical system $(\mathcal{M}, H)$ consisting of a symplectic manifold $\mathcal{M}$ and a Hamiltonian $H$ is free if $\mathcal{M}$ is a symplectic affine space and the motion generated by $H$ is a one-parameter group of affine symplectic transformations, or equivalently $H$ is at most quadratic.

There is a Lie algebra structure on $\Omega^0(\mathcal{M})$ so that the symplectic gradient is a Lie algebra homomorphism to the Lie algebra of vector fields (with Lie
bracket). In other words, this Poisson bracket on functions $O, O'$ satisfies

$$[\xi_O, \xi_{O'}] = \xi_{\{O, O'\}}$$

Because the symplectic gradient has a kernel, this equation does not quite determine the Poisson bracket. But we can define it directly by

$$\{O, O'\} = \xi_O O'$$

A **global symmetry** of the system is a symplectic diffeomorphism of $M$ which preserves $H$. An infinitesimal symmetry is a vector field $\xi$ on $M$ which preserves both the symplectic form $\Omega$ and the Hamiltonian $H$, i.e.

$$L_\xi \Omega = L_\xi H = 0$$

Then the observable $O$ which corresponds to an infinitesimal symmetry satisfies

$$\{H, O\} = 0 \quad \text{(3)}$$

Generally, the time-translation flow on $M$ induces a flow on observable as

$$\dot{O} = \{H, O\}$$

So (3) is equivalent to conservation law for $Q$: $\dot{Q} = 0$. For this reason, the observable $Q$ is called the conserved charge associated to an infinitesimal symmetry. So we see that symmetries lead to conservation laws.

### 3 Lagrangian Mechanics

From a Lagrangian description one recovers the symplectic story.

We study **fields** on a smooth finite-dimensional manifold $M$, which we usually think of as spacetime. Often the spacetime $M$ is equipped with topology or geometric structure—an orientation, spin structure, metric, etc. Attached to $M$ is a space of fields $F$ which are the variables for the field theory.

The evaluation map

$$e : F \times M \longrightarrow X$$

$$(\phi, m) \longmapsto \phi(m)$$

is what distinguishes function spaces $F$ from arbitrary infinite-dimensional manifolds.

$M$ is a submanifold of $F = \text{map}(M^1, X)$. We want to describe this submanifold as the critical manifold of a function

$$S : F \longrightarrow \mathbb{R}$$

If $M$ is the space of solutions to $dS = 0$, such a function $S$ would be called the action of the theory, and the critical point equation the Euler Lagrange equation.
A density on $M^1$ has the form $g(t)\,|dt|$ for some function $g : M^1 \to \mathbb{R}$. The
lagrangian density, or simply lagrangian,

$$L : \mathcal{F} \to \text{Densities}(M^1)$$

attaches to each potential particle motion $x \in \mathcal{F} = \text{Map}(M^1, X)$ a density
$L(x)$ on the line $M^1$. The lagrangian density is well-defined on all of $\mathcal{F}$, but its
integral over the whole line may well be infinite.

The lagrangian for the particle is

$$L(x) = \left[ \frac{m}{2} \dot{x}(t)^2 - V(x(t)) \right] \,|dt|$$

The dependence of the lagrangian on the path $x$ is local in the time variable $t$. For each $x : M^1 \to X$, $L(x)$ is a density on $M^1$. For (finite) time $t_0 < t_1$ the action

$$S_{[t_0, t_1]}(x) = \int_{t_0}^{t_1} L(x)$$

is well-defined, whereas the integral over the whole line may be infinite.

In general, a density on a finite-dimensional manifold $M$ is a tensor field
which in a local coordinate system $x^\mu$ is represented by

$$\ell(x)\,|dx^1 \cdots dx^n|$$

for some function $\ell$.

Denote the set of densities by $\Omega^0(M)$. Then define $\Omega^{1-q}(M)$ to be the set of
twisted $(n-q)$-forms, which is the tensor product of a section of $\wedge^q T M$ and
a density.

We express lagrangian field theory in terms of differential forms on $\mathcal{F} \times M$,
except that we twist by the orientation bundle to use densities on $M$ instead
of forms. So we work in a double complex $\Omega^\bullet,\bullet(\mathcal{F} \times M)$ whose homogeneous
subspace $\Omega^{p,-q}(\mathcal{F} \times M)$ is the space of $p$-forms on $\mathcal{F}$ with values in the space
of twisted $(n-q)$-forms on $M$.

Let $\delta$ be the exterior derivative on $\mathcal{F}$ with values in the space of twisted
$(n-q)$-forms on $M$, and $D = \delta + d$ the total exterior derivative. We have

$$D^2 = d^2 = \delta^2 = 0, \quad d\delta = -\delta d$$

The value of a form $\alpha \in \Omega^{p,q}(\mathcal{F} \times M)$ at a point $m \in M$ and a field $\phi \in \mathcal{F}$
on tangent vectors $\xi_1, \ldots, \xi_p$ to $\mathcal{F}$ is a twisted form at $m$ of $\phi$ and the $\xi_i$. The
form $\alpha$ is local if this twisted form depends only on the $k$-jet at $m$ of $\phi$ and the
$\xi_i$. We will denote the local forms by $\Omega^{p,q}_{\text{loc}}(\mathcal{F} \times M)$

Fix a spacetime $M$ and a space of fields $\mathcal{F}$. A classical field theory is specified
by two pieces of data: a lagrangian density

$$L \in \Omega^{0,0}_{\text{loc}}(\mathcal{F} \times M)$$

and a variational 1-form

$$\gamma \in \Omega^{1,-1}_{\text{loc}}(\mathcal{F} \times M)$$

$\text{4}$
So we can define the total lagrangian to be the sum of lagrangian density and the variational 1-form:

\[ \mathcal{L} = L + \gamma \in \Omega^{[0]}(F \times M) \]

A lagrangian field theory on a spacetime \( M \) with fields \( F \) is a Lagrangian density \( L \in \Omega^{[0]}(F \times M) \) and a variational 1-form \( \gamma \in \Omega^{[1]}_{\text{loc}}(F \times M) \) such that if \( \mathcal{L} = L + \gamma \) is the total lagrangian, then \((D\mathcal{L})^{1,[0]}\) is linear over functions on \( M \).

Newton’s equation \( m\nabla_\dot{x}\dot{x} + \text{grad } V(x(t)) \) is embedded in this formula

\[
(D\mathcal{L})^{1,[0]} = \delta L + d\gamma = -m\langle \nabla_\dot{x}\dot{x} + \text{grad } V, \delta x \rangle \wedge dt \tag{4}
\]

\[
(D\mathcal{L})^{2,[−1]} = \delta \gamma = m\langle \delta \nabla_\dot{x}\dot{x} \wedge \delta x \rangle \tag{5}
\]

\( m\langle \delta \nabla_\dot{x}\dot{x} \wedge \delta x \rangle \) is a 2-form on \( F \) which restricts to the symplectic form in the space \( M \) of solutions to Newton’s law.

So, given a classical field theory, we define the space of classical solutions \( M \subset F \) to be the space of \( \phi \in F \) such that the restriction of \((D\mathcal{L})^{1,[0]}\) to \( \{\phi\} \times M \) vanishes:

\[
(D\mathcal{L})^{1,[0]} = \delta L + d\gamma = 0 \quad \text{on } M \times M.
\]

The associated local symplectic form is

\[
\omega := \delta \gamma \in \Omega^{2,[−1]}_{\text{loc}}(F \times M)
\]

In the lagrangian field theory with Hamiltonian structure, the symplectic form on the phase space \( \mathcal{M} \) is

\[
\Omega = \int_{\{t\} \times N} \omega \in \Omega^2(\mathcal{M})
\]

References

[1] Classical Field Theory and Supersymmetry Daniel S. Freed