**Def.** An action of a Lie algebroid \( A \to M \) along a map \( \mu : S \to M \) is given by a linear map \( \sigma : \mathfrak{p}(A) \to \mathfrak{X}(S) \) such that:

(i) \( \sigma(f_x) = (f \circ \mu) \sigma(x) \quad \forall x, \beta \in \mathfrak{p}(A), f \in \mathcal{D}(S) \)

(ii) \( \sigma([x, \beta]) = [\sigma(x), \sigma(\beta)] \)

Notice that (i) is equivalent to any action is given by a v.f. map

\[
\begin{array}{ccc}
\mu^* A & \xrightarrow{\sigma} & TS \\
\downarrow & & \downarrow \\
S & = & \mathcal{D}(S)
\end{array}
\]

By the next exact, this map satisfies:

\[
\begin{array}{ccc}
\mu^* A & \xrightarrow{\sigma} & TS \\
\downarrow & & \downarrow d\mu \\
A & \xrightarrow{p} & TN
\end{array}
\]

**Lemma** For an action \( \sigma : \mathfrak{p}(A) \to \mathfrak{X}(S) \) the vector fields \( \sigma(x) \neq 0 \) are \( \mu \)-related:

\( \mu_x \sigma(x) = \rho(x) \)

**Proof.** For any \( x, \beta \in \mathfrak{p}(A), f \in \mathcal{D}(S) \)

\[
\begin{align*}
\sigma([x, \beta]) &= \sigma(f \circ (x, \beta)) + \rho(x)(f) \sigma(\beta) \\
&= (f \circ \mu) \sigma([x, \beta]) + \rho(x)(f) \circ \mu \sigma(\beta) \\
&\quad = \sigma(x) \circ \sigma([x, \beta]) + \sigma(x)(f) \circ \mu \sigma(\beta) \\
&\quad = \sigma(x) \circ \rho(\mu) \sigma([x, \beta]) + \sigma(x)(f) \circ \mu \sigma(\beta) \\
&\quad = \sigma(x)(f \circ \mu) = \rho(x)(f) \circ \mu \quad \forall f \quad \Rightarrow \mu_x \sigma(x) = \rho(x)
\end{align*}
\]
For an action $\sigma : \Gamma(A) \to \mathfrak{X}(S)$ on a map $\mu : S \to M$ one has an action via a Lie algebroid

$$\mu^*A \to S$$

where:

- **Anchor**: $p : \sigma : \mu^*A \to TS$

- **Lie bracket**: $[\mu^*A, \mu^*B] := \mu^*[\mu^*A, \mu^*B]$ and extend to any section by Leibniz.

It follows that for a Lie algebroid action:

- **Orbits** := orbits of $\mu^*A \to S$ are locally immersive submanifolds of $S$

- **Isotropy Lie algebroid** := $\ker \sigma_p \cong \text{Isotropy Lie algebroid of } \mu_p$ (left) Groupoid actions

$\mathcal{G} = \Pi \text{ on } \mu : S \to \Pi \Rightarrow \sigma : \mu^*A(0) \to TS$

$$\sigma_p : \mu(p) \to T_pS$$

$$\begin{cases} \mathcal{R}_p : \xi'(\mu(p)) \to S, \xi \mapsto \mathcal{R}_p \xi, \\ \mathcal{E}_p : c_p(\xi')(p) \end{cases}$$

This can also be expressed by:

$$\sigma(\alpha)_p := \left. \frac{d}{dt} \exp(-t\alpha)_p \right|_{t=0}$$

where a bisection $b \in B(\mathcal{G})$ acts by:

$$p \mapsto b([\sigma \circ b]_p(p)) \cdot p$$

If instead right Groupoid action, then

$$\begin{cases} \mathcal{L}_p : \xi'(\mu(p)) \to S, \xi \mapsto \mathcal{L}_p \xi, \\ \mathcal{E}_p : c_p(\xi')(p) \end{cases}$$

Now one has:

$$\sigma(\alpha)_p := \left. \frac{d}{dt} p \cdot \exp(t\alpha) \right|_{t=0}$$

where a bisection acts by:

$$p \mapsto p \cdot b(\mu(p))$$
Examples

1) For a Lie algebra \( g \to \mathfrak{g} \) this notion reduces to the usual notion of infinitesimal \( \mathfrak{g} \)-action on \( S \).

2) Any Lie algebra \( A \to M \) acts on its base \( \mu: M \to A \). If \( A = A(\mathfrak{g}) \to M \) then

\[ \begin{align*} 
\cdot \text{ A acts on } t: & \ G \to M : \quad \sigma(\alpha) = -\bar{\alpha} \\
\cdot \text{ A acts on } & \mathfrak{g} : \ G \to M : \quad \sigma(\alpha) = \bar{\alpha}
\end{align*} \]

These are of course the differentials of the actions of \( G \) on \( M \) and on itself by left/right translations.

3) Given a principal \( G \)-bundle \( \pi: P \to M \) the Atiyah algebra \( A = TP/\mathfrak{g} \to M \) acts on \( \pi: P \to M \):

\[ \sigma: \pi \to TP : \quad \sigma_p(\mathbf{v}) = \mathbf{v} \]

Again this is the differential of the action \( (P \times \mathfrak{g})/G \to M \) on \( \pi: P \to M \).

4) Given any \( \mathfrak{g} \)-bilinear \( \mu: E \to M \) we have an action of \( \mathfrak{g}(E) \to M \) on \( \mu: E \to M \) by setting:

\[ \sigma: \mathfrak{g}(E) \to \mathfrak{X}(E) \quad \sigma(D) = \hat{X}_D^* \]

where \( \hat{X}_D^* \) is the vector field whose flow is \( \psi_D^t \).

In last example, \( \hat{X}_D^* \) is a fibrewise linear vector field on \( E \to M \):

- \( f: E \to M \) is fibrewise linear if:

\[ f(\lambda v) = \lambda f(v), \quad \forall \lambda \in \mathbb{R} \]

\[ \Rightarrow m_\lambda f = \lambda f \quad (m_\lambda(v) = \lambda v) \]
• $X \in \mathcal{X}(E)$ is fiberwise linear if
  \[ \forall f \in C^\infty(E) \Rightarrow X(f) \in C^\infty(E) \]
  \[ \Longleftrightarrow \quad M^X_x X = X \]

Fix contractible open chart $(U, \varphi)$ for $M$ so $E|_U \to U$ has
local basis of sections $\{ e_a \} \mapsto (\varphi^i, \varphi^a)$ coordinates on $E$

$X \in \mathcal{X}^{\text{lin}}(E|_U) \Longleftrightarrow X = X^i(\varphi) \frac{\partial}{\partial \varphi^i} + X^a(\varphi) \frac{\partial}{\partial \varphi^a}$

**Lemma 1.** There is a Lie algebra isomorphism:

\[ \text{Der}(E) \cong \mathcal{X}^{\text{lin}}(E) \]

\[ D \mapsto \tilde{X}^*_D = \left( \frac{\partial}{\partial \varphi^i} \bigg|_{D} \right) \]

In local coordinates:

\[
\begin{cases}
  D(e_a) = D^b_a(e_b) e_b \\
  X_b = X^i_a e_b \frac{\partial}{\partial \varphi^i}
\end{cases}
\]

\[ \tilde{X}^*_D = X^i_a e_a \frac{\partial}{\partial \varphi^i} - D^b_a \frac{\partial}{\partial \varphi^a} \frac{\partial}{\partial \varphi^b} \]

Check the minus sign! (e.g., $[D_1, D_2] \mapsto [\tilde{X}^*_D_1, \tilde{X}^*_D_2]$ only
holds because of the minus sign!) 

**Lemma 2.** The following are equivalent:

(i) An action $\varphi : \mathcal{P}(A) \to \mathcal{X}^{\text{lin}}(E)$

(ii) A map $\nabla : \mathcal{P}(A) \times \mathcal{P}(E) \to \mathcal{P}(E)$, $(a, s) \mapsto \nabla_a s$, satisfying:

\[
\begin{align*}
  (a) & \quad \nabla_a s \text{ is } \mathcal{P}(A)-\text{linear} \\
  (b) & \quad \nabla_a \alpha \text{ is } C^\infty-\text{linear} \\
  (c) & \quad \nabla_a(f s) = f \nabla_a s + \varphi(a)(f) \alpha \\
  (d) & \quad \nabla_{[\alpha, \beta]} s = \nabla_\alpha \nabla_\beta s - \nabla_\beta \nabla_\alpha s
\end{align*}
\]

**Proof:**

\[ \mathcal{X}^{\text{lin}}(E) \ni \gamma(a) \mapsto \nabla_a \in \text{Der}(E) \quad (a), (b) \text{ hold } \]
Def. A map $\nabla : \mathfrak{g}(A) \times \mathfrak{p}(E) \to \mathfrak{p}(E)$ satisfying (a), (b), (c) is called an **A-connection** on the vector bundle $E \to M$. If additionally, it satisfies (d) it is called a **Flat A-connection**.

Note that if $\nabla$ is a flat A-connection we have a map

$$\mathfrak{g}(A) \to \text{Def}_0(E) = \mathfrak{g}(\mathfrak{g}(E)),$$

which is $C^\infty(M)$-linear (by (b)). Hence, it is induced by a v.b. map

$$A \to \mathfrak{g}(\mathfrak{g}(E))$$

$$\downarrow \quad \downarrow$$

$$M \quad \overset{\text{id}}{\to} \quad M$$

This map satisfies:

- $\mathfrak{g}(\mathfrak{g}(E)) \circ \nabla = X \nabla = \mathfrak{g}(A)$ (by (c))

- preserves Lie brackets (by (d))

Hence, it is a Lie algebraic morphism $A \to \mathfrak{g}(\mathfrak{g}(E))$ covering id. Conversely, every such morphism determines a flat connection.

Def. A representation of a Lie algebroid $A \to M$ on a v.b. $E \to M$ is given by any of the following equivalent data:

(a) A linear algebroid action $\mathfrak{g} : \mathfrak{g}(A) \to \mathfrak{g}(\mathfrak{g}(E))$;

(b) A flat A-connection $\nabla$;

(c) A Lie algebroid morphism $\varphi : A \to \mathfrak{g}(E)$ covering $\text{id}_M$.

We say that a rep is **faithful** if $\varphi$ is injective.
Given two reps \((E, V^1), (E_2, V^2)\) of \(A\):
- **Direct sum of Reps:** \(E \oplus E_2 \oplus \nabla_a (s \otimes s_2) = V^1_a \otimes V^2_a s_2\)
- **Tensor product of Reps:** \(E \otimes E_2 \oplus \nabla_a (s \otimes s_2) = V^1_a \otimes s_2 + V^2_a \otimes s_2\)

**Trivial Rep:** \(\mathbb{R} \rightarrow M: \nabla_a (f) = \rho_A(f)\)

\(\Rightarrow\) **Rep(A) is a semi-ring.**

**Examples.**

1. Any **Rep of** \(G = \mathbb{R}\) on \(\mathbb{R} \rightarrow \mathbb{R}\) induces a **Rep of** \(A = A(G)\) on \(E\).

2. **A = \mathbb{A} \ltimes \mathbb{M} \rightarrow \mathbb{M}** acts on Killing coadjoint \(\text{Rep}(A) = \frac{1}{2} \mathbb{A}\)-equivariant vector bundles?\)

3. \((\mathbb{M}, \mathbb{F})\) foliated manifold \(A = T\mathbb{F}\) has a natural Rep on \(\mathbb{U}(\mathbb{F}) \rightarrow \mathbb{F}\):

\[x \in \mathbb{F}, y \in \mathbb{U}(\mathbb{F}) \Rightarrow y \in \mathbb{U}(\mathbb{F}), \bigtriangledown_x y = y \mod \mathbb{U}(\mathbb{F})\]

\[\bigtriangledown_x y := [x, y]\]

This is known as the **Bott Connection.** In foliation theory is called a "partial connection" = \(T\mathbb{F}\)-connection

4. Every Lie algebra \(A \rightarrow \mathbb{M}\) has a canonical Rep on the line bundle \(\Lambda^\text{top} A \otimes \Lambda^\text{top}\mathbb{M} \rightarrow \mathbb{M}\). It is defined by:

\[\nabla_a (\omega_1 \wedge \ldots \wedge \omega_k) = \sum_{j=1}^k \omega_1 \wedge \ldots \hat{\omega}_j \wedge \ldots \wedge \omega_k \otimes [a, \omega_1] + \omega_1 \wedge \ldots \wedge \omega_k \otimes \mathbb{L}_a(\omega_1 \wedge \ldots \wedge \omega_k)\]
(Check this!)

5. Every regular Lie algebraoid $A \to M$ has canonical reps on the isotropy bundle $\mathcal{A}(A) \to M$ and on the normal bundle to the orbit foliation:

$\nabla_A \rho := [\alpha, \rho] \quad \rho \in \mathcal{A}(A) \quad (\mathcal{A}(A) = \text{ker } \rho)$

$\nabla_A \overline{x} := \left[\rho_\alpha(x), \overline{x}\right] \quad \overline{x} \in \mathcal{V}(\mathcal{F}_a) \quad (\mathcal{F}_a = \text{Im } \rho)$

One can put them together into a single rep: $\text{ker } \rho \otimes \text{Im } \rho$ which can be thought of as the adjoint rep of $A$.

For non-regular algebraoids this only exists as a "representation up to homotopy".

6. Let $A \to M$ be any Lie algebraoid. Fix orbit $O \subset M$ Then we have restriction $A|_O \to O$. This as a canonical representations:

i) $\mathcal{A}|_O = (\text{ker } \rho)|_O = \text{ker } \rho|_O$ (special case of 5!)

ii) $\mathcal{V}(O) \to O$ (not special case of 5!)

$\nabla_A \overline{x} = \left[p(\alpha), \overline{x}\right]|_O \quad \alpha \in \mathcal{A}(A), \overline{x} \in \mathcal{V}(O)$

The latter can be thought as the Bott connection on orbit $O$. 