9) Lie algebras of v.b. $E \to M$ a vector bundle

A derivation of $E$ is a pair $(D, X)$ where

- $D : \mathcal{P}(E) \to \mathcal{P}(E)$ linear map
- $X_D \in \mathfrak{X}(M)$

\[ D(\xi S) = \xi D S + X_D(\xi), \quad \forall \xi \in \mathcal{C}(\mathfrak{h}), \forall \xi \in \mathcal{P}(E) \]

If $D_1 = D_2$ then $X_{D_1} = X_{D_2}$, so one usually denotes a derivation simply by $D$. The vector field $X_D$ is called the symbol of $D$.

In $\Der(E) = \{ \text{derivations} \}$:

- Lie bracket:
  \[ [D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1 \]

- The symbol map $\rho : \Der(E) \to \mathfrak{X}(\mathfrak{h})$, a Lie algebra map

Q. Is $\Der(E)$ the space of sections of some vector bundle $A \to M$?

A. Yes! Apply **Serre-Swan Theorem**: Every finitely generated, projective, $\mathcal{C}^\infty(\mathfrak{h})$-module over a connected manifold $M$ is the space of sections of a v.b. Can also use the following instead:

**Exercise.** Show that the Lie algebra of $\GL(E) \to M$, denoted $\mathfrak{g}(E) \to M$, admits a natural linear isomorphism:

\[ \rho(\mathfrak{g}(E)) \cong \Der(E) \]

which takes the Lie bracket and anchor to the commutator and the symbol in $\Der(E)$.  

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\( \nabla \) connection on \( E \) \( \Rightarrow \) \( D := \nabla_x \) is derivation of symbol \( X \) 
\( \Rightarrow \) \( \mathcal{D}(E) \) has surjective anchor, so it is transitive algebroid.

\[ \{ D \in \text{Der}(E) : X_0 = 0 \} = \{ D : \Gamma(E) \to \Gamma(E) | C^\infty(E)-\text{smooth} \} = \Gamma(\mathcal{E}(E)) \]

\( 0 \to \mathcal{E}(E) \to \mathcal{D}(E) \to TM \to 0 \)

Conclusion:
\( \mathcal{D}(E) \cong \mathcal{E}(E) \oplus TM \) (non-canonical/depends on choice of \( \nabla \))

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**Remark.** From our previous discussion:
\[ GL(E) \cong \text{Gauge Groupoid of } \text{Fr}(E) \to M \]
\[ \text{OGL}_m(\mathbb{R}) \]

\( \Rightarrow \) \( \mathcal{D}(E) \cong \text{Atiyah algebroid of } \text{Fr}(E) \to M \)
\[ \text{OGL}_m(\mathbb{R}) \]

\[ = T(\text{Fr}(E)) / \text{GL}_m(\mathbb{R}) \]

10) **Pullback of Lie algebroids.** \( A \to M \) Lie algebroid
\( \mu : N \to M \) smooth map
\[ \mu^! A := A \times_{TN} N \]
\[ \text{Assume this is a vector bundle (e.g., } \mu \text{ flat) } \]

- **Anchor:** \( \rho = p_\mu \mu^! A \to TN \)
- **Lie bracket:** \[ \Gamma(\mu^! A) = \{ (\alpha, X) \in \Gamma(A) \times \Gamma(N) : \rho(\alpha) = \mu^! X \} \]

\[ \{ (x_0, X_1), (x_1, X_2) \} = (x_0, [X_1, X_2])_A \]

Extend to any section by requiring Leibniz
Ranks

- The Lie algebroid of $\Omega^1 G$ is isomorphic to $\mu^1 A(G)$.
- Under appropriate conditions, one can restrict $A \to M$ to a submanifold $i:N \subset M$: $A_N := i^* A$.
- If $O \subset M$ is an orbit of a Lie algebroid, one can always restrict $A$ to $O$, resulting in a transitive algebroid:

$$0 \to O \to A_0 \to O \to 0 \quad \omega |_{O} = \bigcup_{\theta \in O} \Omega^1 A(\theta).$$

- An arbitrary Lie algebroid can be thought of as a collection of transitive algebroids parameterized by its leaves.

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Alternative Description of Lie algebroids:

- Any vector bundle: $\Omega^k(A) := \bigwedge^k (\bigwedge^\infty A^* )$ "A-forms"

- $\Omega^2(A) := (\bigoplus_{k \geq 0} \Omega^k(A), \Lambda)$ is a (graded) algebra.

- $\Omega^2(A)$ is generated by $\Omega^0(A) = C^\infty(M) \otimes \Omega^1(A)$:

$$\omega = \sum_{i=1}^n f_i \theta_i \wedge \ldots \wedge \theta_k \quad \forall f_i \in C^\infty(M)$$

- Any vector bundle map: $\Phi^\omega : \Omega^\omega(A_2) \to \Omega^\omega(A_1)$

$$\Phi^\omega (\alpha_1, \ldots, \alpha_k) := \omega (\Phi(\alpha_1), \ldots, \Phi(\alpha_k)).$$

Proposition. Let $A \to M$ be a vector bundle. There is a $1:1$ correspondence

$$\{ \text{Lie algebroid structures on } A \} \leftrightarrow \{ \text{linear operators } d_A : \Omega^1(A) \to \Omega^2(A) \text{ s.t.} \}

\begin{align*}
& (i) \ d_A(\alpha \wedge \beta) = d_A \alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d_A \beta \\
& (ii) \ d_A^2 = 0
\end{align*}$$
Proof

Given a Lie algebra $(A, [\cdot, \cdot], \rho)$ one defines the $A$-differential

$$d_A: \Omega^1(A) \to \Omega^2(A)$$

by:

$$(d_A \omega)(a_0, \ldots, a_n) := \sum_{i=0} (-1)^i \rho(a_i)(\omega(a_0, \ldots, \hat{a}_i, \ldots, a_n)) +$$

$$\sum_{0 \leq i < j < k \leq n} (-1)^{i+j+k} \omega([a_i, a_j], a_0, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_n)$$

Just like the de Rham differential one shows that (i) and (ii) hold.

Conversely, given $d_A$ satisfying (i) and (ii), we define:

$$\rho: \Omega^1(A) \to \mathcal{X}(M), \quad \rho(\omega)(p) := d_A f(\omega) (f \in \mathcal{C}^\infty(M))$$

$$[,] : \Omega^1(A) \times \Omega^1(A) \to \Omega^2(A), \quad \langle [\omega, \theta], \phi \rangle := -d_A \theta(\omega, \phi) - \rho(\omega)(\phi \theta(p)) +$$

$$\rho(\omega)(\theta \phi(p)) (\theta \in \mathcal{C}^\infty(M))$$

Since $\rho(g \omega) = g \rho(\omega), \forall g \in \mathcal{C}^\infty(M)$, we see that $\rho$

is induced by a bundle map $\rho: A \to TM$.

Definition shows that $[\cdot, \cdot]$ is $A$-bilinear, skew-symmetric.

To check Leibniz:

$$\langle [\omega, fp], \theta \rangle = -d_A \theta(\omega, fp) - \rho(\omega)(\theta (fp)) + \rho(fp)(\theta \omega)$$

$$= -d_A \theta(\omega, f \theta(p)) - \rho(\omega)(f \theta(p)) - f \rho(\omega)(\theta(p)) +$$

$$\rho(fp)(\theta \omega) = \langle f [\omega, fp], \theta \rangle$$

Finally, we need to check Jacobi identity. Since Leibniz holds, it is enough to check this on a local chart $(U, x^i)$ over which $A \to M$ has a local basis of sections $\{e_a\}$. Then one has:

$$\rho(e_a) = B^i_a \frac{\partial}{\partial x^i}, \quad \text{with } B^i_a C^l_{\alpha i}$$

$$[e_a, e_b] = C^c_{ab} e_c$$

Thus:
Jacobi id \iff \sum_{a,b,c} \left[ e_a, \left[ e_b, e_c \right] \right] = 0
\iff \sum_{a,b,c} \left[ e_a, C_{bc}^d \right] = 0
\iff \sum_{a,b,c} \left( p \left( \left[ e_a, \left[ e_b, e_c \right] \right] \right) + C_{bc}^d \left[ e_a, e_d \right] \right) = 0
\iff \sum_{a,b,c} \left( B_a^i \frac{\partial C_{bc}^d}{\partial \xi^i} + C_{ae} C_{bc}^d \right) = 0
\tag{i}

Let \theta^a, \theta^b be dual 1-forms: \theta^a \left( e_b \right) = \delta^a_d. Then:

\cdot d_A \theta^c \left( e_a, e_b \right) \Rightarrow d_A \theta^c = -\frac{1}{2} C_{ab}^c \theta^a \wedge \theta^b
\tag{ii}

\cdot \left( i \right) + \left( ii \right) \Rightarrow 0 = d^2 \theta^c = -\frac{1}{2} d_A \left( C^c_{ab} \theta^a \theta^b \right)
\tag{ii}

\iff \left( \theta^a \right) \tag{ii}

Remark. The equation \rho \left[ \left( [x, y] \right) \right] = \left[ \rho(x), \rho(y) \right] in local coordinates and local sections as above is equivalent to:

C_{ab} B^i_c = B^i_d \frac{\partial B^j_b}{\partial \xi^d} - B^j_b \frac{\partial B^i_a}{\partial \xi^d} \tag{**}

Eqs (i) \& (ii) are the structure equations of a Lie algebra. They characterize locally a Lie algebra. They appear in E. Cartan's work on the "Equivalence Problem."
If \( A = A(g) \), then:

\[
\left( \Omega^K(A), d_A \right) \cong \left( \Omega^K_{\text{inv}}(G), d \right) = \text{left-invariant forms on } G
\]

where a left-invariant form \( \omega \) of degree \( K \) is:

(i) \( A \)-filiated form: \( \omega \in \Lambda^K(\text{Ker} \, d) \)

(ii) left-invariant:

\[
\omega_{|G}(d_h \cdot g_v, \ldots, d_h \cdot g_v) = \omega_v(g_v) = \omega_v, \quad v_1, v_2 \in \text{Ker} \, d
\]

Also \( d \) is the foliated de Rham differential:

\[
(d \omega)(x_0, \ldots, x_k) = \sum_i (-1)^i \omega(x_0, \ldots, \hat{x}_i, \ldots, x_k) + \sum_{i<j} (-1)^{i+j} \omega([x_i, x_j], x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, x_k)
\]

In particular, if \( \Phi: G_1 \to G_2 \) is a morphism, then:

\[
\Phi^* : (\Omega^*(G_1), d) \to (\Omega^*(G_2), d), \text{ map of complexes}
\]

\[
\Rightarrow (\Phi^*)^* : (\Omega^*(A_1), d_{A_1}) \to (\Omega^*(A_2), d_{A_2}), \text{ map of complexes}
\]

This suggests:

**Def.** A **Lie algebraic morphism** is a v.b. map

\[
\begin{array}{ccc}
A_1 & \xrightarrow{\Phi} & A_2 \\
\downarrow & & \downarrow \\
M_1 & \xrightarrow{\phi} & M_2
\end{array}
\]

such that the induced pullback map is a map of complexes:

\[
\left( \Phi^* \right)^* : (\Omega^*(A_1), d_{A_1}) \to (\Omega^*(A_2), d_{A_2})
\]

This allows to formally derive subalgebras:

**Def.** A **Lie subalgebra** of \((A, [\cdot, \cdot], \rho_a)\) is a Lie algebra

\((B, [\cdot, \cdot], \rho_B)\) together with an algebraic morphism \( \Phi: B \to A \) which is

an injective immersion.
Exercise: Show that if $\phi : N_1 \to N_2$ is a diffeo, this definition is equivalent to the old one, i.e.,

$$
\Phi^* d\omega = d_{\Lambda_1} \Phi
$$

iff

$$
\Phi^* \Theta_k (\alpha) = d \phi \circ \phi \circ (\alpha)
$$

$$
\Theta_k (\Phi(\alpha), \Phi(\beta)) = [\Phi(\alpha), \Phi(\beta)]
$$

Hint: $d_{\Lambda_1}$ is determined by its action on 0-forms and 1-forms.

Another byproduct of this discussion is:

Def. The cohomology of the complex $(\Omega^*(A), d_A)$ is called the Lie algebroid cohomology of $(A, \mathcal{E}, \mathcal{F}, d_A)$

Examples

1) $A = TM$: $H^*(A) = H^d_{\text{dr}} (M)$

2) $A = \mathcal{G}$: $H^*(A) = H^* (\mathcal{G})$ (Chevalley-Eilenberg cohom)

3) $A = TF$: $H^*(A) = H^* (\mathcal{F})$ (Poincaré cohomology)

Remark:

- Lie algebroid cohomology is often hard to compute.
- One can show that $(\Omega^*(A), d_A)$ is an elliptic complex iff $A$ is a transitive algebroid. Hence, for transitive algebroids over compact $M$, $H^*(A)$ is finite dimensional.

A Lie algebroid can be thought of as a "geometric Lie algebra" or a "generalized tangent bundle". Here is another illustration of this mixed flavor.
**Exponential Map (version 1)**

\[ G \cong M \text{ Lie groupoid} \rightarrow B(G) = \text{group of bisections} \]

\[ A \rightarrow M \text{ Lie algebraoid} \rightarrow \Gamma(A) = \text{Lie algebra of sections} \]

**Lemma.**

*If \( A = A(G) \), for any \( \alpha \in \Gamma(A) \) the left-invariant u.f.\( \tilde{\alpha} \in \mathfrak{X}(G) \) is complete iff \( \rho(\alpha) \in \mathfrak{X}(K) \) is complete.*

**Proof**

\[ \tilde{\alpha} \neq \rho(\alpha) \text{ are s-related:} \]

\[ d_g s(\tilde{\alpha}_g) = d_g s (d_{\mathfrak{g}L(\alpha_{\mathfrak{g}L(g)})}) = d_{\mathfrak{g}L(\alpha_{\mathfrak{g}L(g)})} = \rho(\alpha_{\mathfrak{g}L(g)}) \]

\[ \Rightarrow \quad s \cdot \tilde{\alpha} = \rho(\alpha) \]

\( s \) is a surjective submersion: \( \tilde{\alpha} \) complete \( \Rightarrow \rho(\alpha) \) complete.

Assume \( \rho(\alpha) \) complete. Let \( g: (a, b) \rightarrow G \) be integral curve of \( \tilde{\alpha} \).

Then \( g(t): s(g(t)) \) is integral curve of \( \rho(\alpha) \) and can be extended to all \( t \in \mathbb{R} \). If \( b < +\infty \), let \( h(t) \) be integral curve of \( \tilde{\alpha} \) with \( h(b) = 1_{\mathfrak{g}(b)} \). Then \( h: [b-\varepsilon, b+\varepsilon] \rightarrow G \), for some \( \varepsilon > 0 \) and we can define:

\[ \tilde{g}(t) = \begin{cases} 
   g(t), & t \in (a, b-\varepsilon) \\
   g(b-\varepsilon) h(b-\varepsilon)^{-1} h(t), & t \in (b-\varepsilon, b+\varepsilon) 
\end{cases} \]

This is an integral curve of \( \tilde{\alpha} \) extending \( g(t) \). Hence \( b < +\infty \).

Similar argument for \( a \).
For any Lie algebraoid $\mathfrak{a}$ we call $\mathfrak{p}(\mathfrak{a})$ complete if $\mathfrak{p}(\mathfrak{a})$ is complete. For example, compactly supported sections are complete, so there are many complete sections.

The set $\mathfrak{p}_c(\mathfrak{a}) \subseteq \mathfrak{p}(\mathfrak{a})$, in general, is not a subspace. The set $\mathfrak{p}_c(\mathfrak{a}) \subseteq \mathfrak{p}(\mathfrak{a})$ is a Lie subalgebra.

**Def.** The exponential map of a Lie groupoid $\exp : \mathfrak{p}_c(\mathfrak{a}) \to \mathcal{B}(\mathcal{G})$

$$\exp(\alpha)(x) := \Phi^1_{\alpha}(x)$$

**Rmk.** By our conventions, $\exp(\alpha)$ is a $t$-parameterized bisection: $t \mapsto \exp(\alpha) = \text{id}$, so $\exp(\alpha) : M \to M$ diffeo.

Just like for a Lie group, the map $\mathbb{R} \to \mathcal{B}(\mathcal{G}), t \mapsto \exp(t\alpha)$ is a group homomorphism.