Def. A **Lie algebroid** is a vector bundle $A \to M$ together with:

- A **Lie bracket** $[\cdot, \cdot]$ on $\mathfrak{P}(A)$
- A bundle map $\rho: A \to TM$ covering $\text{id}_M$ satisfying the **Leibniz identity**:

$$[\alpha_2, f \alpha_1] = f [\alpha_2, \alpha_1] + \rho(\alpha_2)(f) \alpha_1, \quad \alpha_1, \alpha_2 \in \mathfrak{P}(A), \ f \in \mathcal{C}^\infty(M)$$

Rem. For $G = M$ we call $A(G)$, defined in previous lectures, the **Lie algebroid** of $G$. When $A = A(G)$ we say that $A$ is **integrable**.

- $\rho$ is called the **anchor** of $A$. Together with Leibniz, it is what makes a Lie algebroid a geometric object.

**Exercise.** Show that for any Lie algebroid $(A, I, \cdot, \mathfrak{J}_\lambda, \rho)$ the induced map $\rho: \mathfrak{P}(A) \to \mathfrak{X}(M)$ preserves brackets:

$$\rho([\alpha_2, f \alpha_1]) = [\rho(\alpha_2), f \rho(\alpha_1)]$$

What about morphisms?

$$G_1 \xrightarrow{\Phi} G_2 \quad \text{morphisms} \quad \left\{ \begin{array}{l}
t \circ \Phi = \Phi \circ t \\
\Phi_!(1_m) = 1_{\Phi(1_m)}
\end{array} \right. \quad \Rightarrow \quad \begin{array}{l}
A_1 \xrightarrow{\Phi_!} A_2 \\
M_1 \xrightarrow{\Phi_!} M_2
\end{array} \quad \text{v.b. map}
$$

$$\Phi \circ \Phi = \Phi \circ \Phi \Rightarrow \quad \rho_2 \circ \Phi_! = d\Phi \circ \rho_1$$

$$\Phi(gh) = \Phi(g) \Phi(h) \Rightarrow \quad \Phi_! \text{ preserves Lie brackets } (*)$$

If $\Phi$ is a **Diffeomorphism**, then $\Phi_! \mathfrak{P}(A_1) \to \mathfrak{P}(A_2), \alpha \mapsto \Phi_! \Phi^{-1} \alpha \mapsto \Phi_! \circ \Phi^{-1} \alpha$

$$(*) \iff \quad \Phi_!( [\alpha_2, f \alpha_1] ) = [\Phi_!(\alpha_2), \Phi_!(f \alpha_1)]$$

**Issue:** In general, there is no map $\Phi_!: \mathfrak{P}(A_1) \to \mathfrak{P}(A_2)$. We will deal with this later.
Examples

1) **Tangent bundle.** For any manifold $M$
   
   - $A = TM$, $\omega, \rho \in \text{id} : TM \to TM$
   
   \[ [\cdot, \cdot] \text{ is usual Lie bracket on } v.f. \]
   
   - It is easy to check that both $M \times M = M \times \Pi, (M) \Rightarrow M$
   
   have Lie algebraic $\subset TM$ (we say **integrate** $A=TM$)
   
   - For any Lie algebraic, The anchor $\rho : A \to TM$
   
   is a Lie algebraic morphism.

Exercise. If $G \to M$ is Lie groupoid show that groupoid anchor $\Phi = (t,s) : G \to M \times M$ is a groupoid morphism that differentiates to $\rho : A(G) \to TM$ (we say that $\Phi$ **integrates**

The morphism $\rho$)

2) **Lie algebras** $\leftrightarrow$ **Lie algebroids** $\Leftrightarrow$ $M = I \times I$

Exercise: For any Lie algebroid $A \to M$, fixing an $\alpha$:

\[ \mathfrak{g}_\alpha (A) := \ker p_\alpha \subset A_\alpha \]

Show that for if $\alpha, \beta \in \mathfrak{g}_\alpha$ then:

\[ [\alpha, \beta] := [\tilde{\alpha}, \tilde{\beta}] \quad (\alpha, \beta) \]

is well-defined, i.e., independent of choice of extensions.

- $\mathfrak{g}_\alpha (A)$ is the isochory Lie algebra of $A$ at $\alpha$:
  
  - $\mathfrak{g}_\alpha (A) \subset A_\alpha$ is a Lie subalgebroid
  
  - If $A = A(G)$, $\mathfrak{g}_\alpha (A)$ is the Lie algebra of $G_\alpha$. 

3) **Bundle of Lie Algebras** \(\iff\) **Lie Algebroids** \(\forall \mathfrak{p} \equiv 0\)

These do not need to be locally trivial:

- \(\mathfrak{p} = \mathbb{R} \times \mathbb{R}^2, \quad \mathfrak{p} \equiv 0\)
  \[
  e_1(t) = (t, (1,0)) \quad e_2(t) = (t, (0,1)) \\
  [e_1, e_2] = t e_1
  \]

- \(\mathfrak{p} = \mathbb{R} \times \mathbb{R}^3, \quad \mathfrak{p} \equiv 0\)
  \[
  e_1, e_2 \equiv 0 \quad [e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_2] = e_1
  \]

- \(\mathfrak{p} \equiv 0\)
  \[
  A_t = \begin{cases} 
  \text{so}(3), & t > 0 \\
  \text{so}(2), & t = 0 \\
  \text{sl}(2), & t < 0 
  \end{cases}
  \]

There is a bundle of Lie groups \(G \xrightarrow{\mathfrak{p}} \mathfrak{p}\) integrating this bundle of Lie algebroids. If we require \(G\) to have connected fibers, \(G\) is **non-Hausdorff**.

4) **Involutive Distributions**. \(D \subset TM\) a vector subbundle such that:

\[X, Y \in \mathfrak{X}(D) \implies [X, Y] \in \mathfrak{X}(D)\]

Then \(A = D\) is Lie algebroid \(\forall\):

- **Anchor**: \(\mathfrak{p} : A \hookrightarrow TM\) (inclusion)
- **Bracket** = usual Lie bracket or v.f.

Recall Frobenius Theorem:

\[
\{\text{involutive distributions}\} \iff \{\text{Foliations } F \}\quad \text{on } M
\]

\[D = TF\]

**Exercise.** \(\Pi, (M, F) \neq \text{Hol}(M, F)\) both have Lie algebroid isomorphic to \(TF\).
**Remark.** For a general Lie algebroid $A \rightarrow M$

$$\mathfrak{g}(A) = \text{Ker} p = \bigcup_{x \in M} \text{Ker} p_x$$

is a bundle of Lie algebras, but not smooth. We call $A$ a **regular Lie algebroid** if $p$ has constant rank so $\mathfrak{g}(A) \rightarrow M$ is bundle of Lie algebras. In this case we have a short exact sequence of Lie algebroids:

$$0 \rightarrow \text{Ker} p \rightarrow A \rightarrow \text{Im} p \rightarrow 0$$

where $\text{Im} p \subset TM$ is an integrable distribution. The foliation $\mathcal{F}$ corresponding to $\text{Im} p$ is called the **orbit foliation** of the regular algebroid $A$.

For general Lie algebroid $p : A \rightarrow M$ one still has an orbit foliation:

- It is the unique partition of $M$ into connected, regular, immersed submanifolds $F_a = \{ O_i : i \in I \}$ with $\mathcal{T} O_i = \text{Im} p_x , \forall x \in O_i$.
- $x, y \in M$ belong to same orbit iff $\exists$ smooth path $a : I_0, I_1 \rightarrow A$ whose base path connects $x \neq y$:

$$\gamma : [0,1] \rightarrow M , \ \gamma(t) = p(a(t)) , \ \gamma(0) = x , \ \gamma(1) = y$$

And:

$$p(a(t)) = \frac{d}{dt} \gamma(t) , \ \forall t \in [0,1].$$

- If $A = A(G)$ for some Lie groupoid $G \rightarrow M$, then

  orbits of $A(G) = \text{connected components or orbits of } G \rightarrow M$

We will not prove this result. See references.

A Lie algebroid is called **transitive** if $\text{Im} p = TM$; sequence becomes:

$$0 \rightarrow \text{Ker} p \rightarrow A \xrightarrow{p} TM \rightarrow 0$$
5) **Atiyah algebroid.** To any principal bundle $\pi : P \rightarrow M$, 

$$TP \rightarrow P \sim A := TP/K \rightarrow M = P/K \sim TM$$

$$d\pi : TP \rightarrow TM \sim \rho : A \rightarrow TM$$

$$P(A) \cong \mathcal{E}(P)^K \xrightarrow{\rho} [\cdot, \cdot] : P(A) \times P(A) \rightarrow P(A)$$

where:

$$\mathcal{E}(P)^K := \{ X \in \mathcal{E}(P) : K \cdot X = X, \forall \kappa \in K \}$$

This is a transitive Lie algebroid. Also:

$$X \in \mathcal{E}(P)^K, \quad d\pi(X) = 0 \iff X : P \rightarrow K, \quad K\text{-equivariant}$$

for adjoint action on $K$

$$\iff \text{sections of adjoint bundle}$$

$$P[K] := (P \times K)/K$$

So the Atiyah algebroid is associated to an exact sequence:

$$0 \rightarrow P[K] \rightarrow TP/K \rightarrow TM \rightarrow 0$$

**Exercise.** Show that the Lie algebroid of the gauge groupoid

$$\tilde{G} = (P \times P)/K \Rightarrow M$$

is the Atiyah algebroid.

Conclude that if a transitive Lie algebroid $A \rightarrow M$ over a connected base is integrable then $A \cong$ Atiyah algebroid of some principal bundle.
6) Prequantization Algebra. Fix $\omega \in \Omega^2_{cl}(M)$.

- $A_\omega := TM \oplus R_M$ (where $R_M := \pi \times M$)
- Anchor: $p : p_{R_2} : A_\omega \to TM$
- Lie bracket on $\mathfrak{p}(A_\omega) = \mathfrak{X}(M) \times C^\infty(M)$

\[
[(X(f), \psi(y)) := ([X, \psi], X(g) - \psi(f) + \omega(x, y))
\]

Transitive algebraoid by Atiyah sequence:

\[
0 \to R_M \to TM \oplus R_M \to TM \to 0
\]

**Prequantization Problem.** Given a closed 2-form $\omega$ is there a principal $\mathbb{S}^1$-bundle $\pi : P \to M$ with connection $\Theta \in \Omega^1(P)$ such that $d\Theta = \pi^*\omega$?

**Exercise.** Show that if $\pi : P \to M$ is such a prequantization bundle, then its gauge groupoid $\text{Gauge}(\pi) \cong M$ has Lie algebroid $A_\omega$:

- $\omega$ prequantizable $\implies$ $A_\omega$ integrable

\[
\begin{align*}
\uparrow & \quad \uparrow \\
\text{Pre}_\mathbb{R}(\omega) \subset \mathbb{R} \text{ discrete} & \quad \text{Pre}_\mathbb{S}(\omega) \subset \mathbb{R} \text{ discrete} \\
\{ \int \omega \in \mathbb{H}_2(M, \mathbb{R}) \} & \quad \Rightarrow \quad \{ \int \omega \in \mathbb{H}_2(M, \mathbb{R}) \}
\end{align*}
\]

**Examples:**
- $M = \mathbb{T}^2 \times \mathbb{T}^2$, $\omega = \rho_\mathbb{S}^2 \omega_1 + \sqrt{2} \rho_\mathbb{S}^2 \omega_2$ with $\omega / M \in \mathbb{S}^2(\mathbb{T}^2)$, $\int \omega \in \mathbb{T}^2$,

\[
\text{Pre}_\mathbb{R}(\omega) = \langle 4, 1, \sqrt{2} \rangle \subset \mathbb{R} \quad \Rightarrow \quad \text{not prequantizable}
\]

\[
\text{Pre}_\mathbb{S}(\omega) = \{ 0, 1 \} \subset \mathbb{R} \quad \Rightarrow \quad A_\omega \text{ is integrable}
\]
7) **Vector fields.** Given \( X \in \mathfrak{X}(M) \)
   
   \[ A = \mathbb{R}_M \rightarrow M \]
   
   **Anchor:** \( \rho (\alpha , \lambda ) := \lambda X_\alpha \)
   
   **Lie Bracket:** \( \Gamma (\mathbb{R}_M) \cong C^\infty (M) \)
   
   \[ [f, g] := f X(g) - g X(f) \]

   *Any Lie algebroid structure on \( \mathbb{R}_M \rightarrow M \) is of this form*

   \( X := \rho (e) \quad e(x, \lambda) = (x, \lambda) \)

   *The flow groupoid \( D(X) \cong M \) has Lie algebra \( A = \mathbb{R}_M \).*

8) **Action algebroids.** \( A : \mathfrak{g} \rightarrow \mathfrak{X}(M) \) infinitesimal action

   \[ A = M \times \mathfrak{g} \rightarrow M \]
   
   **Anchor:** \( \rho : M \times \mathfrak{g} \rightarrow TM, (\alpha, v) \mapsto \alpha(v) \)
   
   **Lie Bracket:** \( \Gamma (\mathfrak{g}) = C^\infty (M, \mathfrak{g}) \)
   
   \[ [f, g](\alpha) = [f(\alpha), g(\alpha)] + (\mathcal{L}_\alpha f)(g) - (\mathcal{L}_g f)(\alpha) \]

   *The Lie algebroid of \( X \in \mathfrak{X}(M) \) is a special case.*

   *The Lie groupoid associated w/ an action \( G \times M \) has Lie algebra \( \mathfrak{g} \) the one associated w/ the corresponding infinitesimal action.*

   *A Lie algebra action does not need to integrate to a Lie group action. But the action algebroid is always integrable!"