I.4. Definition & Examples (cont.)

Last Time: After choosing base point:

Transitive Lie groupoids $\sim$ principal bundles

For any Lie groupoid $G \rightarrow M$ we have isotropy bundle

$$\text{Iso}(G) = \bigcup_{x \in M} G_x$$

But this is not a (smooth) bundle of groups.

**Proposition**

If $G = (P \times P)/K$ is the gauge groupoid of $P \rightarrow M$, then $\text{Iso}(G)$ is an associated bundle for $K \ltimes K$ by conjugation. In particular, for any transitive groupoid $G \rightarrow M$:

(i) Isotropy groups are all isomorphic
(ii) $\text{Iso}(G) \rightarrow M$ is a Lie groupoid

**Proof:** If $G$ is the gauge groupoid:

$$G = (P \times P)/K, \quad \text{Iso}(G) = \bigcup_{x \in M} \{ (p_x, p_x) : \pi(p_x) = \pi(p_x) = x \}$$

$$= \{ (p, pg) : \pi(p) = x, \ g \in K \}$$

$$P \times K \xrightarrow{\Phi} P \times P \xrightarrow{\Phi} (P, pg)$$

$$\downarrow$$

$$G = (P \times P)/K, \quad \text{Iso}(G) \xrightarrow{\Phi} (P, pg)$$

$$\text{(here } K \ltimes K \text{ by inner action)}$$

$$g \cdot K = k' g k$$
is $K$-equivariant:

$$\Phi((p,g)\cdot K) = \Phi(pk, k'^*gk) = (pk, pk^*k'^*gk) = (pk, pgk) = \Phi(p, g)\cdot K$$

$$\Rightarrow \Phi: (P \times K)/K \to (P \times P)/K$$

This is an embedding as image $\text{Iso}(G)$.

10) **General Linear Groupoids.** $\pi: E \to M$ vector bundle

$$GL(E) = \{ (y, A, x) \mid A: E_x \to E_y \text{ linear isomorphism} \}$$

\[\Downarrow\]

$M$

- If $M = \mathbb{R}^n \Rightarrow E = V$ is a vector space $\Rightarrow$ Lie group $GL(V)$
- $GL(E) \simeq M$ is transitive groupoid

**Exercise.** For a vector bundle $E \to M$ one has the bundle of frames $(\mathbb{R} = \text{rank} E)$:

$$\text{Fr}(E) := \{ u: \mathbb{R}^n \to E_x \mid \text{see } M, \text{ linear isomorphism}\}$$

This is a principal $GL_n(\mathbb{R})$-bundle. Show that $GL(E) \simeq M$ is canonically isomorphic to the gauge groupoid of $\text{Fr}(E) \to M$ $\text{GL}_n(\mathbb{R})$
11) Restrictions. If $G \rightarrow M$ is a Lie and $N \subset M$ is a submanifold

$$G \mid _N = \mathcal{E}(N) \cap \mathcal{E}(N)$$

is not Lie in general

Need conditions on $N$, e.g., $(t,s): G \rightarrow M \times N \neq N \times N$. But other conditions work, e.g., $N$ is union of orbits of $G$ (we say $N$ is "saturated"). In particular, for any orbit

$$G \mid _G \rightarrow G$$

is a (transitive) Lie groupoid

**Conclusion:**
A Lie groupoid can be thought of a collection of transitive Lie groupoids (= principal bundles) that are glued nicely.

12) Pullbacks. Restriction is special case of pullback under a map $\varphi: N \rightarrow M$:

$$\varphi^*G := N \times \mathcal{E} \times N = \{ (y,g,x) : \varphi(y) \in \varphi(G) \}$$

is not Lie in general

$\varphi^*G \rightarrow N$ is Lie groupoid whenever $\varphi^*G \cap N \times \mathcal{E} \times N$ is a submanifold.

One has a morphism of Lie groupoids:

$$\varphi^*G \rightarrow G$$

**Exercise:** Show that $\varphi^*G$ is a Lie groupoid whenever $\varphi$ is a submersion.
13) **Čech Groupoids.** \( U = \{ U_i : i \in I \} \) open cover of \( M \)

\[ N := \bigsqcup_{i \in I} U_i \quad \text{(disjoint union)} \rightarrow M \]

\[ \gamma = (M \rightrightarrows M) \text{ identity groupoid (one arrow for each object)} \]

\[ \gamma_U := \varphi^* \gamma = N \quad \text{Čech groupoid} \]

\[ \gamma_U = \bigsqcup_{i,j} U_i \cap U_j = \{(i, e, j) : e \in U_i \cap U_j \} \]

\[ \bigsqcup_i U_i = \{(i, e) : e \in U_i \} \]

**Remark:** If cover is not countable this violates our conventions.

14) **Tangent Groupoids.** For any Lie Groupoid \( \mathcal{G} \) apply \( \text{Tangent Functor:} \)

\[ \text{dim} : \Gamma(\mathcal{G} \times \mathcal{G}) \rightarrow \Gamma \mathcal{G} \]

\[ \Gamma \mathcal{G} \times \Gamma \mathcal{G} \]

\[ \text{di} : \Gamma \mathcal{G} \rightarrow \Gamma \mathcal{G} \quad \text{du} : \text{T} \mathcal{G} \rightarrow \Gamma \mathcal{G} \]

**Remark:** There is also a \( \mathcal{G} \) Groupoid as well as direct sum \( \oplus \Gamma \mathcal{G} \) and \( \oplus \Gamma \mathcal{G} \) (later in course). These will be relevant to understand geometric structures on groupoids.
**Groups vs. Groups:**

**Def:** A bisection of $G = M$ is a submanifold $B \subset G$ such that $s|_B : B \to M$ and $t|_B : B \to M$ are diffeomorphisms.

Equivalently, a bisection is a map $b : M \to G$ such that $s \circ b = \text{id}_M$ and $t \circ b : M \to M$ is a diffeomorphism.

Bisections can be multiplied:

\[
b_1 \circ b_2 (x) := b_1 (t_2 (b_2 (x))) \cdot b_2 (x)
\]

This makes the space $B(G)$ of bisections into a "Lie group". But this is 0-dimensional and can be very wild.

**RMK.** One can also define local bisection:

- A submanifold $B \subset G$ such that $s|_B : B \to U$ and $t|_B : B \to V$ are diffeoonto open sets $U, V \subset M$.

\[
\Leftrightarrow \text{ a map } b : U \to G \text{ such that } s \circ b = \text{id}_U \text{ and } t \circ b : U \to V \text{ is a diffeo}
\]
Proposition:
Every $g \in G$ belongs to the image of some local bisection.

Proof.
Choose a subspace $L \subset T_g G$ complementary to both
$Ker \, d_x s$ and $Ker \, d_x g t$. Choose a submanifold $g \in B \subset G$ with
$T_g B = L$. If $B$ is small enough it is a bisection. \(\blacksquare\)

Proposition
Let $G = M$ be a Lie groupoid.
(i) $\tilde{s}'(\alpha) \cap \tilde{s}'(\gamma)$ are closed embedded submanifolds of $G$
(ii) The isotropy groups $G_x$ are Lie groups
(iii) $T : \tilde{s}'(\alpha) \to O_x$ is a principal $G_x$-bundle
(iv) The orbits $O_x$ are immersed submanifolds in $M$

To prepare for proof and for future use:

**Left translations:**

\[ \begin{array}{ccc}
  s \in G & \xrightarrow{h} & gh \\
  L_g : \tilde{s}'(s) & \to & \tilde{s}'(t(g))
\end{array} \]

is a diffeomorphism with inverse $L_g^{-1}$

**Right translations:**

\[ \begin{array}{ccc}
  s \in G & \xrightarrow{h} & hg \\
  R_g : \tilde{s}'(t(g)) & \to & \tilde{s}'(s)
\end{array} \]

is a diffeomorphism with inverse $R_g^{-1}$

\[ T_g \, \tilde{s}'(s) = K u \, d_x t \quad \text{and} \quad T_g \, \tilde{s}'(g) = K u \, d_x s \]
Proof of Proposition:

Fix $s$-Fiber $S'(s)$:

Claim: $D_s := K_u d_x s n K_u d_x t$ is a (constant rank) distribution on $S'(s)$

Indeed, we have for any $g \in S'(s)$

$$d_x s \circ K_u d_x t : K_u d_x t \to K_u d_x t$$

is isomorphism.

So $L_g \circ s$ in $E'(l)$ \Rightarrow $d_x s \circ L_g(D_{sl}) = D_s$

Hence, picking a basis $\{v_1, ..., v_m\}$ for $D_{sl}$ we obtain a basis of vector fields $\{x_1, ..., x_m\}$ on $S'(s)$ spanning $D$:

$$x; g \mapsto d_x s \circ L_g (v_i)$$

This proves the claim.

$D$ is involutive distribution in $S'(s)$; it coincides with kernel of the differential of smooth map $t : S'(s) \to M$.

Frobenius Theorem $\Rightarrow S'(s) \cap E'(l)$ are submanifolds (= connected components are leaves)

Note: Since source/target fibers are Hausdorff, 2nd countable, we can apply Frobenius. Also, $S'(s) \cap E'(l)$ are closed in $S'(s)$ \Rightarrow leaves are closed, embedded $\Rightarrow S'(s) \cap E'(l)$ are closed embedded submanifolds $\Rightarrow$ (i) holds.

(ii) $G = S'(s) \cap E'(l) \subset G$ are closed, embedded and are groups.

Need to check multiplication/inversion are smooth.
Embedding

\[ G \times G \to G, \quad i : G \to G \text{ are smooth} \]

(iii) \( S^1 \times G \to S^1, \quad (g, h) \to gh \) (right action)

- Free action
- Orbits are fibers of \( t : S^1 \to M \Rightarrow \) action is proper
- Principal right action \( \Rightarrow t : S^1 \to S^1/G \approx O \)

\( O \approx S^1/G \) (bijection)

\[ G \text{ has smooth structure} \]

\[ S^1 \stackrel{t}{\longrightarrow} S^1/G \rightarrow M \]

**Remark:** An immersion \( i : N \to M \) is called **regular** if

For any map \( f : P \to N \) the composition \( f \circ i \) is smooth if \( f \) is smooth.

- Every embedding is a regular immersion
- The irrational line in the torus

\[ \mathbb{R} \to S^1 \times S^1, \quad t \mapsto (e^{it}, e^{it}) \quad (\lambda \neq 0) \]

is a regular immersion which is not an embedding

- A non-regular immersion:
In general, a subset $N \subseteq M$ can have different smooth structures such that inclusion $N \subseteq M$ is immersion.

However, a set $N \subseteq M$ can have at most one smooth str. Such that $N \subseteq M$ is a regular immersion, and that smooth structure is the unique one making the inclusion an immersion.

Alternative notations

Regular immersion = Weakly embedded = Initial submanifold

Exercise (somewhat hard)

The orbits of a Lie groupoid are regularly immersed

Hint: Look at the proof that the leaves of a foliation are regular immersed submanifolds (see, e.g., Warner "Foundations of Differentiable Manifolds and Lie Groups")