Last time:

- $\mathcal{M}$ = category of smooth manifolds

- Fibered category over $\mathcal{M}$: a functor $\pi: G \to \mathcal{M}$ satisfying:
  1. For every $f: X' \to X \in G$ over $X$, pullback exists:
     $$C' = f^* C = C \mid X'$$

  2. Existence of unique lifts:
     $$\exists g \text{ s.t. } \pi(g) = f$$

- Fiber over $X$: $\mathcal{C}_X = \{ (C, g) : \pi(C) = X, \pi(g) = \text{id}_X \}$ (def)

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Fibered categories generalize presheaves

In one direction, given a presheaf (i.e., a contravariant functor):

$$P: \mathcal{M} \to \text{Sets}$$

one defines a fibered category:

$$G := \begin{cases}
\text{Obj} = \{ (X, \alpha) : X \in \mathcal{M}, \alpha \in P(X) \} \\
\text{Arr} = \{ (X, \alpha) \xrightarrow{f} (Y, \gamma) \mid f: X \to Y, P(f)(\gamma) = \alpha \} \\
\end{cases}$$

$$\pi: G \to \mathcal{M} : \text{ forgetful functor}$$

**Rank:** Note that this is a discretely fibered category, i.e., fibers $G_X$ are identity categories (as pullbacks are unique).
In the other direction, given fibered category $\pi: \mathcal{C} \to \mathcal{M}$ make a choice of pullbacks:

1. For each $C \in \mathcal{C}_X$ map $f: X \to X$ choose a lift $\tilde{f}: C' \to C$

With notation $C' = f^* C$, by property (ii), if $C, C' \in \mathcal{C}_X$ and $g: C \to C'$

Then:

$$
\begin{array}{c}
C_1 \\
g
\end{array} 
\begin{array}{c}
f^* C_1 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
C_2 \\
g
\end{array} 
\begin{array}{c}
f^* C_2 \\
\end{array}
$$

We obtain a contravariant map

$$P: \mathcal{M} \to \text{Gpds}$$

where:

- $P(X) = C_X$ (a groupoid)
- $P(f: X_1 \to X_2) = f^*: C_{X_2} \to C_{X_1}$ (a groupoid morphism)

But this is only a pseudo functor: by axiom 2, there is a

$$
(f \circ f')^* \cong f^* \circ f'^*
$$

where $\tau_{f,f}$ is a unique natural transformation satisfying some coherence conditions. One calls $P$ a Lax presheaf of groupoids. One finds that:

- Different choices of pullbacks $\Rightarrow$ equivalent Lax presheaves or groupoids
- Lax presheaf of groupoids $\Rightarrow$ fibered category (similar to above)
- Equivalent Lax presheaves of groupoids $\Rightarrow$ Equivalent fibered categories
  (we derive equivalence later)

**Remark:** A choice of pullbacks for $\pi: \mathcal{C} \to \mathcal{M}$ is also called a cleavage.
Examples

1) The fibered category $\mathcal{M}$ has unique cleavage. The associated presheaf is the strict presheaf:
$$U \mapsto C^0(U, \mathcal{M}) \quad \text{(identity groupoid)}$$

More generally:

strict presheaves of groupoids $\leftrightarrow$ discretely fibered categories (i.e., unique pullbacks)

2) $\pi : \mathcal{B}G \rightarrow \mathcal{M}$ has a natural cleavage: pullbacks of principal $G$-bundles.
The associated lax presheaf of groupoids is
$$X \mapsto \begin{cases} \text{Principal } G \text{-bundles over } \mathcal{M} \mathcal{3} \\ (f : X \rightarrow X_2) \mapsto \hat{f}^* \text{ Pullback of principal } G \text{-bundles} \end{cases}$$

This is not a strict presheaf:
$$(f_1 \circ f_2)^* \neq f_2^* (f_1^* P)$$

3) For any Lie group $G = X$ we can define a strict presheaf of groupoids:
$$U \mapsto \text{Groupoids with} \begin{cases} \text{Objs: } C^0(U, X) \\ \text{Mor: } C^0(U, G) \\ C^0(U, G) \xrightarrow{f^*} C^0(U, G) \end{cases}$$

By correspondences above this defines a discrete fibered category. When $G$ is the identity groupoid $X = \mathcal{X}$ we recover $X$.

Note: In general, this fibered category is $\neq \mathcal{B}G$
Prestacks & Stacks: Descent

So far we have not used covering families, i.e., the Grothendieck topology. These allows us to introduce "gluing axioms" similar to sheaves, and define stacks. Recall:

- A covering family of \( X \) is \( \{ U_i, f_i : X \} \) w/ \( f_i \) étale and \( U_f(U_i) : X \).

Consider a presheaf over \( 
\mathcal{M} \), i.e., a contravariant functor
\[ P : \mathcal{M} \rightarrow \text{Sets} \]

Given \( f : U \rightarrow V \in P(V) \), we use the usual notations:
\[ \alpha |_{U} := P(f)(\alpha) \]

Also, for a covering family \( \{ f_i : U_i \rightarrow X \} \) we set:
\[ U_{ij} := U_i \times _X U_j, \quad U_{ij} := U_i \times _X U_j \times U_k, \text{ etc.} \]

Note that these are all good pullbacks. Recall:

Def:

(i) A presheaf \( P \) over \( 
\mathcal{M} \) is **separated** if
\[ \{ f_i : U_i \rightarrow X \} \text{ covering family } \Rightarrow \alpha = \alpha' \]

\[ \alpha, \alpha' \in P(X), \quad \forall i \colon \alpha |_{U_i} = \alpha' |_{U_i} \]

(ii) A presheaf \( F \) over \( 
\mathcal{M} \) is a **sheaf** if
\[ \{ f_i : U_i \rightarrow X \} \text{ covering family } \Rightarrow \exists \alpha \in P(X) \text{ w/ } \alpha |_{U_i} = \alpha_i \]

Since fibrero categories can be thought as lax presheaves, one can impose similar properties. These lead to prestacks & stacks.
First we define the analog of separated:

**Def.**: A fiber category \( \pi: C \to M \) is called a **prestack** if for some cleavage, for any \( C, C' \in \mathcal{C} \), any covering family \( \{ f_i: U_i \to X \} \) and isomorphisms \( \phi_i: C|_{U_i} \to C'|_{U_i} \),

\[
\phi_i = \phi_0 \circ (\pi|_{C|_{U_i}}) \Rightarrow \exists \text{ isomorphism } \phi: C \to C'.
\]

**Rmk.**: Without a choice of cleavage, the condition is that one can always fill the diagram.

**Examples**

1) For a Lie group \( G \), \( BG \to M \) is a prestack: The condition amounts to the gluing axiom for maps of spaces applied to principal bundles.

Similarly, principal \( G \)-bundles with connection form a prestack \( BG^\nabla \to M \): principal bundle connections can also be glues.

More generally, for any Lie groupoid \(BG \to B\tilde{G}\) are prestacks.

2) If \( \pi: G \to M \) is a discretely fibered category then it is a prestack if and only if the corresponding presheaf \( P: \mathcal{M} \to \text{Sets} \) is separated.

In particular, for any manifold, \( M \) is a prestack and for any Lie groupoid \( G \), \( X \to C^\infty(X,G) \) gives a prestack.
**Def:** A prestack $\pi: C \to \mathcal{M}$ is called a **stack** if for some choice of cleavage, for any $X \in \mathcal{M}$, any covering family $\{f_i: U_i \to X\}$, any $C_i \in C_{U_i}$ and $\phi_{ij}: C_i|_{U_{ij}} \to C_j|_{U_{ij}}$, satisfying the cocycle condition

$$\phi_{kj} \circ \phi_{ij} = \phi_{ki} \quad (\text{in } C_{U_{ijk}})$$

There exists $C \in C_X$ and isomorphisms $\phi_i: C|_{U_i} \to C_i$ such that

$$\phi_{ij} \circ \phi_i = \phi_j \quad (\text{in } C|_{U_{ij}})$$

**Rmk:** Without a choice of cleavage, the condition says that given data:

There is $C \in C_m$ and arrows $C_i \to C$ filling diagram:

**Rmk:** The data $\{C_i, \phi_{ij}\}$ satisfying cocycle condition is call **descent data**. The condition for a prestack to be a stack is called the **descent condition**.
Examples:

1) A discretely fibered category is a stack iff the corresponding prestack is a sheaf. In particular, for any manifold $M$ is a stack.

The fibered category associated to $X \to \mathcal{C}(X, G)$ is not, in general, a stack.

2) For $\pi: BG \to M$ descent data amounts to a collection of principal $G$-bunlles $P_i \to U_i$ and transition functions giving a $G$-cococone. Then there exists a principal $G$-bundle $P \to M$ and isomorphisms $f_i^* P \cong P_i$ compatible with transition maps. Hence, $BG$ is a stack.

**RMK:** Given a pre-stack there is a stackification. If one stackifies the fibered category associated to $X \to \mathcal{C}(X, G)$ one obtains $BG$.

**Maps of Stacks**

**Def.** Let $\pi_1: C_1 \to M$ and $\pi_2: C_2 \to M$ be fibered categories,

i) A map of fibered categories is a functor $\Phi: C_1 \to C_2$ such that $\pi_2 \circ \Phi = \pi_1$.

ii) A 2-isomorphism of fibered categories between $\Phi: C_1 \to C_2$ and $\tilde{\Phi}: C_1 \to C_2$ is a natural isomorphism $\gamma: \Phi = \tilde{\Phi}$ of $\pi(X) = C_X$.

iii) An equivalence of fibered categories is a map $\Phi: C_1 \to C_2$ admitting a quasi-inverse $\tilde{\Phi}: C_2 \to C_1$ (so $\tilde{\Phi} \circ \Phi \cong \text{id}_{C_1}$, $\Phi \circ \tilde{\Phi} \cong \text{id}_{C_2}$).

A map, 2-isomorphism, or equivalence of stacks is just a map, 2-isomorphism or equivalence of the underlying fibered categories.
Hence, fibered categories and stacks are both 2-categories. The later is denoted $\mathbf{St}(\mathbf{M})$.

Properties:

(i) If $\Phi : C_1 \to C_2$ is a map of fibered categories then $\Phi$ is an equivalence iff for every $X \in \mathbf{M}$ the restriction to the fibers $\Phi |_X : C_1 |_X \to C_2 |_X$ is an equivalence of categories.

(ii) If $\Phi : C_1 \to C_2$ is an equivalence of fibered categories then $C_1$ is a prestack (resp. stack) iff $C_2$ is a prestack (resp stack).

Examples:

1) Given $\phi : M \to N$ we obtain a map of stacks:

$$\phi : M \to N \quad \begin{cases} (\ell : X \to M) \mapsto \phi \circ \ell : X \to N \\
X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \quad \xrightarrow{\phi \circ \ell} \\
X_1 \phi \circ f_1 \xrightarrow{X_2 \phi \circ f_2} \end{cases}$$

Every map of stacks $\Phi : M \to N$ is of this form. Note that

- $M$ is discretely fibered \Rightarrow \begin{cases} \text{No non-trivial } 2\text{-isomorphisms} \\
\text{between maps} \end{cases}

So we have a full embedding of 2-categories:

$$\mathbf{M} \to \mathbf{St}(\mathbf{M}) \quad \begin{cases} M \mapsto M \\
\phi \mapsto \phi \end{cases}$$

Remark: One can also show that for any stack $\Pi : G \to \mathbf{M}$ there is a canonical equivalence of groupoids:

$$G_X \simeq \text{Hom}(X, G)$$

This example and Remark formalize the idea that we can understand a (generalized) space by looking at all maps into the space.
2) Let $\Phi: G \to \mathcal{F}$ be a morphism of Lie groupoids. One obtains a map of stacks $\Phi_\ast: B\mathcal{G} \to B\mathcal{F}$ as follows:

$$
\begin{array}{c}
\xymatrix{ G^P \ar[d] \ar[r] & (\mathcal{H} \times P)/\mathcal{G} \ar[d] \\
\mathcal{G} \ar[r] & \mathcal{H} \\
M \ar[r] & N }
\end{array}
$$

$$(f: P_1 \to P_2) \mapsto \Phi_\ast(f)([h, p]) = [h, f(p)]$$

**Exercise:** Show that $\Phi_\ast: B\mathcal{G} \to B\mathcal{F}$ is an equivalence of stacks iff $\Phi: G \to \mathcal{F}$ is a Morita map.

**Def.** Let $\Phi: G_1 \to G_2$ be a map of fibered categories over $\mathcal{M}$

(i) $\Phi$ is a monomorphism if for every $X \in \mathcal{M}$ the restriction to the fibers $\Phi_\ast(G_1)_X \to (G_2)_X$ is fully faithful.

(ii) $\Phi$ is an epimorphism if for every $X \in \mathcal{M}$ and $C_2 \in (G_2)_X$ there is a covering family $\{p_i: U_i \to X\}$ and $C_1 \in (G_1)_U$ such that $\Phi(C_1) \cong C_2 | U_i$.

**Rmk:** If $\Phi_\ast(G_1)_X \to (G_2)_X$ is ess. surjective for every $X \in \mathcal{M}$, then (ii) holds for trivial covering family $\{\text{id}: X \to X\}$. So (ii) is weaker than this condition. Hence, an equivalence of fibered categories is both a monomorphism and an epimorphism but not the converse.

**Exercise:** Let $\phi: M \to N$ be a smooth map. Show that for the map of stacks $\Phi: M \to N$:

(i) $\phi$ is always a monomorphism;

(ii) $\phi$ is epimorphism iff $\phi$ is a surjective submersion.