Principal $G$-bundles

Recall a principal $G$-bundle is given by a $G$-space $P$ with a $G$-invariant submersion $\pi: P \to M$ such that:

\[(x) \quad G \times P \to P \times P, \quad (g, p) \mapsto (gp, p) \quad \text{is a diffeomorphism} \]

Remark: (x) says that the action correspondence $G \times P \to P$ is isomorphic to the submersion correspondence $P \times P \to P$, which is a way of expressing that action is proper and free (since action $G \times P \to P$ is proper and free of isotropy).

Examples

1) For a Lie group $G = \{x\}$ this recovers usual notion of principal $G$-bundle.

2) Any Lie groupoid $G \to M$, left action on itself gives a principal $G$-bundle:

\[
\begin{align*}
G \times M & \to M \\
G & \to M
\end{align*}
\]

3) For a general principal $G$-bundle $P \to X$, each fiber $\tilde{\pi}(x)$ is isomorphic to a source fiber $\tilde{\sigma}(p)$ where $p = \mu(x), \nu \in \tilde{\sigma}$.

\[
\tilde{\sigma}(p) \to \tilde{\pi}(x), \quad g \mapsto g \nu
\]

4) If $G = G \times M \to M$ a principal $G$-bundle is just an ordinary principal $G$-bundle $\pi: P \to X$ together with a $G$-equivariant map $\mu: P \to M$. 

Morphisms:

A morphism or principal $G$-bundle is a map between principal $G$-bundles

$$\Phi : P_1 \to P_2$$

which is $G$-equivariant:

$$\Phi(gu) = g \Phi(u)$$

In particular:

$$P_1 \xrightarrow{\Phi} P_2 \quad P_1 \xrightarrow{\Phi} P_2 \quad \pi_1 \xrightarrow{\Phi} \pi_2 \quad X_1 \xrightarrow{\phi} X_2$$

For a smooth map $\phi : X_1 \to X_2$.

Pullbacks:

- $P \to X$ principal $G$-bundle $\to$ principal $G$-bundle: $\phi^*P \to Y$
- $\phi : Y \to X$

The map $\Phi : \phi^*P \to P$ is a morphism of principal $G$-bundles.

Proposition:

Every morphism of principal $G$-bundles

$$P_1 \xrightarrow{\Phi} P_2 \quad \pi_1 \xrightarrow{\Phi} \pi_2 \quad X_1 \xrightarrow{\phi} X_2$$

induces an isomorphism:

$$P_1 \xrightarrow{\phi^*} \phi^*P_2 \quad \pi_1 \xrightarrow{\phi^*} \pi_2 \quad X_1 \xrightarrow{1 \phi} X_1$$

Proof: A morphism covering $1d$ is an isomorphism.
Local triviality

A principal $G$-bundle $P \xrightarrow{\pi} X$ is trivial if it is isomorphic to pullback of the unit principal $G$-bundle $G \times M$.

Lemma. For a principal $G$-bundle $P \xrightarrow{\pi} X$ the following are equivalent:

(i) $P \xrightarrow{\pi} X$ is trivial

(ii) There exists a morphism $\Phi : P \rightarrow G$

(iii) $P \xrightarrow{\pi} X$ has a section

Proof:

(i) $\Rightarrow$ (ii)

\[
\begin{array}{c}
P \xrightarrow{\Phi} G \\
\downarrow \quad \downarrow \\
X \xrightarrow{id} X \xrightarrow{\pi} M
\end{array}
\]

(i) $\Rightarrow$ (iii)

$\Phi^* G = X \times_G G \\
\text{and} \\
\pi \downarrow \quad \downarrow \\
X \xrightarrow{\Phi^*} G$

(iii) $\Rightarrow$ (i)

For any principal $G$-bundle $\pi : P \rightarrow X$ is a surjective submersion, so it admits local sections.

$\Rightarrow$ principal $G$-bundles are locally trivial

\[
\begin{array}{c}
P \xrightarrow{\pi} X \\
\downarrow \quad \downarrow \\
M \xrightarrow{s^*_a} U_a \\
\Rightarrow \\
P_{|U_a} = \phi^*_a G
\end{array}
\]
**Cocycle description**

Given principal $G$-bundle $\pi: P \to X$ over $X$ by open sets $\{U_a\}$ where there exist local sections $s_a: U_a \to P$

- $\phi_a: \pi \circ s_a: U_a \to M$, $P|_{U_a} \cong \phi_a^* G$

- On $U_{ap} := U_a \cap U_p$:

$$\phi_a^* G|_{U_{ap}} \cong \phi_p^* G|_{U_{ap}}$$

where $g_{pa}: U_{ap} \to G$ are analogs:

$$\phi_a(g) \to \phi_p(g_{pa}(g))$$

- On triple intersections:

$$\theta_{ap}(g_{pa}(g)) = g_{ap}(g) \quad (g \in U_{ap})$$

A $G$-cocycle is a family $(\phi_a, g_{ap})$ with $\phi_a: U_a \to M, g_{ap}: U_{ap} \to G$:

so $g_{pa} = \phi_a$, to $g_{pa} = \phi_p$ (on $U_{ap}$)

$\theta_{ap} g_{pa} = g_{ap} \quad (\phi \in U_{ap})$

Two $G$-cocycles $(\phi_a, g_{ap}) \neq (\tilde{\phi}_a, \tilde{g}_{ap})$ are equivalent if $\exists \lambda_a: U_a \to G$

so $\lambda_a = \phi_a$, to $\lambda_a = \tilde{\phi}_a$ (on $U_a$)

$$\tilde{g}_{pa} = \lambda_p \cdot g_{pa} \cdot \lambda_a^{-1} \quad (\text{on } U_{ap})$$

After refinement, this gives equivalence relation and one finds:

Principal $G$-bundles/iso $\leftrightarrow G$-cocycles/equiv.

**Rem:** One can also describe generalized maps and Morita equivalences using principal $G$-bundles (see Bibliography).
Differentiable Stacks

A differentiable stack is a (very general) notion of singular space, generalizing manifolds.

A differentiable stack is a Morita equivalence class of Lie groupoids. There is a more conceptual way of approaching them based on Grothendieck's philosophy of the "functor of points":

- A manifold $M$ is completely determined, up to canonical isomorphism, by the set of all smooth maps $X \to M$, where $X$ is a manifold. Equivalently, by the set of all smooth maps $\mathbb{R}^n \to M$ ($n=0,1,...$).

Formally, this means replacing $M$ by the representable functor:

$$ M : \text{Manifolds} \to \text{Sets} \quad \begin{cases} M(X) = \{ f : X \to M \} \\ M(X \times X') = \{ f \mapsto f \circ g \} \end{cases} $$

A singular space is a more general "functor" $\text{Manifolds} \to \text{Sets}$ which is not necessarily representable (i.e., equivalent to some $M$). This philosophy is completed by observing that:

Exercise: Show that there is a bijection:

$$ C^\infty(M_1, M_2) \leftrightarrow \text{Nat}(M_1, M_2) $$

(This is a version of Yoneda's Lemma).

We are now going to formalize this and provide the connection with Lie groupoids.
Notation:
- $\mathcal{M}$ is category of $C^\infty$-manifolds & $C^\infty$-maps
- Given $X \in \mathcal{M}$ a covering family of $X$ is any family
  \[ \{ U_i, f_i : X \} \] where $f_i$ are étale & $\bigcup_i f_i(U_i) = X$

Rmk:
1) Covering families define a unique Grothendieck topology on $\mathcal{M}$, called the étale topology. A category equipped with a Grothendieck topology is called a site. In what follows $\mathcal{M}$ can be replaced by any site. One then obtains topological stacks, algebraic stacks, etc., by replacing $\mathcal{M}$ by Top or Sch.

2) $\text{Obj}(\mathcal{M})$ is not a set. One can replace $\mathcal{M}$ by:
- $\mathcal{M}_{\text{emb}}$ = w/ objects embedded submanifolds in some $\mathbb{R}^n$
- $\mathcal{R}$ = w/ objects = \{ $\mathbb{R}^i$, $\mathbb{R}^1$, $\mathbb{R}^2$, $\ldots$ \}
- $\text{Euc} = \text{w/ objects disjoint unions of open subsets in some } \mathbb{R}^n$

Defn: A category fibred in groupoids $\pi : \mathcal{G} \to \mathcal{M}$ is a functor from some category satisfying:

(i) For every $f : X' \to X \in \mathcal{G}$ over $X$, there exists $g : C' \to C$ in $\mathcal{G}$ with $\pi(g) = f$.

\[ \begin{array}{ccc}
\text{C'} & \xrightarrow{\pi} & X' \\
\downarrow g & & \downarrow f \\
\text{C} & \xrightarrow{\pi} & X = \pi(C)
\end{array} \]

(ii) Given a diagram:

\[ \begin{array}{ccc}
C_1 & \xrightarrow{g_1} & C_2 \\
\downarrow g & & \downarrow g_2 \\
C_1' & \xleftarrow{g} & C_2'
\end{array} \quad \Rightarrow \quad \begin{array}{ccc}
X_1 & \xrightarrow{f_1} & X_2 \\
\downarrow f & & \downarrow f_2 \\
X_1' & \xleftarrow{f} & X_2'
\end{array} \]

There is a unique lift $g$. \]
Rules:
- We have not used yet conclusive families (i.e., the Grothendieck topology).
  By (ii) The object \( C \)' in (i) is unique up to a unique isomorphism.
  One calls \( C \)' a pullback of \( C \) via \( f: X \rightarrow X \) and one often writes
  \[ C' \cong C|_{X} = f^{*}C \]
- Fixing \( X \in \mathcal{M} \), we have the fiber over \( X \), which is
  the subcategory \( G_{X} \subset G \) with:
  \[
  \text{Obj} \ (G_{X}) = \{ \ C \in \text{Obj} \ (G) : \pi(C) = X \} \\
  \text{Arr} \ (G_{X}) = \{ \ f \in \text{Arr} \ (G) : \pi(f) = \text{id}_X \}
  \]

Exercise: Using (ii), show that fibers \( G_{X} \) are groupoids, i.e., every arrow in \( G_{X} \) has an inverse.

Abreviation: Tiberia category = category fiber over in groupoids

Examples:
1) Fix \( M \in \mathcal{M} \). Let \( G = \mathcal{M} \) be the category

\[
\text{Obj} \ (\mathcal{M}) = \{ \ f: X \rightarrow M \} \\
\text{Arr} \ (\mathcal{M}) = \{ \ \begin{array}{c}
X_1 \xymatrix{ & X \ar[d]_{f_1} \ar[r]^{f_2} & M \ar[l]_f \\
X_2 \ar@{.>}[ur]_{g} & 
\end{array} \}
\]

It is a fibered category for the forgetful functor:

\[
\pi: \mathcal{M} \rightarrow \mathcal{M} \times \mathcal{M}
\]
Both axioms hold:

1) Given \( (g: X' \to X) \in \text{Ar} (M) \) \& \( (f_2: X \to M) \in \text{Obj} (M) \)

An object over \( M \), we have the pullback:

\[
\begin{array}{ccc}
X' & \xrightarrow{s} & X \\
\downarrow & & \downarrow \\
M & \xrightarrow{\pi} & X \\
\end{array}
\]

\[ g \circ f_2 \]

In this example:

pullbacks are unique on fibres and identity objects

**Def.** A **discrete fibre category** over \( M \) is a fibre category \( \pi: G \to M \) such that \( G_x \) is an identity category for all \( x \in M \).

2) Let \( G \) be a Lie group and \( G = BG \) be the category:

\[ \text{Obj} (BG) = \text{principal } G \text{-bundles; } p: P \to X \]

\[ \text{Ar} (BG) = \text{morphisms of principal } G \text{-bundles} \]

\[ P_1 \xrightarrow{\gamma} P_2 \]

\[ P_1 \downarrow \quad \downarrow P_2 \]

\[ X_1 \xrightarrow{g_1} X_2 \]

The forgetful functor \( \pi: BG \to M \) is a fibre category.

One checks that (i) and (ii) hold. Note that pullbacks are not unique, there are only unique up to a unique isomorphism.

**Exercise:** Show that principal \( G \)-bundles in connection also give a fibre category \( \pi: BG \to M \).
3) More generally, any Lie groupoid $G \rightrightarrows M$ defines a fibered category
$$\pi : BG \to M$$
with:
$$\text{Obj} (BG) = \{ \text{principal } G\text{-bundles } \xrightarrow{\pi} X \}_{M}$$
$$\text{Arr} (BG) = \{ \text{morphisms of principal } G\text{-bundles } \}_{\pi}$$
$\pi$ = Forgetful function: $\pi (P) = X$

3) Let $G = \mathcal{F}_G$ be the category:
$$\text{Obj} (\mathcal{F}_G) = \{ \text{fiber bundles } p : E \to X \text{ by fiber a Riemann surface }$$
$$\text{of genus } g \text{ with complex structure smoothly varying on fibers} \}$$
$$\text{Arr} (\mathcal{F}_G) = \{ \text{commutative diagrams} \}_{\mathcal{F}_G}$$

$$\begin{array}{ccc}
E_1 & \longrightarrow & E_2 \\
\downarrow{P_1} & & \downarrow{P_2} \\
X_1 & \longrightarrow & X_2
\end{array}$$
with:
$$E_1 \longrightarrow X_1 \times E_2$$
$$X_1 \longrightarrow X_2$$
a conformal isomorphism.

The Forgetful function $\pi : \mathcal{F}_G \to M$ is a fibered category.

Remark: Often fibred categories arise as in previous example from moduli problems. Then one thinks of the fibred category $\pi : G \rightrightarrows M$ as:
- An object in $G$ over $M$ is a $G$-family parameterized by $M$.
- Aim is to classify all objects over $pt \in M$. 