Back to orbifolds

Given an orbifold \( X \),

To each orbifold atlas \( U = \{ (U_i, G_i, \phi_i) : i \in I \} \)
\[
U = \bigsqcup_{i \in I} U_i, \quad \phi = \{ \phi_i \} : U \to X
\]
\[
\overline{U}(U) = \{ \psi \in \text{Diff}_{\text{loc}}(U) : \phi \circ \psi = \phi \mid_{\text{Dom}(\psi)} \}
\]
This is a pseudo-group over \( U \). So we have effective etale groupoids:
\[
\Gamma(U) \equiv \Gamma(\overline{U}(U)) = U
\]

Exercise: If \( M \) is a manifold and \( U \) is an atlas, show that \( \Gamma(U) \) is the cover groupoid.

Proposition: For any orbifold atlas \( U \) of \( X \):

(i) \( \Gamma(U) \) is a proper, effective, etale groupoid.

(ii) If \( U' \) is another orbifold atlas of \( X \), then \( \Gamma(U) \) and \( \Gamma(U') \) are Morita equivalent.

Proof:

(i) Let \( (p, q) \in U_i \times U_j \subset U \times U \). We claim that:

- \( \exists K \ni (p, q) \) compact neighborhood \( u_j \mid U_j \subset U \) compact.

This implies that \( U_j \mid U_j \subset U \times U \) is proper.

- \( \phi_j(p) \neq \phi_i(q) \) this is clear since \( X \) is locally compact Hausdorff and \( G_i \) are finite.
If \( \phi_i(p) = \phi_j(q) = \infty \). By compatibility of charts, there properties, one can find open \( p \in V \in U \); \( \phi \) embeddings of orbifold charts \( \lambda: (V, (G_i)_{V}, \phi_i|_V) \to (U, (G_0), (G_0)) \) so that \( \phi \circ \lambda = \phi_i \mid V \), \( \lambda(p) = q \). Note that then \( \lambda \in \Phi(U) \).

We may assume that \( (G_i)_{V} \to (G_i)_p \) by eventually shrinking \( V \). Then \( \lambda(V) \) is \( (G_{ij})_{\lambda(V)} = (G_{ij})_{\lambda(V)} \)-stable, and it follows from properties of charts:

\[
(\text{Eq. 3}) \quad (\lambda(V) \times V) = \{ \gamma \in (G_{ij})_{\lambda(V)} : \gamma \in (G_{ij})_{\lambda(V)} \times \gamma V \}
\]

Since \( G_{ij} \) are finite, claim follows.

(i) If \( U \not\sim V \) are equivalent atlases, so \( U \cup V \) is also an atlas. Then there are groupoid morphisms:

\[
\begin{align*}
\Gamma(U) & \longrightarrow \Gamma(U \cup V) \\
\Gamma(V) & \longrightarrow \Gamma(U \cup V)
\end{align*}
\]

These are Morita maps since they preserve 1-data.

(ii) On the other hand:

Proposition For any proper, effective, groupoid \( G = \mathcal{M} \) there is a canonical orbifold structure on \( X = \mathcal{M}/G \) such that for any orbifold atlas \( U \), \( \Gamma(U) \not\sim G \) are Morita equivalent.

Proof:

For each \( p \in \mathcal{M} \) one can find a saturated neighborhood \( U_p \subset \mathcal{M} \\) and a \( G_p \)-invariant neighborhood \( \sigma \in V_p \subset T_p \mathcal{M} \) such that:

\[
G \mid_{U_p} = G_p \times V_p
\]
Then the collection \((V_p, \phi_p, \phi_p)\) with \(\phi_p : V_p \rightarrow U_p \rightarrow M/G\)
is an orbifold atlas \(\mathcal{U}\) for \(M/G\).

Notice that:
\[
\mathcal{U} = \bigcup_{p \in \mathcal{G}} U_p, \quad i : U \rightarrow M, \quad \mathcal{G}_U : i^* \mathcal{G} \rightarrow \mathcal{G}
\]
is a Morita map. On the other hand:
\[
\mathcal{G}_U \rightarrow \mathcal{G}(U), \quad g \mapsto \text{gen}(g, b), \quad b \text{ local direction}
\]
is also a Morita map, so:
\[
\mathcal{G} \cong \mathcal{G}(U).
\]

\begin{proof}

\(\text{Thm.}\) For any Lie groupoid \(\mathcal{G}\) the following are equivalent:

(i) \(\mathcal{G}\) is Morita equiv. to a proper effective étale copa

(ii) \(\mathcal{G}\) is Morita equiv. to a groupoid associated with an orbifold atlas

(iii) \(\mathcal{G}\) is Morita equiv. to the holonomy copa of a foliation with compact leaves with finite holonomy

(iv) \(\mathcal{G}\) is Morita equiv. to the action copa of an proper effective Lie group action with finite isotropy

\end{proof}

\begin{proof}

(i) \(\Rightarrow\) (ii): This follows from previous props.

(iii) \(\Rightarrow\) (i): \(\text{Hol}(\mathcal{G}, \mathcal{F}) = \mathcal{G} \ltimes_{\mathcal{M}} \text{Hol}(\mathcal{G}, \mathcal{F})|_{T} = T\) for a complete, transversal \(T\), later copa satisfies (i)

(iv) \(\Rightarrow\) (i): If \(\mathcal{G} \ltimes \mathcal{M}\) is proper, effective, with finite isotropy. Then connected components of orbits form a foliation. If \(T\) is complete transversal, then \(\mathcal{G} \ltimes \mathcal{M} \ltimes (\mathcal{G} \ltimes \mathcal{M})|_{T} = T\), later copa satisfies (i).

\end{proof}
(iii) $\Rightarrow$ (iv): We saw that we can find compact, connected Lie group action $K \times N$, which is effective and has finite isotropy, so that $X = M/K$. The orbifold $\overline{K \times N} = M$ is Morita Equiv. to $\mathfrak{P}(U)$ for an orbifold atlas $U$ and satisfies (iv). The orbits of $K$ form a foliation $(M, \mathcal{F})$ and $\text{Hol}(M, \mathcal{F}) = K \times N$, so (iii) also holds. 

This all discussion suggests:

Def: Let $X$ be a top. space.

(i) An orbifold atlas for $X$ is a pair $(G, \phi)$ where $G$ is a proper Lie groupoid with finite isotropy and $\phi: M \to X$ is a map inducing a homeomorphism $M/G \cong X$.

(ii) Two orbifold atlas $(G_1, \phi_1)$ and $(G_2, \phi_2)$ are equivalent if there exists a Morita equivalence giving comm. diagram

\[
\begin{array}{ccc}
G_1 & \xrightarrow{\phi_1} & M/G_1 \\
\downarrow & & \downarrow \phi_1 \\
X & \xrightarrow{=} & X
\end{array}
\]

(iii) An orbifold structure on $X$ is an equivalence class of orbifold atlases.

Recall that:

$G$ proper & finite isotropy $\Rightarrow G$ Morita equivalent to proper, étale orb.

Def: An effective orbifold $X$ is an orbifold structure on $X$ admitting an atlas $(G, \phi)$ where $G$ is proper, effective, étale orb.
Note that:

- Effective orbifolds are classical orbifolds with classical atlas.
- Orbifolds cannot be defined using classical atlas (properties of atlas collapse without effective assumption).

**Prop:** Any orbifold structure on $X$ has an underlying classical (effective) structure.

**Proof:**

$G$ proper étale $\Rightarrow$ Eff$(G)$ proper, effective, étale.

**Example:**

Let $G \times M \to M$ be an action of a finite group (possibly ineffective). Then $\tilde{G} = G \times M \to M$ defines an orbifold structure on $M / G$. By factoring the kernel $K$ of the action:

$$K = \{ g \in G : g \cdot p = p, \forall p \in M \}$$

we obtain an eff. action $G / K \times M$, and $G / K \times M \to M$ is the classical orbifold structure on:

$$M / G = M / (G / K).$$

An extreme case is when $K = G$, i.e. a trivial action.

The underlying classical orbifold is a manifold.

- What do we gain with the Groupoid approach to orbifolds?

1) It solves some issues
2) It is conceptually simpler
3) It extends to even more singular spaces
1) Let $X$ be an orbifold. What is a suborbifold $Y \subset X$?

Classically, there are problems. Take $\mathbb{Z}_2 \ltimes \mathbb{R}^2$, $(x,y) \to (x,y)$

$= \ X = \mathbb{R}^2/\mathbb{Z}_2$

Is the set $\mathbb{R}^2/\mathbb{Z}_2$ a suborbifold?

- We can think of it as a 1-dim manifold (=orbifold with no isotropy). But what happens to isotropy groups?

- We can think of it as a 1-dim orbifold with isotropy $\mathbb{Z}_2$ at every point. Not an effective orbifold.

- Now-classically:
  - $G = \mathcal{M}$ an orbifold atlas for $X$.
  - $N \subset \mathcal{M}$ closed submanifold such that $G|_N = N$ is a suborbifold of $G = \mathcal{M}$.

$\Rightarrow \ Y = N/G_N \hookrightarrow X = \mathcal{M}/G_M$ is suborbifold.

In example: $G = \mathbb{Z}_2 \ltimes \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$\begin{align*}
\{1\} = \mathbb{Z}_2 \times \{0\} \subset \mathbb{R}^2 \\
\{\lambda\} = \mathbb{Z}_2 \times \{\lambda\} \subset \mathbb{R}^2
\end{align*}$

2) Homotopy groups of an orbifold:

- $\mathcal{G} = \mathcal{M}$ a groupoid representing $X$.

- Nerve of $\mathcal{G}$: is the simplicial manifold:

\[
\mathcal{G}^{(0)} \xrightarrow{\partial_0} \mathcal{G} \xrightarrow{\partial_1} \mathcal{G} \xrightarrow{\partial_2} \cdots \mathcal{G}^{(n)} = \prod_{i=0}^{n-1} \mathcal{G}^{(i)}
\]
Face maps: \( d_i : G^{(i)} \rightarrow G^{(i+1)} (i = 0, \ldots, m) \)

\[
d_i (g_0, \ldots, g_m) = \begin{cases} 
(g_1, \ldots, g_m), & i = 0 \\
(g_0, g_1, \ldots, g_{m-1}, g_m), & 1 \leq i \leq m-1 \\
(g_0, \ldots, g_{m-1}), & i = m
\end{cases}
\]

Degeneracies: \( s_i : G^{(i)} \rightarrow G^{(i+1)} (i = 1, \ldots, m+1) \)

\[
s_i (g_0, \ldots, g_m) = (g_0, \ldots, g_{i-1}, 1, g_i, \ldots, g_m)
\]

As for any simplicial set we have its (fat) geometric realisation,

\[
||G|| := \left( \bigsqcup \mathbb{R}^n \right) / \sim \quad (\text{with quotient topology})
\]

where:

- \( \Delta_m = \{(t_0, \ldots, t_m) : t_i \geq 0, \sum_{i=0}^m t_i = 1\} \)
- \( \partial_i : \Delta_{m-1} \rightarrow \Delta_m \quad (i = 0, \ldots, m) \) face maps:

\[
\partial_i (t_0, \ldots, t_{m-1}) = (t_0, \ldots, t_{i-1}, 0, t_{i}, \ldots, t_{m-1})
\]

\( \sim \) is equivalence relation generated by:

\[
(cl_i (g), t) \sim (g, \partial_i (t))
\]

If \( \phi : G_1 \rightarrow G_2 \) is a monita map one gets a simplicial map \( \phi' : G' \rightarrow G'' \) and hence a continuous map

\[
||\phi|| : ||G_1|| \rightarrow ||G_2||
\]

One can show that this gives an isomorphism of homotopy groups.

**Thm:** If \( G_1 \) and \( G_2 \) are monita equivalent then \( ||G_1|| \) and \( ||G2|| \) are weak homotopy equivalent.

One can use \( G' \neq ||G|| \) to attach geom. \& topological invariants to the orbifold \( X \) represented by the atlas \( (G, \phi) \):
1) The orbifold homotopy groups:
   \[ \pi_n^{orb}(X,x) := \pi_n(\|G\|_1, 101) \]

2) For any ring \( R \), the singular cohomology of \( X \):
   \[ H^n(X, R) := H^n(\|G\|_1, R) \]

3) The de Rham cohomology of \( X \):
   - \( \Omega^n(X) := \{ \omega \in \Omega^n(M) : \delta \omega - t^* \omega = 0 \} \)
   - \( d : \Omega^n(X) \rightarrow \Omega^{n+1}(X) \)

   **De Rham Theorem:**
   \[ H^i(\Omega(X), d) \cong H^i_0(X, R) \]

4) Riemannian Metric on \( X \):
   - \( g \) : metric on \( G \) making \( \rho \) \& \( \tau \) Riem. sub
   - \( i : G \rightarrow G \) an isometry.

   There is a Gauss-Bonnet Theorem, etc.

**Example:**

Can use \( \pi^0_{orb} \) to find obstacles to be a global quotient:

**Prop:** If \( \pi^0_{orb}(X) \neq 1 \), then \( X \) is not a global quotient.

**Sketch of proof:**

\( G \subset M \) is effective, proper action on finite cohomology, \( X = M/G \)

Then \( \exists \) long exact seq in homotopy,

\[ \cdots \rightarrow \pi_n(G) \rightarrow \pi_n(M) \rightarrow \pi_n^0(X) \rightarrow \pi_{n-1}(G) \rightarrow \cdots \]

When $G$ is finite this gives:

\[
\begin{align*}
\pi^\text{orb}_m(X) &\sim \pi_m(\mathcal{M}), \quad m \geq 2 \\
1 &\rightarrow \pi_1(\mathcal{M}) \rightarrow \pi^\text{orb}_1(X) \rightarrow G \rightarrow 1
\end{align*}
\]

But if $X$ is not smooth, then $G \neq 1$, so $\pi^\text{orb}_1(X) \neq 1$.

**Corollary**

If $\pi_G \mathcal{N}$ is a proper effective action with finite isotropy on a 1-connected manifold $\mathcal{N}$, then $X = \mathcal{N}/\pi_G$ is not a global quotient.

**Proof:**

The long exact sequence (1) gives $\pi^\text{orb}_1(X) = 1$.

**Tear Drop:**

$S^1 \times S^2 \ni (\theta, \omega) \in \{ (z, \omega) \in \mathbb{C}^2 : |z|^2 + |\omega|^2 = 1 \} \Rightarrow X = S^3/\mathbb{Z}_m$ not global quotient.

\[
\begin{align*}
\theta \cdot (z, \omega) &= (e^{i\theta} z, e^{i\theta} \omega) \
\mathbb{Z}_m &
\end{align*}
\]