Last time: Main orbifolds

\( X \) is topological space, 2nd countable, Hausdorff, with orbifold atlas, i.e., family of pairwise compatible orbifold charts covering \( X \)

- **Orbifold charts** \( (U, G, \phi) \):
  
  \[ U \subset \mathbb{R}^m \text{ connected } \# \text{ open, } G \subset \text{Diff}(U) \text{ finite} \]

  \[ \begin{array}{ccc}
    U & \xrightarrow{\phi} & X \\
    \downarrow & & \downarrow \\
    U/G & \xrightarrow{\hat{\phi}} & X
  \end{array} \]

  \( \hat{\phi} \) open embedding

- **Embedding of orbifold charts** \( \lambda: (U, G, \phi) \to (V, H, \psi) \)
  
  \[ \lambda: U \to V \text{ embedding s.t. } \begin{array}{ccc}
    U & \xrightarrow{\phi} & X \\
    \downarrow & & \downarrow \\
    V & \xrightarrow{\psi} & X
  \end{array} \]

- **Compatible orbifold charts** \( (U_1, G_1, \phi_1) \neq (U_2, G_2, \phi_2) \)
  
  \( \forall x \in \phi_1(U_1) \cap \phi_2(U_2) \), \( \exists \) chart \( (V, H, \psi) \) with \( x \in \psi(V) \) and embeddings \( \lambda_i: (V, H, \psi) \to (U_i, G_i, \phi_i) \)

Given \( x \in X \) and chart \( (U, G, \phi) \) with \( \phi(p) = x \):

- Faithful rep: \( G_p \to \text{GL}(m, \mathbb{R}), \quad g \mapsto d_p \lambda_g \)

- \( x \in X \) is called singular if \( G_p \neq 1 \)

- \( \text{Isq}_x(X) = \) isotropy type = conjugacy class of image of \( G_p \) in \( \text{GL}(m, \mathbb{R}) \)

\[ \Sigma X \text{ is singular locus } = \{ x \in X : \text{Isq}_x(X) \neq 1 \} \]
Crash Course on proper actions

\[ G \times M \text{ action fixing } \varphi_0 \in M \Rightarrow \begin{cases} G \times T_{\varphi_0} M \text{ (linear action)} \\ g \cdot v = d_{\varphi_0} \lambda_g(v) \end{cases} \]

**Bochner Linearization Theorem**

If \( G \) is compact Lie group acting on \( M \) at fixed pt \( \varphi_0 \in M \),

then action can be linearized around \( \varphi_0 \): \( \exists \) open, \( G \)-invariant \( \varphi_0 \in U \subset M \), \( \forall v \in T_{\varphi_0}M \) and a \( G \)-invariant diffeo \( \phi : U \to V \), \( \phi|_{\varphi_0} = \varphi_0 \).

**Idea of Proof:** (Dixmier-Kolk)

1) Every neighborhood \( \widetilde{U} \ni \varphi_0 \) contains a \( G \)-invariant neighborhood \( U \).

2) Choose embedding \( \phi : U \to T_{\varphi_0}M \), \( \phi(\varphi_0) = 0 \), \( d_{\varphi_0} \phi = \text{Id} \).

Define "average":

\[ \overline{\phi}(x) := \int \limits_{g \in G} d_{\varphi_0} \lambda_g(\phi(\lambda^{-1}_g(x))) \, \, d\mu_G \]

where \( \mu_G \) is left-invariant volume form on \( G \) (Haar measure). Then \( \overline{\phi} : U \to T_{\varphi_0}M \) is \( G \)-equivariant:

\[ \overline{\phi}(\lambda_g(x)) = d_{\varphi_0} \lambda_g(\overline{\phi}(x)) \]

3) At \( \varphi_0 \):

\[ d_{\varphi_0}(d_{\varphi_0} \lambda_g \circ \phi \circ \lambda_g^{-1}) = d_{\varphi_0} \lambda_g \circ d_{\varphi_0} \lambda_g^{-1} = \text{Id} \]

\[ \Rightarrow d_{\varphi_0}(\overline{\phi}) = \text{Id} \]

So there exists \( G \)-invariant neighborhood where \( \overline{\phi} \) is diffeo.

**If** \( G \) being compact, we can choose a \( G \)-invariant inner product \( (\cdot,\cdot)_G \) on \( T_{\varphi_0}M \): if \( (\cdot,\cdot) \) is any inner product on \( T_{\varphi_0}M \):

\[ (v,w)_G = \int \limits_{g \in G} (g v, gw) \, \, d\mu \]
In particular, if $GGM$ effective action of compact Lie group $\omega_1$ fixed point, so embedding $G \subset O(m,\mathbb{R})$

**Remark:** Compactness of $G$ is crucial; choose diffeo $\phi : \mathbb{R} \to \mathbb{R}$

such that $\phi(m) = x$, $x \in \mathbb{R}$, $\phi'(x) < 1$, $\phi(n) > 0$. Then

$G \subset \mathbb{R}$, $m \cdot x := \phi^{-1}(x)$. Not linearizable

Recall $GGM$ is proper in $G \times M \to M \times M$, $(g, p) \mapsto (g \cdot p, p)$ is a proper map. This implies:

- $G_p \subset G$ are compact subloops
- $O_p \subset M$ are embedded submanifolds
- $M/G$ is Hausdorff

**Examples:**

1. Actions of finite groups, more generally compact groups
2. $GL(m,\mathbb{R}) \times \mathbb{R}^m$ is not proper (isotropy group at origin is not compact)

**Local Linear Model:** around orbit $g \cdot o \in M/G$

$M^{lo}_{p} := (G \times V_p(o_{p}))/G_p$

where:

- $G_p \subset G, V_p(o_p) = T_pM/T_pO$
- $g \cdot [v] := [\partial_p \lambda_g(v)]$ well-defined since $\lambda_g(G_p) \subset O_p$

We have $G$-action on local model:

$G \cdot M^{lo}_{g}, g \cdot [g', v] := [g'g, v]$
Thm. If \( G \times M \rightarrow M \) is a proper action, for any orbit \( O_p \) there exists a saturated open set \( U \subset O_p \subset M \) and a \( G \)-equivariant diffeomorphism \( \varphi : U \rightarrow M^{\text{loc}}_{O_p} \) with \( \varphi(p) = [e, O_p] \).

Idea of Proof: (Doucet-Oemra & Kolk)

1) 3 slices through \( p \), i.e., \( G \)-invariant submanifolds \( S \subset M \) with \( \text{dim} S = \text{codim} O_p \) s.t.:
   - \( T_q M = T_o q + T_q S \), \( \forall q \in S \);
   - \( g \in G, q \in S \), \( gq \in S \Rightarrow g \in G_p \)

2) Given slice \( S \), we have embedding
   \[
   (G \times S)/G_p : \rightarrow M, \ [g, s] \rightarrow gs
   \]
   This is \( G \)-equivariant for \( G \)-action:
   \[
   g \cdot [g', s] = [gg', s]
   \]

3) Apply Bochner to \( G_p GS \):
   \( S \sim T_p S \sim U_p (O_p) \)

Exercise: Show that if \( \{ U_i \}_{i \in I} \) is a \( G \)-invariant open cover of \( M \) there exist a partition of unity \( \{ \varphi_i \}_{i \in I} \) consisting of \( G \)-invariant functions subordinated to this cover.
Remark: Defining $f: M/\mathcal{G} \to \mathbb{R}$ smooth if for $M \to \mathbb{R}$, this exercise says that $M/\mathcal{G}$ admits smooth partitions of unity.

**Corollary** Given a proper action $G \times M$ there exists a Riemannian metric $g$ on $M$ which is $G$-invariant.

**Sketch of proof:**

1) On $G$ there is left-invariant Riemannian metric $g^L$ invariant also under right $G_p$-action:
   - choose any left-invariant metric $g$ average over $G_p$ ($G_p$ is compact)

2) On $V_p(G_p)$ choose $G_p$-invariant inner product $(\cdot, \cdot)_g$

3) $g^L \times (\cdot, \cdot)_g$ metric on $G \times V_p(G_p)$ \\
    $\Rightarrow$ $G$-invariant metric on $M^p$

4) Use $G$-invariant partition of unity to glue $G$-invariant metrics on local models.

**Remark:** Conversely, given a Riemannian manifold $(M, g)$ the group of isometries $\text{Iso}(M, g)$ is a Lie group which acts properly on $M$. Hence, if $G \times M \to M$ is an action by isometries which is effective then $G$ acts properly on $M$.

(Effective) proper actions $\leftrightarrow$ actions by isometries
More Examples of Orbifolds

Proper actions w/ finite isotropy: \( G \times M \to M \). Then \( X = M/G \) has a natural orbifold structure of \( \dim M - \dim G \).

- By factoring kernel of action \( \Rightarrow \) assume action is effective \( \Rightarrow G_p \in \mathcal{U}(G_p) \) is effective.

- Local normal form \( \Rightarrow \forall G_p \in M/G \) has neighborhood \( (U_p(G_p), G_p, \phi_p) \) where

\[
\phi_p : U_p(G_p) \to M/G, \quad \nu \mapsto \pi(\phi(\nu, e. \nu))
\]

where \( \pi : M \to M/G \) \& \( \phi : U \to M^\times \).

Exercise: These charts are pairwise compatible so they give an orbifold atlas for \( M/G \).

We will see that the converse holds:

Thm: Every orbifold \( X \) is isomorphic to \( M/G \) where \( G \) is compact Lie group acting on \( M \) with finite isotropy groups.

Exercise: Let \( G \in M \) be an effective action of a compact, connected Lie group with finite isotropy groups. Show that the orbits form a foliation \( F \) of \( M \); codim \( F = \dim G \) whose leaves are compact with finite holonomy.

Hint: Work on local model.
Foliation \((M, \mathcal{F})\) with finite holonomy & compact leaves:

The leaf space \(X = M/\mathcal{F}\) has a natural orbifold structure of \(\text{clim} \ X = \text{codim} \ \mathcal{F}\).

Local model around a leaf \((M^{\text{lin}}, \mathcal{F}^{\text{lin}})\):

- \(L_0\) leaf of \((M, \mathcal{F})\) with holonomy group \(G^{\mathcal{F}_0} = \text{Hol}(L_0)\)
- \(\tilde{L}_0 \to L_0\) holonomy cover: \(\tilde{L}_0 = L_0/K\) where
  \[ G^{\mathcal{F}_0} \to K = K_{\mathcal{F}_0}(\text{Hol}_{\mathcal{F}_0} : \pi_1(L, \mathcal{F}_0) \to \text{Hol}_{\mathcal{F}_0}(L_0)) \]
- \(G^{\mathcal{F}_0} G \subset U_{\mathcal{F}_0}(L_0) : \delta : (v) = [d_{\mathcal{F}_0} \text{hol}(v)]\)

Then

\[ M^{\text{lin}} = (\tilde{L}_0 \times U_{\mathcal{F}_0}(L_0))/G^{\mathcal{F}_0}, \quad \mathcal{F}^{\text{lin}} = \text{pr}(\tilde{L}_0 \times \text{lin}) \]

Reeb Stability Theorem

Let \(L_0\) be a compact leaf of \((M, \mathcal{F})\) with finite holonomy group. There exists a saturated neighborhood \(L_0 \subset U_{\mathcal{F}_0}\) and an diffeomorphism \(\phi : (U, \mathcal{F}_0|_U) \to (M^{\text{lin}}, \mathcal{F}^{\text{lin}})\)

If \((M, \mathcal{F})\) has all leaves compact with finite holonomy

\[ \Rightarrow L_0 \in M/\mathcal{F} \text{ has neighborhood } U/\mathcal{F} \to U_{\mathcal{F}_0}(L_0)/G^{\mathcal{F}_0}, \]

\[ \Rightarrow \text{Orbifold charts } (U_{\mathcal{F}_0}(L_0), G^{\mathcal{F}_0}, \phi_{\mathcal{F}_0}) \text{ for } X = M/\mathcal{F} \text{ where:} \]

\[ \phi : U_{\mathcal{F}_0}(L_0) \to M/\mathcal{F}, \quad v \mapsto \tilde{\phi}(\text{pr}(v)) \]

Exercise: Show that these charts are compatible so they form an orbifold atlas for \(X = M/\mathcal{F}\).