I.1 Lie Groupoids: Definition & Examples

**Definition:** A Lie groupoid $G = (M, s, t)$ is a groupoid where $s$ and $t$ are smooth maps, $s, t : G 	o M$ are submersions, and $(M, s, t)$ are smooth.

**Notation:** We have spaces of composable arrows:

$$G^{(0)} = M, \quad G^{(1)} = G, \quad G^{(2)} = G \times_G G = \{(g, h) : s(g) = t(h)\}$$

$$\cdots G^{(n)} = G \times_G \cdots \times_G G = \{(g_1, \ldots, g_n) : s(g_i) = t(g_{i+1})\}$$

$s$ and $t$ are submersions $\Rightarrow G^{(1)}$ is a manifold.

In particular, it makes sense to say $M : G^{(2)} \to G$ is smooth.

**Definition:** A morphism from a Lie groupoid $G = (M, s, t)$ to a Lie groupoid $H = (N, s, t)$ is a pair of smooth maps $f : G \to H$ and $f : M \to N$ which are compatible with the structure maps.

Compatibility $\equiv (f, f)$ functor:

- If $g \xrightarrow{s} \alpha e \in G$ Then $f(\alpha) \xrightarrow{f(s)} f(\alpha) \in H$
- If $(g, h) \in G^{(2)}$ Then $f(gh) = f(g)f(h)$
- If $e \in M$, Then $f(1e) = 1f(e)$
- If $g \xrightarrow{t} \alpha e \in G$ Then $f(g^{-1}) = f(g)^{-1}$

The last property follows from the others.
**Convention:**

Manifolds are assumed Hausdorff and 2nd countable.

We do not assume this for the space of arrows $G$.

But we still assume that $M$ and the fibers of $s$ must be Hausdorff and 2nd countable (see examples).

**Remark:** Because $s'(x)$ and $t'(y)$ are closed, embedded, Hausdorff and 2nd countable, for most purposes one can work with $G$ as if it was Hausdorff and 2nd countable.

**Exercise:** Show that for a Lie groupoid $G = M$:
- $m : G^{(1)} \to G$ is a submersion
- $i : G \to G$ is a diffeo
- $u : M \to G$ is an embedding, which is closed if $G$ is Hausdorff

**Proposition**

Let $G = M$ be a Lie groupoid.

(i) $s'(x) \cap t'(y)$ are closed embedded submanifolds of $G$.

(ii) The isotropy groups $G_x$ are Lie groups.

(iii) $t : s'(x) \to O_x$ is a principal $G_x$-bundle.

(iv) The orbits $O_x$ are immersed submanifolds in $M$.

**Explanation about (iii):**
Examples:

1) Lie groups $\rightarrow$ Lie groupoids over $M = \mathbb{R}^3$

   One orbit / one isotropy group

2) Bundles of Lie groups $\rightarrow$ Lie groupoids with $G = \mathbb{R}^3$

   Orbits = pts of $M$  Isotropy groups = fiber of $t = \ast$

Very special case: identity groupoid $\rightarrow$ \( \begin{array}{c} M \\ \downarrow \id \end{array} \)

**Remark.** A bundle of groups need not be locally trivial neither as a bundle nor as a group bundle:

\[
G = \mathbb{R} \times \mathbb{R}^2 \\
M = \mathbb{R}
\]

\[
\begin{cases} 
  t = 0, & G_0 \text{ is abelian} \\
  t \neq 0, & G_t \text{ is non-abelian}
\end{cases}
\]

\[
\mathbb{R}^2 \rightarrow \Delta = \{ (t, \frac{m}{t}) : m \in \mathbb{Z}, t \neq 0 \} \cup \{(0,0)\} \subset \mathbb{R}^2
\]

\[
\begin{array}{c}
\downarrow \text{p}_2 \\
\downarrow \text{p}_1 \\
\mathbb{R}^2 \\
\downarrow \text{p}_1 \\
\mathbb{R}
\end{array}
\]

\[
\begin{array}{c}
\rightarrow \quad G = \mathbb{R}^2 / \Delta \\
\downarrow s \\
M = \mathbb{R}
\end{array}
\]

\[
\begin{cases} 
  t = 0, & G_0 = \mathbb{R} \\
  t \neq 0, & G_t = \mathbb{S}^1
\end{cases}
\]

3) **Pair Groupoids.** For any set $M$:

\[
\begin{array}{c}
M \times M \\
t = \text{p}_1, \quad \text{p}_2 = t
\end{array}
\]

One orbit / isotropy groups are all trivial
4) **Submersion Groupoids.** For a submersion $\mu: M \to N$

\[
\begin{array}{ccc}
M \times M & \xrightarrow{(\mu,\mu)} & M \\
\downarrow & & \downarrow \\
\mu & \xrightarrow{\sim} & \mu
\end{array}
\]

Orbits = Fibers of $\mu$ / Isotropy Groups are Trivial.

**Remark:** If $\mu = \text{id}: M \to M$ we recover the pair groupoid.

5) **Equivalence Relations.** Any equivalence $R \subset M \times M$
defines a subgroupoid or the pair groupoid:

\[
\begin{array}{c}
R \\
pr_1 \downarrow \downarrow pr_2 \\
M
\end{array}
\]

This is a Lie groupoid if $R \subset M \times M$ is an immersed submanifold
and $pr_1, pr_2$ restrict to submanifolds. We say that $R$ is **smooth**

For any Lie groupoid $G \rightrightarrows M$ one has a Lie groupoid morphism, called the **anchor** of $G$:

\[
\Phi: G \xrightarrow{(\iota,\iota)} M \times M
\]

The image of $\Phi$ is the equivalence relation groupoid
associated with orbit equivalence relation (not Lie, in general.)

**Exercise:** Show that a Lie groupoid $G \rightrightarrows M$ is isomorphic
to an equivalence relation groupoid iff its isotropy groups
are all trivial.
**Def.** A Lie subgroupoid or \( G \rightarrow M \) is a Lie groupoid \( H \rightarrow N \) together with a Lie groupoid morphism:

\[
\begin{array}{ccc}
G & \rightarrow & G \\
H & \downarrow & H \\
N & \rightarrow & M
\end{array}
\]

which is an injective immersion. If \( N = M \) we call the Lie subgroupoid wise.

- An equivalence relation is the same thing as a wise Lie subgroupoid or \( M \times M \).
- An isotropy group \( G_x \rightarrow G \) is a Lie subgroupoid which is not wise.

6) **Action Groupoids.** Any Lie group action

\[
G \times M \rightarrow M , (g, x) \mapsto g \cdot x
\]

\[
\begin{array}{cccc}
G \times M & \rightarrow & M \\
(\cdot, x) & \mapsto & (h, y) \cdot (g, x) = (hg, x) \\
& & \text{if } y = g \cdot x
\end{array}
\]

Orbits = orbits of action

Isotropy = isotropy groups of action

7) **Flow of a vector field.** For \( X \in \mathfrak{X}(M) \) take flow:

\[
\phi^t_x \quad \text{with domain } \mathcal{D}(X) \subseteq \mathbb{R} \times M \quad \text{(open set)}
\]

\[
\begin{array}{ccc}
\mathcal{D}(X) & \rightarrow & \phi^t_x(x) \\
(t, x) & \mapsto & (t, y) = (s+t, x) \\
& & \text{if } y = \phi_x^t(x)
\end{array}
\]

It seems there might be a typo in the last line of the flow diagram.
• Orbits = orbits of vector field

• Isotropy group of $\alpha$ \( \in \mathbb{R} \)

\[
\begin{cases}
\mathbb{R} & \text{if } \alpha \text{ is zero} \\
\mathbb{Z} & \text{if } \alpha \text{ lies in periodic orbit} \\
\mathbb{I} & \text{otherwise}
\end{cases}
\]

**Remark:**

$X$ is complete $\iff D(X) = \mathbb{R} \times M \Rightarrow$ Flow defines $\mathbb{R}$-action on $M$

so flow groupoid becomes action groupoid

8) Homotopy groupoid. For any manifold $M$

\[
\begin{array}{c}
\Pi_1(M) \\
\uparrow \\
M
\end{array}
\xymatrix{
\mathbb{R} & \Pi_1(M) \\
\mathbb{R} & M
}

\[ [\gamma] \cdot [\gamma_0] = [\gamma \cdot \gamma_0] \]

\[
\delta \circ \gamma_1(t) =
\begin{cases}
\gamma_2(t) & 0 \leq t \leq 1/2 \\
\gamma_1(2t - 1) & 1/2 \leq t \leq 1
\end{cases}
\]

This is a Lie groupoid. Assume $M$ connected:

\[
\tilde{M} \equiv \{ [\gamma] \mid \gamma : [0,1] \to M, \gamma(0) = \alpha_0 \} \xrightarrow{\pi} M
\]

\[
\pi : \tilde{M} \to M
\]

Universal covering space

\[
\Pi_1(M, \alpha_0) \subseteq \tilde{M}
\]

- $\tilde{M}$ is a smooth manifold
- $\Pi_1(M, \alpha_0) \to \tilde{M}$ is proper & free action

\[
\Pi_1(M) \cong (\tilde{M} \times \tilde{M}) / \Pi_1(M, \alpha_0)
\]
Conclusion:

\[ \tilde{M} \times \tilde{M} \xrightarrow{\sim} \pi_1(n, x) = \Pi_1(M) \cong (\tilde{M} \times \tilde{M}) / \pi_1(M, x) \]

Pair Groupoid

Proper & free

Action by Groupoid automorphisms

Orbits = connected components of \( M / \text{isotropy at } x = \pi_1(M, x) \)

Exercise:

\[
\begin{align*}
\{ & G = M \text{ Lie groupoid}, \\
& G \triangleleft K \text{ Lie group action by group automorphisms} \\
\Rightarrow & G / K \text{ is a Lie groupoid} \\
\Updownarrow & M / K
\end{align*}
\]

9) **GAUGE Groupoid.** \( P \triangleright G \) principal \( G \)-bundle

\[
\begin{array}{c}
(P \times P) / G \\
\downarrow \\
M
\end{array}
\]

(quotient or pair groupoid \( P \times P = P \) by )

Diagonal action of \( G \): \( (p, q)g = (p, qg) \)

\[
\begin{array}{c}
[p, q] \\
\pi(q) \\
\pi(p)
\end{array}
\]

One orbit / isotropy groups \( \sim G \)

Remark: When \( M \) is connected, \( \Pi_1(M) \) is an example of a Gauge groupoid (associated with \( \tilde{M} \triangleleft \pi_1(M, x) \) )
Def. A groupoid is called transitive if it has only one orbit.

- Gauge groupoid of $\mathbb{P} \rightarrow M$ is transitive

- $G \leq M$ transitive $\Rightarrow t : S'(x_0) \rightarrow M$ is principal $G_x$-bundle

If $G \leq M$ is transitive then:

$$
\begin{array}{ccc}
(S'(x_0) \times S'(x_0))/G_{x_0} & \longrightarrow & G \\
\downarrow & & \downarrow \\
M & \longrightarrow & M
\end{array}
$$

is a Lie groupoid isomorphism.