II - Singular Spaces

** Aim:** Differential Geometry on spaces which are **singular**, i.e., which are not smooth manifolds.

- Often such singular spaces arise as quotient spaces of smooth manifolds, but to be able to work on them we need to keep track of extra structure.

- Our point of view:
  
  singular spaces $\cong$ orbit spaces of Lie groupoids

where:

1. Extra structure is encoded by groupoids (e.g., isotropy groups) but groupoids also contain irrelevant extra information.
2. Two groupoids can present the same singular space.

(i) & (ii) $\implies$ Morita equivalence of Lie groupoids

As a warm-up we consider a special case of singular spaces, namely

** Orbifolds**

** Idea:** An orbifold is a topological space $X$ where each $x \in X$ has a neighborhood $V_x \cong U/G$ with $U \subset \mathbb{R}^n$ open and $G \subset \text{Diff}(U)$ a finite group
To formalize this:

**Def:** Let $X$ be a topological space.

(i) An **orbifold chart** of dimension $M$ for $X$ is a triple $(U, G, \phi)$ where $U \subset \mathbb{R}^n$ is a connected open set $G \subset \text{Diff}(U)$ is a finite subgroup and $\phi : U \to X$ is a $G$-invariant open map inducing a homeomorphism $U/G \to \phi(U) \subset X$.

(ii) An **embedding of orbifold charts** $(V, H, \psi) \to (U, G, \phi)$ is an embedding $\lambda : V \to U$ such that $\lambda \downarrow U \to \phi$.

(iii) Two orbifold charts $(U_1, G_1, \phi_1) \neq (U_2, G_2, \phi_2)$ are said to be **compatible** if for any $x \in \phi_1(U_1) \cap \phi_2(U_2)$ there exists an embedding of orbifold charts $\lambda_i : (V, H, \psi) \to (U_i, G_i, \phi_i)$ with $x \in \psi(V)$.

(iv) An **orbifold atlas** of dimension $M$ for $X$ is a collection of pairwise compatible orbifold charts of dimension $M$, $\mathcal{U} = \left\{(U_i, G_i, \phi_i) : i \in I\right\}$ with $X = \bigcup_{i \in I} \phi_i(U_i)$. Two orbifold atlas $\mathcal{U}_1$ and $\mathcal{U}_2$ for $X$ are compatible if $\mathcal{U}_1 \cup \mathcal{U}_2$ is an orbifold atlas.

(v) An **orbifold of dimension** $M$ is a pair $(X, U)$ where $X$ is a second countable, Hausdorff topological space and $U$ is a maximal orbifold atlas.
Rmars

1) Any orbifold atlas \( U \) defines an orbifold (\( U \) is contained in a unique maximal orbifold atlas)

2) Every orbifold is locally compact and paracompact

3) A smooth function \( f: X \to \mathbb{R} \) is a continuous map such that for any orbifold chart \( (U, \mathcal{G}, \phi) \), \( f \circ \phi: U \to \mathbb{R} \) is smooth

4) Similarly, a smooth map \( f: X \to Y \) between to orbifolds is a continuous map such that for each \( U \in \mathcal{U} \) there are orbifold charts \( (U, \mathcal{G}, \phi) \) with \( \phi(U) \) and \( (V, \mathcal{H}, \psi) \) with \( \psi(V) \), and a smooth map \( \overline{f}: U \to V \), such that \( f \circ \phi = \psi \circ \overline{f} \)

Notation: For any manifold \( M \) and \( G \subset \text{Diff}(M) \) we write:

\[
\lambda_g: M \to M, \quad \omega \mapsto g \omega \quad \text{(action by } g \in G) \\
\Sigma_g := \{ \omega \in M : g^\omega = \omega \} \\
\Sigma_G := \bigcup_{g \neq e} \Sigma_g = \{ \omega \in M : G_\omega \neq 1 \} \\
G_G := \{ g \in G : gS = S \}
\]

A subset \( S \subset M \) is called \( G \)-stable if either:

\[ gS = S \quad \text{or} \quad gS \cap S = \emptyset \]

Exercise:

\( G \)-stable sets are the connected components of \( G \)-invariant sets. If \( G \) is finite, the open \( G \)-stable sets give a base for topology of \( M \).
We will look at finite subgroups $G \subset \text{Diff}(M)$ and we will show that:

- If $(U, G, \phi)$ is orbifold chart and $V \subset U$ is a $G$-stable open subset, then $(V, G_V, \phi|_V)$ is an orbifold chart compatible with $(U, G, \phi)$.
- Given two orbifold charts $(U, G, \phi), (V, H, \psi)$ and $x \in \phi(p) \cap \psi(q)$, one has that:
  
  (i) $p \in \Sigma_G$ iff $q \in \Sigma_H$.
  
  (ii) There are Faithful representations
  
  $G_p \to \text{GL}(n, \mathbb{R})$, $g \mapsto d_p \lambda_g$ 
  
  $H_q \to \text{GL}(n, \mathbb{R})$, $h \mapsto d_p \lambda_h$

  and images are conjugate subgroups.

**Def.** For an orbifold $X$:

(i) $x \in X$ is called a **singular point** if for some chart $(U, G, \phi)$

$x \in \phi(p)$ with $p \in \Sigma_G$. The **singular locus** of $X$ is denoted $\Sigma_X$.

(ii) The **isotropy type** of $x$ is the conjugacy class in $\text{GL}(n, \mathbb{R})$ of the image $G_p \to \text{GL}(n, \mathbb{R})$ for some chart $(U, G, \phi)$, and is denoted $\text{ISO}_x(X)$.

Hence:

$$\Sigma_X = \{ x \in X : \text{ISO}_x(X) \neq 1 \}.$$  

We will see that $\Sigma_X \subset X$ is a closed subset with empty interior.
Examples

1. Smooth manifolds = orbifolds w/ empty singular locus
   orbifolds w/ $\text{Is}_0(X) = 1$, $\forall x$

2. If $G \subset \text{Diff}(M)$ is a finite group then $X = M/G$ with
   quotient topology has a natural orbifold structure, with:
   $\Sigma_x = \{ x \in G : G_x \neq 1 \}$, $\text{Is}_0(X) \cong G_0$

   To construct orbifold chants on $\pi : M \to M/G$, given $p_0 \in M$
   there exists a chart $(V, \psi)$ for $M$ centered at $p_0$ such that:
   - $V$ is $G_{p_0}$ invariant;
   - $g \in G \cap V = \emptyset$ is $g \notin G_{p_0}$.

   We obtain $G_{p_0} \cong H \subset \text{Diff}(\psi(V))$, so we can define an
   orbifold chart $(\psi(V), G_{p_0}, \pi \circ \psi)$.

   An orbifold isomorphic to $M/G$ for some finite group
   $G \subset \text{Diff}(M)$ is called a global quotient or a good orbifold.

Ex: $M = \mathbb{S}^2$ = $\mathbb{C}$ : $121 = 1 \bar{3}$, $G = \mathbb{Z}_2 \mathbb{G}$, complex conjugation

Then: $X = \mathbb{S}^2/\mathbb{Z}_2 \cong \{-1, 1\}$

   $\Sigma_x = \{ -1, 1 \}$

   $\text{Is}_0(X) \cong \text{Is}_{-1, 1}(X) = \mathbb{Z}_2$

Ex: $M = \mathbb{S}^2$, $G = \mathbb{Z}_k$ act by rotations $\frac{2\pi}{k}$

Then: $X = \mathbb{S}^2/\mathbb{Z}_k \cong \mathbb{S}^2$

   $\Sigma_x = \{ \frac{2\pi}{k}, P \leq \}$

   $\text{Is}_0(X) \cong \text{Is}_{\frac{2\pi}{k}}(X) = \mathbb{Z}_k$
3. An orbifold which is not a global quotient:

Take $X = \mathbb{S}^2$ has a topological space. Consider two orbifold charts:

(a) $(D, 1, \phi)$: $\phi : D \cong \mathbb{S}^2 \setminus \{p_n\}$

(b) $(D, Z_m, \psi)$: $Z_m = \frac{1}{m}$ relations by $\frac{2\pi}{m} f \in \mathbb{R}^2$

$\psi : D \to D/\mathbb{Z}_m \cong \mathbb{S}^2 \setminus \{p_n\}$

These charts are compatible: If we consider the orbifold chart $(D - 103, 1, \mathbb{I})$:

$\mathbb{I} : D - 103 \to \mathbb{S}^2 \setminus \{p_n, p_3\}$

We have maps:

$(D - 103, 1, \mathbb{I}) \xrightarrow{id} (D, 1, \phi)$

$(D - 103, 1, \mathbb{I}) \xrightarrow{(D, Z_m, \psi), \ z \to z^m}$

The first map is an embedding. The second map is not cover. So restricting to subsets $U_d \subset D - 103$ we obtain embedding of orbifold charts.

Gaining compatibility on the charts.

This orbifold has singular set $\Sigma_x = \{p_n\}$

and isotropy $\text{Iso}^+_p(X) = \mathbb{Z}_m$.

Similarly, one can construct a 2-dim orbifold $X = \mathbb{S}^2$ with two singular pts and isometries $\mathbb{Z}_m \neq \mathbb{Z}_m$. It is a global quotient iff $m = m$.

One can show that orbifold structures on $\mathbb{S}^2$ with 3 or more singular points are always good.