Last time: \( \mathcal{F}_a \subset \mathcal{P}(A) \) foliation determined by \( A \)-path homotopy

When is \( A \) integrable?

\[ \iff \text{When is the leaf space } \mathcal{G}(A) = \mathcal{P}(A) / \mathcal{F}_a \text{ smooth?} \]

An obstruction to integrability:

Assume \( A \) integrable

\[ \Rightarrow \mathcal{G}(A)_x \text{ is a Lie group w/ Lie algebra } \mathcal{Q}_x = \ker \mathcal{P}_x \]

\[ \Rightarrow q_x: \mathcal{G}(Q_x) \rightarrow \mathcal{G}(A)_x \text{ is a covering map} \]

**Lemma:** \( \tilde{N}_x = \ker q_x \subset \mathcal{G}(Q_x) \) is a discrete subgroup of \( \pi_1(G/G_0) \)

**Proof:** \( q_x \) is covering \& homomorph \( \Rightarrow \tilde{N}_x \) is discrete and normal.

Every normal, discrete subgroup \( D \) of a connected Lie group \( G \)

is contained in \( \pi_1(G) \):

\[ g \in G, \; d \in D. \text{ Choose path } g(t) \in G \; w/ \; g(0) = e, \; g(t) = g. \text{ Then:} \]

\[ g(t) \cdot d \cdot g(t) \in D \text{ must be constant } \iff g \cdot d \cdot g^{-1} = g \cdot d \cdot g^{-1} = g \cdot d \in \pi_1(G). \]

\[ \Rightarrow g \cdot d \cdot g^{-1} = g \cdot d \cdot g^{-1} = g \cdot d \Rightarrow d \in \pi_1(G). \]

Hence:

\[ \mathcal{G}(A)_x = \mathcal{G}(Q_x) / \tilde{N}_x \; \& \; \pi_1(\mathcal{G}(A)_x) = \tilde{N}_x \]

\[ + \; S(\alpha) \rightarrow \mathcal{Q}_x \text{ principal } \mathcal{G}(A)_x \text{- bundle w/ } S(\alpha) \text{- connection, so:} \]

\[ \vdots \rightarrow \pi_1(\mathcal{Q}_x) \xrightarrow{2_1} \pi_1(\mathcal{G}(A)_x) \rightarrow 1 \rightarrow \pi_1(\mathcal{Q}_x) \rightarrow \pi_0(\mathcal{G}(A)_x) \rightarrow 1 \]
It follows that we have short exact sequence:

\[ \pi_2(G_\alpha) \xrightarrow{\vartheta_\alpha} G(Y_\alpha) \xrightarrow{q_\alpha} G(A)_\alpha \rightarrow \pi_1(G_\alpha) \rightarrow 1 \quad (*) \]

where \( \text{Im} \vartheta_\alpha = \widetilde{N}_\alpha \).

\( (*) \) still exists in the non-integrable case!!

This leads to:

**Main Obstruction to Integrability:**

If \( A \) is integrable, \( \widetilde{N}_\alpha = \text{Im} \vartheta_\alpha \subset G(Y_\alpha) \) is discrete.

We will see that \( \widetilde{N}_\alpha \) can be computed in many cases. First we see how it can be defined for any Lie algebra

\[ \text{Proposition} \quad \text{For any Lie algebra} \ A, \text{there is a short exact sequence of groups:} \]

\[ \pi_2(G_\alpha) \xrightarrow{\vartheta_\alpha} G(Y_\alpha) \xrightarrow{q_\alpha} G(A)_\alpha \xrightarrow{P_\alpha} \pi_1(G_\alpha) \rightarrow 1 \]

where:

(i) \( P_\alpha \) maps \([a] \mapsto [\Delta a] \)

(ii) \( q_\alpha = G(i) \), \( i : G_\alpha \rightarrow A \)

(iii) \( \vartheta_\alpha \) maps \([6] \) to \([a] \) where \( a : I \rightarrow G_\alpha \) is \( A \)-path homotopy to \( G_\alpha \) via \( A \)-path homotopy covering \( 6 \).

**Remk.** What we are doing is working out explicitly the first terms of long exact sequence of \( \tilde{S}^1(\alpha) \rightarrow G_\alpha, \ G(\alpha) \rightarrow G(A) \). Since \( \tilde{S}^1(\alpha) \) can be very pathological, we are not allowed to use the result that principal bundle is a Serre Fibration. We defer the proof for later.
**Defn.** The map $\Theta_x: \pi_x(0_x) \to G(G_x)$ is called the monodromy map of $A$ and $\tilde{\mathcal{N}}_x(A) := \text{Im} \Theta_x$ is called the monodromy group at $x \in N$.

Note that $\tilde{\mathcal{N}}_x(A)$ is a normal subgroup of $G(G_x)$. But it may fail to be closed (obstruction!). Still:

**Lemma** $\tilde{\mathcal{N}}_x(A) \subset \mathcal{Z}(G(G_x))$

**Proof.**

Each $g \in \tilde{\mathcal{N}}_x(A)$ is represented by an $A$-path $a: I \to G_x$ which is $A$-path homotopic to $0_x$. Working on orbit $0_x$, we have $\text{Rep}(A_B)$ on isotropy $G_B$ ("Bott connection")

\[ \nabla_x p = [\alpha, p] \quad (\alpha \in \text{P}(A_B), p \in \mathcal{P}(A_B)) \]

This $\text{Rep}(A_B)$ restricts to $\text{Ad}: G_x \to G(G_x)$. Then $\text{Ad}^{\text{rep}}$ gives $\text{Ad}$ on $G(G_x)$. Hence, since $a \cdot 0_x$:

\[ \begin{align*}
\cdot \quad & \tau_a = \text{Ad}_g: G_x \to G_x \\
\cdot \quad & \tau_a = \tau_{0_x} = \text{Id} \quad \Rightarrow \quad \text{Ad}_g = \text{Id} \iff g \in Z
\end{align*} \]

Note that $\mathcal{Z}(G(G_x))$ integrates $\mathcal{Z}(G_x)$ but may fail to be connected. Passing to connected component or identity:

\[ \exp: (\mathcal{Z}(G_x), +) \rightarrow (\mathcal{Z}(G(G_x)), \circ) \]

This leads to a version of monodromy living in $G_x \subset A_x$, which is better for computations.
Proposition: Set:
\[ N\alpha (A) = \exp(\tilde{\tilde{\alpha}}(A) \cup \tilde{\tilde{Z}}) \]

TFAG:
(i) \( \tilde{\tilde{\alpha}}(A) \subset G(G) \) is closed
(ii) \( \tilde{\tilde{\alpha}}(A) \subset G(G) \) is discrete
(iii) \( N\alpha (A) \subset G\alpha \) is closed
(iv) \( N\alpha (A) \subset G\alpha \) is discrete

Proof: For a 1-connected Lie group \( G \) with torsor \( G \)
\[ \exp : G \rightarrow G \]
restricts to a group isomorphism
\[ \exp : Z(G) \rightarrow Z(G) \]
Since \( \pi_2(\tilde{\tilde{G}}_n) \) is countable, \( \tilde{\tilde{\alpha}}(A) \) and \( N\alpha (A) \) are countable and the equivalences follow.

\[ \square \]

Note that:
- \( \tilde{\tilde{\alpha}}(A) = \{ g \in G(A) : g \sim 0\} \)
- \( N\alpha (A) = \{ v \in G\alpha : v \sim 0\} \)

Exercise: Show that if \( x, y \) belong to some orbit \( O \) of \( A \)
Then \( N\alpha (A) \cong N\alpha (A) \) and \( \tilde{\tilde{\alpha}}(A) \cong \tilde{\tilde{\alpha}}(A) \) (canonically!)
Moreover, there is a bundle isomorphism \( T : \tilde{\tilde{G}}_0 \rightarrow \tilde{\tilde{G}}_0 \)
such that \( T : N\alpha (A) \rightarrow N\alpha (A) \).

Hint: Use the Bott \( A_0 \)-connection on \( \tilde{\tilde{G}}_0 \) as in proof above
Set: \[ N(A) = \bigcup_{x} N_x(A) \subseteq \text{ker} p \subseteq A \]

**Theorem (Crainic - F)**

A Lie algebraoid \( A \) is integrable iff there exists an open \( U \subseteq A \) containing zero section \( O_{m,o} \) s.t.
\[ N(A) \cap U = \{0\}_{m,o} \quad (\star) \]

**Remarks**

- Fixing \( x \in \mathbb{R} \), \( U_x = \cup_{r} B_{x,r} \) is open in \( G_x \), so (\( \star \)) gives \( N(A) \cap U_x = \{0\} \Rightarrow N_x(A) \subseteq B_x \) are discrete
- Condition (\( \star \)) says that \( N_x(A) \) are "uniformly" discrete
- IF \( x,y \) belong to same orbit: \( N_x(A) = N_y(A) \) and \( N_x \) is discrete iff \( N_y \) is discrete (Exercise above)
- IF \( A \) is transitive, \( N_x(A) \subseteq B_x \) discrete \( \Rightarrow (\star) \) (Again by Exercise above!)
- For general \( A \), we can think that condition (\( \star \)) has two components:
  
  (a) Along leaves: \( N_x(A) \subseteq B_x \) is discrete
  
  (b) Transverse to leaves: \( d(N_x(A) - \{0\}, O_x) \) stays bounded away from 0 when \( x \) varies in transverse direction.

**Corollary:** Any Lie algebraoid \( A \) with trivial monomorphism groups \( N_x(A) \) is integrable, for example, this happens if:

(i) \( A \) has trivial center, \( \forall x \in \mathbb{R} \);

(ii) Orbits \( O_x \) have finite \( \pi_x \), \( \forall x \in \mathbb{R} \);

(iii) For every orbit \( O_{cm} \), there is a splitting \( \sigma : T_{0} \rightarrow A_{0} \) of the anchor \( p : A_{0} \rightarrow T_{0} \) preserving Lie bimodules.
Still we would like to compute the monodromy groups.
We will see that next lecture.

**Proof or Proposition:**

We want to show exactness of sequence of group homomorphisms:

\[
\pi_2(G_x) \xrightarrow{\partial_x} G(G_x) \xrightarrow{q_x} G(A)_x \xrightarrow{\rho_x} \pi_1(G_x) \longrightarrow 1
\]

First one needs to check that maps \( \partial_x \) and \( q_x \) are well-defined:

- \( q_x : G(G_x) \to G(A)_x \): This is just \( G(i) \) where \( i : G_x \subset A \)

- \( \partial_x : \pi_2(G_x) \to G(G_x) \): Let \( \sigma : I \times I \to O_x \) with \( \sigma(\partial(\text{Id} \times \text{Id})) = x \)

Then \( \partial_x[\sigma] = [a] \) where \( a : I \to G_x \) is \( A \)-path homotopic to \( O_x \) via \( \overline{\sigma} : T(I \times I) \to A \) \( \text{covering } \sigma \).

**Exercise:** Show that \( \overline{\sigma} \) exists (Hint: See proofs last time and use splitting of monon).

Now similarly to covering homotopy theory, one shows:

(i) If \( a : I \to A \) has base path \( \overline{a} \), and \( \sigma : I \times I \to O_x \) is homotopy starting at \( \overline{a} \), \( \exists \) \( A \)-path homotopy \( \overline{\sigma} \) covering \( \sigma \) and starting at \( a \).

(ii) Two \( G_x \)-path \( a_0, a_1 : I \to G_x \) are \( G_x \)-homotopic iff \( \exists \) \( A \)-path homotopy whose base path \( \sigma : I \times I \to \Theta_x \) is the trivial class \( [\text{Id}] \in \pi_2(G_x) \).

\[ \text{Im} \, \partial_x = \ker q_x \] is obvious from definition. For \( \sigma \) let \( a : I \to G_x \) represent \( [a] \in G(G_x) \) in \( \ker q_x \). This means \( \exists \overline{\sigma} \) \( A \)-path homotopy giving \( a \) \( O_x \). But since \( \gamma \) of this homotopy defines \( [\gamma] \in \pi_1(G_x) \) \( \ker \theta_x = [a] \).
\[ \text{In} \ g_n = \text{Ker} \ p_n: \ C \text{ is obvious from definitions. For } \gamma \text{ let } \]
\[ [a] \in g(A), \text{ be in } \text{Ker} \ p_n, \text{ i.e., } \gamma_n \text{ is contractible in orbit } \Theta_n. \]
Choose path-homotopy \( \delta: I \times I \to \Theta_n \) such that \( \delta_n \equiv \delta. \) Can define A-path homotopy \( \delta: T(I \times I) \to A \) covering \( \delta \) with \( \delta_n(t, s) = \gamma_n(s). \)

Then \( \delta_n(t, s) \) is \( \delta_n \)-path which is A-path homotopic to \( \gamma_n \).
So:
\[ q_n([\delta]) = [a], \delta = [a] \]

- \( p_n \) is surjective: Any loop \( \gamma: I \to \Theta_n, \delta(s) \equiv \gamma_0 \) is the base path of an A-path \( a: I \to A \) (e.g., use splitting \( \delta: T0 \to A_0 \) or Theorem).