**Math 595 - Lecture 13**

**Geodesics**

**Def.** Given an A-connection $\nabla$ on $A$ a **geodesic** for $\nabla$ is an A-path $a: I \to A$ such that $Da = 0$.

In general, geodesics exist only for a small time:

$$\nabla_{\cdot} x^i = \Gamma^i_{\cdot s} a^s, \quad a(t) = a^i(t) \alpha_i(\gamma_\alpha(t))$$

$$\psi(x_i) = B^s_i \frac{\partial}{\partial x^s}, \quad \gamma_\alpha(t) = (\gamma_\alpha^s(t)), ..., \gamma_\alpha^m(t)$$

$$\begin{cases} \frac{dx^k(t)}{dt} = -\Gamma^k_{ij} (\gamma^j_\alpha(t)) a^i(t) a^j(t) \\ \frac{dx^s(t)}{dt} = B^s_i (\gamma^i_\alpha(t)) a^i(t) \end{cases}$$

So geodesics are integral curves of vector field

$$X^\nabla = B^s_i (x^i) \frac{\partial}{\partial x^s} - \Gamma^s_{ij} (x^j) \frac{\partial}{\partial x^s}$$

This is called the **geodesic spray** of $\nabla$.

**Proposition.** The geodesic spray is well-defined (independent) of choice of coordinates and satisfies:

(i) $d^n a(X^\nabla_a) = \psi(a)$, $\forall a \in A$

(ii) $(d^n t)_* X^\nabla = \frac{1}{t} X^\nabla$, $\forall t > 0$

Conversely, any vector field satisfying (i) and (ii) is the geodesic spray of a unique torsion-free connection.
Sketch of proof:

Given vector field:

$$X = U^s(x, s) \frac{\partial}{\partial x^s} + U^k(x, s) \frac{\partial}{\partial x^k}$$

(i) + (ii) $\Rightarrow$

$$\begin{cases} 
U^s(x, s) = \mathcal{B}^s_i(x) \xi^i \\
U^k(x, s) = \mathcal{C}^k_{i\delta}(x) \xi^i \xi^\delta 
\end{cases}$$

To find $\nabla_a x^i = \Gamma^i_{\delta} a^\delta$ with zero tension and $X^a X = X$ solve the system:

$$\begin{align*}
\Gamma^i_{\delta} - \Gamma^k_{i\delta} &= C^k_{i\delta} \\
\Gamma^i_{\delta} + \Gamma^i_{\delta} &= U^k_{i\delta}
\end{align*}$$

Using partition of unity $\Rightarrow \nabla a X^V = X$.

\[\blacksquare\]

Corollary: Given an $A$-connection $\nabla$ on $A$ there is a unique connection $\nabla$ with same geodesics and zero tension.

Exponential map

$$\nabla = A \text{-connection on } A \quad \exp_{\nabla} : V \rightarrow \mathcal{M}$$

$\phi^{t}_{X^V} = \text{flow of geodesic spray}$ $\exp_{\nabla}^{a} : \mathcal{P}(\phi^{t}_{X^V}(p))$

Exercise: Show that $\exp_{\nabla}^{a} : \mathcal{M} \rightarrow \mathcal{M}$ is a submersion onto an open neighborhood $U \subset \mathcal{O}_{x}$. This is not quite the "right" exponential map.

Prop: Let $G \approx M$ Lie groupoid with $\text{Lie}(G) = A$. There is a map:

$$\exp_{\nabla} : V \rightarrow G \quad \mathcal{O}_n \subset V \subset A \text{ open}$$

which is a local diffeomorphism if $V$ is sufficiently small neighborhood of $\mathcal{O}_n$. 
Proof

The left-invariant vector fields generate all vector fields tangent to $t$-fibers $\Rightarrow$ a unique Kähler-connection $\tilde{\nabla}$ s.t.:

$$\tilde{\nabla}_\alpha \beta = \tilde{\nabla}_\beta \alpha$$

One can think of $\tilde{\nabla}$ has a (smooth) family of ordinary connections on $t$-fibers.

$$\exp_\nabla : V \to G, \quad \exp_\nabla |_{V \cap \mathcal{A}_x} = \exp_{\hat{\nabla}} |_{\mathcal{A}_x}$$

Note that:

$$\exp_\nabla = \exp_{\hat{\nabla}} \quad \text{to} \quad \exp_\nabla = \exp_{\hat{\nabla}}$$

Remark: As for Lie Groups, one can express the groupoid structure in "exponential coordinates". The formulas now depend on choice of $\nabla$, so there is no "universal" Baker-Campbell-Hausdorff formula.

Still it is possible to use this approach to show that to every Lie algebroid there is a "Local Lie groupoid" integrating it.

**A-path homotopy Groupoid (a.k.a. Weinstein Groupoid)**

**Aim**: Given algebroid $A \to M$ construct groupoid:

$$\mathcal{G}(A) := \frac{A\text{-paths}}{A\text{-path homotopy}}$$

**Lemma 1**

A-path homotopy is an equivalence relation.
**Proof**

- Reflexive: \( \varepsilon \)-constant A-path homotopy \( \Rightarrow \) \( a_0 \sim a_0 \)

- Symmetry: \( a_0 \sim a_2 \) via \( \overline{\varphi} \) then \( a_1 \sim a_0 \) via \( \overline{\varphi} \)

\[
\overline{\varphi}(t, \varepsilon) = \varphi(t, t-\varepsilon) dt - \overline{\varphi}(t, 1-\varepsilon) d\varepsilon
\]

- Transitivity: \( a_0 \sim a_1 \) via \( \overline{\varphi} \) and \( a_1 \sim a_2 \) via \( \overline{\varphi} \) then

\[
a_0 \sim a_2 \text{ via: } \overline{\varphi}_2(t, \varepsilon) = \begin{cases} 
\varphi_0'(t, 2\varepsilon) dt + 2 \varphi_0(1, 2\varepsilon) d\varepsilon & 0 \leq t \leq 1 \\
\varphi_0(t, 2\varepsilon - 1) dt + 2 \varphi_0(t, 2\varepsilon - 1) d\varepsilon & \frac{1}{2} \leq \varepsilon \leq 1
\end{cases}
\]

with \( \varphi : [0,1] \rightarrow [0,1] \) reparametrization in \( \varepsilon \)-direction with

\[
\begin{align*}
\varphi(0) & = 0 \text{ if } 0 \leq \varepsilon \leq \frac{1}{2} \\
\varphi(1) & = 1 \text{ if } \frac{1}{2} \leq \varepsilon \leq 1
\end{align*}
\]

**Lemma 2** If \( \phi : [0,1] \rightarrow [0,1] \) is a reparametrization then \( a \sim a^* \) and \( A \)-path homotopy.

**Proof.** Define \( \overline{\varphi} : T(I \times I) \rightarrow A \) by:

\[
\overline{\varphi}(t, \varepsilon) = ((1-\varepsilon) + \varepsilon \phi(t)) \alpha((1-\varepsilon) t + \varepsilon \phi(t)) dt + (-t + \phi(t)) \alpha((-1) t + \varepsilon \phi(t)) d\varepsilon
\]

Need to check this is A-path homotopy:

\[
\begin{align*}
\overline{\varphi}(t, 0) & = \alpha(t) \\
\overline{\varphi}(t, 1) & = \phi(t) \alpha(\phi(t)) = \alpha^*(t) \\
\overline{\varphi}_2(0, \varepsilon) & = \phi(0, \varepsilon) = \overline{\varphi}_2(t, 0)
\end{align*}
\]

If \( a^* \in \pi(A) \) is any section extending \( \alpha(t) \), Then:

\[
\begin{align*}
\alpha_{t_1, e} = ((1-e) + e \phi(t)) \alpha((1-e) t + e \phi(t)) \\
\beta_{t, e} = (-t + \phi(t)) \alpha((-1) t + e \phi(t))
\end{align*}
\]

are extensions of \( \overline{\varphi} \) and \( \overline{\varphi}_2 \) which satisfy:

\[
\left( \frac{d}{dt} \beta_{t, \varepsilon} - \frac{d}{d\varepsilon} \alpha_{t, \varepsilon} \right)_{\varepsilon(t, \varepsilon)} = -\left[ \alpha_{t, \varepsilon}, \beta_{t, \varepsilon} \right]_{\varepsilon(t, \varepsilon)}
\]
- \( P(A) = \{ A\text{-paths} \} \) "\( \sim \)" A-path homotopy

- \( G(A) := P(A)/\sim \rightarrow M \) w/ structure maps

Source/ target maps: \( s([a]) = \gamma_a(0), \quad t([a]) = \gamma_a(a) \)

Unit map: \( U(\infty) = [O_\infty] \)

Invers map: \( \iota([a]) = [\overline{a}] \)

Multiplication: Fix a reparameterization \( \phi \) \( \phi^w(0) = \phi^w(1) = 0 \)

\[ [a_1] \cdot [a_2] := [a_1 \cdot a_2] \]

Topology: On \( P(A) \) consider \( C^1 \)-topology:

\[ d(a,a') = \max \left\{ \sup_{t \in [0,1]} d^A(a(t),a'(t)) , \sup_{t \in [0,1]} d^1(a(t),a'(t)) \right\} \]

where \( d^A \) and \( d^1 \) are distances in \( A \) and \( TA \). Then consider quotient topology on \( G(A) \).

**Theorem**

\( G(A) \) is a \( t \)-simply connected topological groupoid, independent of choice of reparameterization \( \phi \). Whenever \( A \) is integrable, \( G(A) \) has a compatible smooth structure such that \( \gamma \) is a Lie groupoid integrating \( A \).

**Remark**:

1. \( G(A) \) is called the Weinstein groupoid of \( A \).
2. \( G(\cdot) \) is a functor: if \( \phi : A_1 \rightarrow A_2 \) is a Lie algebraoid morphism, then we obtain a morphism of topological groupoids

\[ \phi : G(A_1) \rightarrow G(A_2), \quad [a] \mapsto [\phi \circ a] \]

If \( A_1, A_2 \) are integrable, this is a smooth morphism integrating \( \phi \).
$G(A)$ is topological gap: structure maps are continuous since they are continuous at the level on $A$-paths. One still needs to check that source/target are open maps. This follows from:

**Lemma**: $P(A) \to P(A)/_p = G(A)$ is open map.

Given $D \subset P(A)$ open, we need to check that its saturation

$$\hat{D} = \{ a \in P(A) : a' \in D \}$$

is open. This follows by showing that if $a_0, a_1$, there exists a homotopy $T: P(A) \to P(A)$ with $T(a_0) = a_1$.

Let $a_0, a_1 \in P(A)$ such that:

$$a_0(t_0, e) = a_0, (t_0, e)$$

$$a_1(t_0, e) = a_1, (t, e)$$

So that

$$\frac{d}{dt} (\beta(t, e)) - \frac{d}{de} (\alpha(t, e)) = -\left[ \alpha(t, e), \beta(t, e) \right]$$

We can assume that $\beta(t, e)$ is compactly supported (since $\delta(E_{\pm})$ is compact).

Given $A$-path $\tilde{a}: I \to A$ let $\tilde{\alpha}^0 \in P_c(A)$ be time-orientable section with compact support:

$$\tilde{\alpha}^0(\varphi_k(I)) = \tilde{a}(I)$$

Let $\tilde{a}(t, e)$ be the solution of ode:

$$\begin{cases}
\frac{d}{dt} \tilde{a}_t(e) = \frac{d}{dt} \beta(t, e) + [\tilde{\alpha}(t, e), \beta(t, e)] \\
\tilde{a}_{t_0} = \tilde{\alpha}^0
\end{cases}$$
Then if
\[ \tilde{\gamma}(t,\varepsilon) := \Xi_{t,1}(\gamma(t)) \]
we see that
\[ \tilde{\gamma} = \tilde{\gamma}_{t,\varepsilon}(\tilde{d}(t,\varepsilon)) \, dt + \beta_{t,\varepsilon}(\tilde{d}(t,\varepsilon)) \, d\varepsilon \]
is an A-homotopy spanning \( A \). We then set:
\[ T(A) := \tilde{\gamma}_{t,1}(\tilde{d}(t,\varepsilon)). \]

- \( G(A) \) as 1-connected t-fibers (= s-fibers)

Fix \( x \in M \) and let \( \gamma : [a,b] \rightarrow G(A) \) be a loop in \( G(A) \) based at \( A \).
\( a : I \rightarrow A \) is a family of A-paths with \([a_0] = [a_b] = 1_x \) (no assumption on \( S \)-dependence). We can assume that \( a_0(t) = a_1(t) = 0_x \).

Then we define:
\[ H : I \times I \rightarrow G(A) \]
\[ (s,\varepsilon) \mapsto \tilde{d}(s,\varepsilon) \]

Claim: \( H \) is a path-homotopy in \( G(A) \) between \( s \mapsto 0_x \) and \( 1_x \)

- \( H \) is continuous;
- For fixed \( s, H(s,\varepsilon) \) is continuous because
  \[ \{0\} \rightarrow P(A), \varepsilon \rightarrow \tilde{d}_s(\varepsilon) \]
is continuous
- For fixed \( \varepsilon, H(s,\varepsilon) \) is continuous because
  \[ \Xi : P(A) \rightarrow P(A), a \mapsto \tilde{d}(a_s) \]
is continuous and satisfies \( a_0 \cup a_1 \Rightarrow \tilde{d}(a_0) \sim \sim \tilde{d}(a_1) \)
  \[ \Rightarrow s \mapsto H(s,\varepsilon) = \Xi H(s,\varepsilon) \]
is continuous
- Boundary conditions:
  \[ H(s,0) = [0_x] = 1_x \]
  \[ H(s,1) = [a_s] = 0_x \]
  \[ H(0,\varepsilon) = [\varepsilon a_0(\varepsilon)] = [0_x] = 1_x \]
  \[ H(1,\varepsilon) = [\varepsilon a_1(\varepsilon)] = [0_x] = 1_x \]