1. Prove the following statements or give a counterexample:
   (a) If $H_1$ and $H_2$ are Lie subgroups of $G$ then $H_1 \cap H_2$ is a Lie subgroup of $G$;
   (b) If $\phi : G \to K$ is a Lie group homomorphism then $\phi(G)$ is a Lie subgroup of $K$;
   (c) If $\phi : G \to K$ is a Lie group homomorphism and $H \subset K$ is a Lie subgroup, then $\phi^{-1}(H)$ is a Lie subgroup of $G$.

2. Show that $\exp : \mathfrak{gl}(2, \mathbb{C}) \to GL(2, \mathbb{C})$ is surjective. On the other hand, show that the matrix
   \[
   \begin{pmatrix}
   -2 & 0 \\
   0 & -1
   \end{pmatrix}
   \]
   is not in the image of $\exp : \mathfrak{gl}(2, \mathbb{R}) \to GL(2, \mathbb{R})$.

3. Let $G$ be a Lie group. Show that there exists a neighborhood of the identity $e \in U \subset G$ such that if $H \subset U$ is a subgroup then $H = \{e\}$ (“a Lie group has no small subgroups”).

4. Let $\nabla$ be a linear connection on a Lie group $G$. We say that $\nabla$ is left invariant if the connection is preserved by left translations:
   \[
   \nabla_{(L_g)_*} V (L_g)_* W = (L_g)_* (\nabla V W), \quad \forall V, W \in \mathfrak{X}(G), \ g \in G.
   \]
   Show that:
   (a) The assignment defined by:
   \[
   B(X,Y) := \nabla_X Y, \quad X, Y \in \mathfrak{g}
   \]
   establishes a 1:1 correspondence between left invariant connections $\nabla$ on $G$ and bilinear maps $B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$;
   (b) If a left invariant connection $\nabla$ corresponds to the bilinear map $B$, a curve $t \mapsto \exp(tX)$ is a geodesic for $\nabla$ if and only if $B(X, X) = 0$.
(c) $G$ has a unique connection $\nabla$ which invariant under left translations, right translations and inversion;

(d) Show that the connection in (c) has vanishing torsion ($T = 0$) and parallel curvature ($\nabla R = 0$).

(HINT: For (c), consider the bilinear form $B : g \times g \to g$ given by $B(X, Y) = \frac{1}{2}[X, Y]$. For (d), it is enough to compute $T$ and $\nabla R$ on left invariant vector fields.)

5. Let $G$ be a Lie group with Lie algebra $g$. A Riemannian metric $\eta$ on $G$ is called left invariant if left translations are isometries, i.e.,

$$\eta_{gh}(dL_g \cdot v, dL_g \cdot w) = \eta_h(v, w),$$

for all $g, h \in G$ and $v, w \in T_h G$. Show that:

(a) The assignment defined by:

$$(X, Y) := \eta(X, Y), \quad X, Y \in g$$

establishes a 1:1 correspondence between left invariant Riemannian metrics $\eta$ on $G$ and inner products $(\cdot, \cdot) : g \times g \to \mathbb{R}$;

(b) If $G$ is connected, a left invariant Riemannian metric $\eta$ is also right invariant if and only if $(\cdot, \cdot)$ is ad-invariant, i.e.,

$$([X, Y], Z) + (Y, [X, Z]) = 0, \quad \forall X, Y \in g;$$

(c) If $\eta$ is a bi-invariant Riemannian metric on $G$ then the geodesics through $e$ are the curves $t \mapsto \exp(tX)$, with $X \in g$;

(HINT: Consider the Levi-Civita connection $\nabla$ associated with $\eta$ and use the previous problem.)

REMARK: We will see that if $G$ is a compact Lie group then it admits a bi-invariant Riemannian metric. Since $G$ is compact, this metric is necessarily complete, so it follows from (c) that the exponential map $\exp : g \to G$ is surjective.