(1) Note for this one: in the case where all subgroups involved are closed Lie subgroups, these proofs may be much easier using the closed subgroup theorem; however, as shown below, they hold in general as well!

(a) Let \( \text{Lie}(H_1) = \mathfrak{h}_1 \) and \( \text{Lie}(H_2) = \mathfrak{h}_2 \). It is easy to check that, since \( \mathfrak{h}_1 \) and \( \mathfrak{h}_2 \) are subalgebras of \( \text{Lie}(G) = \mathfrak{g} \), \( \mathfrak{h}_1 \cap \mathfrak{h}_2 \) is also a Lie subalgebra. Let \( K \) be the maximal leaf through the identity of the distribution generated by \( \mathfrak{h}_1 \) and \( \mathfrak{h}_2 \). On a small neighborhood \( V \) of \( 0 \in \mathfrak{h}_1 \cap \mathfrak{h}_2 \), \( \exp : V \to \exp(V) := U \) is a diffeomorphism. However, by the naturality of \( \exp \), since \( \mathfrak{h}_1 \cap \mathfrak{h}_2 \) is a subset of \( \mathfrak{h}_1 \) and \( \mathfrak{h}_2 \), \( U \subset H_1 \cap H_2 \). Therefore, as \( U \) generates \( K \), \( K \subset H_1 \cap H_2 \).

Now, note that via the diffeomorphisms \( L_g \) for any \( g \in H_1 \cap H_2 \), we may extend both the topology and smooth structure of \( K \) to every set \( gK \) for \( g \in H_1 \cap H_2 \). In our hypothesized Lie subgroup structure, \( K \) will serve as the connected component of the identity while each \( gK \) will serve as a coset. It is clear this defines a smooth structure for \( H_1 \cap H_2 \) but it remains to be demonstrated that this induced topology (i.e., the topology of some number of disjoint copies of \( K \)) is second-countable.

Again, take \( V \) a small neighborhood of \( 0 \in \mathfrak{h}_1 \cap \mathfrak{h}_2 \) for which \( \exp : V \to \exp(V) := U \) is a diffeomorphism. Without loss of generality, by shrinking \( V \) as necessary, we may assume that \( V = V_1 \cap V_2 \) for \( V_i \), a neighborhood of \( 0 \) in \( \mathfrak{h}_i \) on which \( \exp \) is a local diffeomorphism. So because \( \exp \) is a diffeomorphism on \( V_i \), \( \exp(V_1) \cap \exp(V_2) \). It follows that, for any open subset \( W \) of \( K \) containing \( e \), there exist open \( W_1 \subset H_1 \), \( W_2 \subset H_2 \) such that \( W_1 \cap W_2 \) is open in \( K \) and \( e \in W_1 \cap W_2 \subset W \). Via left translations, this holds for every \( g \in H_1 \cap H_2 \). Thus for \( B \) any bases of \( H_1 \) and \( H_2 \), \( B := \{ B_1 \cap B_2 \cap K \mid B_i \in B_i \} \) is a basis for the topology of \( H_1 \cap H_2 \).

In particular, since \( H_1 \) and \( H_2 \) are second countable, we may choose \( B \) to be countable and therefore \( H_1 \cap H_2 \) admits a countable basis.

(b) This one is fairly simple once we know that the quotient of a Lie group by a closed Lie subgroup is again a Lie group. But as we haven’t proven this in class yet, we may instead proceed as follows: let \( \varphi : \mathfrak{g} \to \mathfrak{k} \) be the map of Lie algebras induced by \( \phi : G \to K \). As \( \varphi \) is a map of Lie algebras, it is easy to check that \( \mathfrak{h} := \varphi(\mathfrak{g}) \) is a Lie subalgebra of \( \mathfrak{k} \). Let \( H \) be the maximal leaf of the distribution generated by \( \mathfrak{h} \) through \( e \in K \). Near \( e \in H \), we have via the naturality of \( \exp \) that \( \exp(\varphi(v)) = \phi(\exp(v)) \). So \( H \) is in the image of \( \phi \) in a neighborhood of the identity and, as \( \phi \) is a homomorphism, this implies that \( H \subset \phi(G) \).

As in the previous problem, we may extend this to topology and a smooth structure for all of \( \phi(G) \) but must show then that, with respect to this induced topology, \( \phi(G) \) is second countable.

So, let \( V_G \subset \mathfrak{g} \) and \( V_H \subset \mathfrak{h} \) be two neighborhoods of \( 0 \) for which \( \exp : V_G \to \exp(V_G) := U_G \) and \( \exp : V_H \to \exp(V_H) := U_H \) are diffeomorphisms. Without loss of generality, by shrinking \( V_G \) as necessary, we may assume that \( \varphi(V_G) = V_H \).
Then via the naturality of exp, \( \phi(U_G) = U_H \). It follows that \( \phi \) maps \( G_0 \) onto \( H \). Similarly, \( \phi \) maps each connected component \( gG^o \) onto the connected component \( \phi(g)H \) of \( \phi(G) \). Thus, \( \phi(G) \) has at most the same number of connected components as \( G \). As \( G \) is a Lie group, it has only countably many connected components. This is enough to conclude \( \phi(G) \) is second countable.

(c) Again, let \( \varphi : \mathfrak{g} \to \mathfrak{k} \) be the map of Lie algebras induced by \( \phi : G \to K \). Then for \( \text{Lie}(H) = \mathfrak{h} \), it is not hard to check that, since \( \varphi \) is a map of Lie algebras, \( \varphi^{-1}(\mathfrak{h}) \) is a Lie subalgebra of \( \mathfrak{g} \). Let \( A \) be the maximal leaf through the identity of the distribution generated by \( \varphi^{-1}(\mathfrak{h}) \). As above, using the naturality of exp, we may conclude that there is a neighborhood \( U \) of \( e \in A \) with \( U \subset \phi^{-1}(H) \) and therefore, since \( \phi \) is a homomorphism, \( A \subset \phi^{-1}(H) \). Indeed, \( A \subset \phi^{-1}(H^o) \) and \( \phi \) maps \( A \) onto \( H^o \). As before, we topologize and give \( \phi^{-1}(H) \) a smooth structure via diffeomorphisms applied to \( A \).

To check for second countability, we will show that there are countably many components in \( \phi^{-1}(hH^o) \) for each \( h \in H \). Since there are only countably many cosets \( hH^o \) of \( H \), this would imply that, in total, there are only countably many cosets of \( \phi^{-1}(H) \).

So, fix \( h \in H \) and \( h_1 \in \phi^{-1}(hH^o) \). Note that, since \( \ker(\varphi) \subset \varphi^{-1}(\mathfrak{h}) \), \( \ker(\phi)^o \subset A \). It follows that \( h_1 \ker(\phi)^o \subset h_1A \). Suppose \( h_2 \) is also in \( \phi^{-1}(hH_0) \). Then we must have \( h_2 = h_1k \) for some \( k \in \ker(\phi) \). So \( h_2A = (h_2h_1^{-1})h_1A = h_1kA \) must contain fully contain \( h_1(k \ker(\phi)^o) \), for some coset \( k \ker(\phi)^o \) of \( \ker(\phi) \). That is, each coset \( h_2A \) of \( \phi^{-1}(hH^o) \) fully contains a translate of a coset of \( \ker(\phi) \). Therefore, there at most as many cosets \( h'A \) in \( \phi^{-1}(hH^o) \) as there are cosets of \( \ker(\phi) \). As we already know that \( \ker(\phi) \) is a closed Lie group and therefore only has a countable number of cosets, the result follows.

(2) First, let’s show \( \text{exp} : \mathfrak{gl}(2, \mathbb{C}) \to GL(2, \mathbb{C}) \) is surjective. Fix \( A \in GL(2, \mathbb{C}) \) and let \( \lambda_1, \lambda_2 \in \mathbb{C}\setminus\{0\} \) be the eigenvalues of \( A \). First, assume \( \lambda_1 \neq \lambda_2 \). Then \( A \) is diagonalizable, so for \( D \) the matrix

\[
D = \begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix}
\]

there exists a matrix \( P \in GL(2, \mathbb{C}) \) with \( A = PDP^{-1} \). Recall now that the matrix exponential map satisfies \( e^{PDP^{-1}} = Pe^{P}P^{-1} \). So to show that \( A \) is in the image of \( \text{exp} \), it is enough to show that \( D \) is in the image of \( \text{exp} \).

But, for any diagonal matrix \( D' \in \mathfrak{gl}(2, \mathbb{C}) \),

\[
\text{if } D' = \begin{bmatrix}
z_1 & 0 \\
0 & z_2
\end{bmatrix}, \text{ then } \exp(D') = \begin{bmatrix}
e^{z_1} & 0 \\
0 & e^{z_2}
\end{bmatrix}
\]

As \( \text{exp} : \mathbb{C} \to \mathbb{C} \) has image \( \mathbb{C}\setminus\{0\} \) and \( \lambda_1 \lambda_2 \neq 0 \), \( D \) must be in the image of \( \text{exp} : \mathfrak{gl}(2, \mathbb{C}) \to GL(2, \mathbb{C}) \) and so, as explained above, \( A \) is also in the image of \( \text{exp} \).

So suppose \( \lambda := \lambda_1 = \lambda_2 \). Then we may have that the Jordan normal form to \( A \) is of the form

\[
J = \begin{bmatrix}
\lambda & 1 \\
0 & \lambda
\end{bmatrix}
\]
For any \( \sigma, \tau \in \mathbb{C} \), the matrices \[
\begin{pmatrix}
\sigma & 0 \\
0 & \tau
\end{pmatrix}
\] and \[
\begin{pmatrix}
0 & \sigma \\
0 & 0
\end{pmatrix}
\] commute, (i.e., their Lie bracket is zero), therefore, we must have that

\[
\exp \left( \begin{pmatrix}
\sigma & 0 \\
0 & \tau
\end{pmatrix} + \begin{pmatrix}
0 & \sigma \\
0 & 0
\end{pmatrix} \right) = \exp \left( \begin{pmatrix}
\sigma & 0 \\
0 & \tau
\end{pmatrix} \right) \exp \left( \begin{pmatrix}
0 & \sigma \\
0 & 0
\end{pmatrix} \right) = \begin{pmatrix}
e^\tau & e^\sigma \\
e^\sigma & e^\tau
\end{pmatrix}
\]

So for \( \tau \) satisfying \( e^\tau = \lambda \), it follows that \( J = \exp \left( \begin{pmatrix}
\lambda^{-1} & 0 \\
0 & \lambda
\end{pmatrix} \right) \). As in the case where \( J \) was diagonal, it follows that \( A \) is in the image of \( \exp \).

Now, let’s look at the real case. Note that the natural inclusion \( GL(2, \mathbb{R}) \to GL(2, \mathbb{C}) \) makes \( GL(2, \mathbb{R}) \) into a (closed) Lie subgroup of \( GL(2, \mathbb{C}) \) (thinking about all objects involved as real Lie groups!). It follows by the naturality of \( \exp \) that \( \exp : \mathfrak{gl}(2, \mathbb{C}) \to GL(2, \mathbb{C}) \) takes \( \mathfrak{gl}(2, \mathbb{R}) \subset \mathfrak{gl}(2, \mathbb{C}) \) to \( GL(2, \mathbb{R}) \subset GL(2, \mathbb{C}) \).

It follows from above that \( \exp \) takes a matrix with eigenvalues \( \lambda_1, \lambda_2 \) to a matrix with eigenvalues \( e^{\lambda_1} \) and \( e^{\lambda_2} \). So \( \exp \) cannot map a matrix with real eigenvalues to a matrix with negative eigenvalues. Recall however that a matrix \( A \in \mathfrak{gl}(2, \mathbb{R}) \) may have complex eigenvalues \( \lambda_1 \) and \( \lambda_2 \), but they must satisfy \( \sum_1 = \lambda_2 \). So if \( B \in GL(2, \mathbb{R}) \) is the image of a matrix \( A \in \mathfrak{gl}(2, \mathbb{R}) \) with complex eigenvalues \( \lambda, \bar{\lambda} \), then \( B \) has eigenvalues \( e^\lambda, e^{-\lambda} \). Therefore, the eigenvalues of \( B \) satisfy \( |e^\lambda| = |e^{-\lambda}| \). So the matrix stated in the statement of the problem (or indeed any matrix in \( GL(2, \mathbb{R}) \) with two different negative eigenvalues) cannot be in the image of \( \exp \).

(3) Let \( \mathfrak{g} := Lie(G) \) and let \( V \) be an open neighborhood of \( 0 \in \mathfrak{g} \) such that \( \exp \mid_V \) is a diffeomorphism onto its image. Without loss of generality, we may assume that \( V \) is bounded and \( V \) is star-shaped with respect to \( 0 \) (i.e., \( V \) is closed under scalar multiplication by scalars \( \lambda \in [0,1] \)). Let \( V' = \{ v \in V \mid 2v \in V \} \) and let \( U := \exp(V') \).

Suppose \( H \subset U \) is a Lie subgroup. Fix \( h \in H \) and assume \( h \neq e \). Then by design, there exists non-zero \( X \in V' \) with \( \exp(X) = h \). Since \( V \) is bounded, there exists a largest integer for \( n \) such that \( 2^n X \in V \). By definition, we must have \( 2^n X \in V \setminus V' \). Since \( \exp \) is a bijection on \( V \), \( \exp(V \setminus V') \cap U = \emptyset \). But \( \exp(2^n X) = h^{2^n} \in H \). Thus, we must have \( H = \{ e \} \).

(4) Left invariant connections:

(a) Fix a basis \( \{ X_1, \ldots, X_n \} \in T_1 G \). Then, via left translations, this defines a basis to the tangent space at every point \( g \in G \). So the left invariant vector fields \( \{ X_1, \ldots, X_n \} \in \mathfrak{g} \) define a \( C^\infty(G) \)-basis for \( \mathcal{X}(G) \). That is, for every \( Y \in \mathcal{X}(G) \), there are unique functions \( f_1, \ldots, f_n \in C^\infty(G) \) for which \( Y = \sum f_i X_i \).

Now, let \( Y = \sum f_i X_i \) and \( Z = \sum g_i X_i \). Then, purely via the required properties of a linear connection, recall we have that any connection \( \nabla \) must satisfy

\[
\nabla_Y Z = \sum_i f_i \nabla_{X_i} Z = \sum_i f_i \left( \sum_j (g_j \nabla_{X_i} X_j + X_i(g_j) X_j) \right)
\]

So to define \( \nabla \), it is enough to determine \( \nabla_{X_i} X_j \in \mathcal{X}(G) \) for every \( i \) and \( j \) and, in turn, these choices determine \( \nabla \) uniquely.

So, in the case where \( \nabla \) is left-invariant, it is clear that \( \nabla_{X_i} X_j \in \mathfrak{g} \) (i.e., \( \nabla_{X_i} X_j \) is left-invariant). As \( \nabla \) is linear in both input slots, the definition \( \nabla_{X_i} X_j \in \mathfrak{g} \)
for each pair of basis elements $X_i$ and $X_j$ of $\mathfrak{g}$ extends to a unique bilinear map $B_\nabla : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

On the other hand, given a bilinear map $B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ $B$ defines a left-invariant connection $\nabla^B$ via $\nabla^B_{X_j}X_i := B(X_i, X_j)$ (as explained above). It is clear that for any bilinear $B$ and left-invariant connection $\nabla$, $B_\nabla = B$ and $\nabla^B_\nabla = \nabla$.

For below: note that for any $X \in \mathfrak{g}$, $X = \sum c_i X_i$ for constants $c_i$; thus, for $X, Y \in \mathfrak{g}$, $\nabla_X Y = B_\nabla(X, Y)$.

(b) Recall that a geodesic for a connection $\nabla$ is a path $\gamma(t)$ satisfying $\nabla_\gamma(t)\dot{\gamma}(t) = 0$.

So for $\nabla$ corresponding to bilinear $B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $\gamma(t) := \exp(tX)$ is a geodesic if and only if $\nabla_X X = B(X, X) = 0$.

(c) If $\nabla$ is left invariant, right invariant, and inversion invariant, we can show that, for $\gamma : [0, 1] \to G$ with $\gamma(t) := \exp(tX)$, parallel transport along $\gamma(t)$ is given by $d_e(\exp(tX/2) \circ \exp(tX/2))$. Assuming this is true, we have that, for $X, Y \in \mathfrak{g}$,

$$\nabla_X Y = \frac{d}{dt} \bigg|_0 d_e(\exp(tX/2) \circ \exp(tX/2))(Y) = \frac{d}{dt} \bigg|_0 d_{\exp(tX/2)}R_{\exp(tX/2)}(\exp(tX/2))(Y) = [X/2, Y] = \frac{1}{2}[X, Y]$$

(this is the formula that allows us to recover a connection from its parallel transport).

So, let’s show that parallel transport along $\gamma(t)$ is given by $d_e(\exp(tX/2) \circ \exp(tX/2))$. As $\nabla$ is left invariant, right invariant, and inversion invariant, then the map $\Phi(g) := (\exp(tX/2) \circ \exp(tX/2))(g^{-1})$ is an affine transformation (i.e., it preserves $\nabla$). Now, note that $d_{\exp(tX/2)}\Phi = -\text{id}$ (this is not so hard to check as any curve through $\exp(X/2)$ is tangent to a curve of the form $\exp(X/2) \cdot \exp(tY)$).

So, for $v \in T_eG$, let $v(t)$ denote the parallel transport of $v$ along $\gamma(t)$. Note that $\Phi(\gamma(t)) = \exp((1 - t)X) = \gamma(1 - t)$. Then since $\Phi$ is affine, we have that $d_{\exp(tX)}\Phi(v(t))$ is the parallel transport of $d_e\Phi(v)$ along $\Phi \circ \gamma = \gamma(1 - t)$. For $t = 1/2$, we have that $d_{\exp(tX)}\Phi(v(1/2)) = -v(1/2)$. Via the properties of parallel transport, we must have that $d_{\exp(tX)}\Phi(v(t)) = -v(1 - t)$. So $v(1) = -v(0) = -d_e\Phi(v) = d_e(\exp(tX/2) \circ \exp(tX/2))$.

It is easy to check that the connection generated by $B(X, Y) := \frac{1}{2}[X, Y]$ is also right-invariant and inversion invariant.

(d) As above, since $\{X_i\}$ yield a $C^\infty(G)$-basis of $\mathcal{X}(G)$ (for $\{X_i\}$ a basis of $\mathfrak{g}$), it is enough to compute $T \nabla$ and $\nabla R$ for left-invariant vector fields.

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = \frac{1}{2}([X, Y] - [Y, X]) - [X, Y] = 0$$

To compute $\nabla R$, let’s first compute $R$:

$$R(X, Y)(Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z = \frac{1}{4}[X, [Y, Z]] - \frac{1}{4}[Y, [X, Z]] - \frac{1}{2}[[X, Y], Z]$$

$$= \frac{1}{4}[X, [Y, Z]] + \frac{1}{4}[Y, [Z, X]] + \frac{1}{2}[Z, [X, Y]]$$

$$= \frac{1}{4}[Z, [X, Y]]$$

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Then:
\[ \nabla^R(W, X, Y, Z) = \nabla_W(\nabla^R(X, Y, Z)) - \nabla_W(\nabla^R(X, Y)) - \nabla_W(\nabla^R(X)) - \nabla_W(\nabla^R(Y)) + \nabla_W(\nabla^R(Z)) \\
= \frac{1}{8}([W, [Z, [X, Y]]] - [Z, [W, X]] - [Z, [W, Y]] - [W, Z, [X, Y]]) \\
= \frac{1}{8}([W, [Z, [X, Y]]] - [W, Z, [X, Y]] + [Z, [W, Y]]) \\
= \frac{1}{8}([W, [Z, [X, Y]]] - [Z, [W, Y]] - [W, Z, [X, Y]]) \\
= \frac{1}{8}([W, [Z, [X, Y]]] + [Z, [X, Y]] - [W, Z]) - [X, Y]) \\
= \frac{1}{8}([X, Y], [W, Z]) - [W, Z, [X, Y]]) = 0 \\
\]

(5) Left invariant metrics:

(a) First, recall that, for any Riemannian metric \( \eta \) on \( G \), \( \eta_g \) is an inner product on \( T_e G \) for every \( g \in G \). In particular, it defines an inner product \( (\cdot, \cdot) = \eta_e \) on \( T_e G \).

On the other hand, given an inner product \( (\cdot, \cdot) \) on \( \mathfrak{g} \), we may define a left-invariant Riemannian metric \( \eta_g(v, w) := (dL_{g^{-1}}(v), dL_{g^{-1}}(w)) \). It is clear this establishes a bijection between left-invariant Riemannian metrics and inner products on \( \mathfrak{g} \).

(b) If \( \eta \) is bi-invariant, then \( (dL_g(dR_{g^{-1}}(v)), dL_g(dR_{g^{-1}}(w))) = (Ad_g(v), Ad_g(w)) \) for all \( g \in G \). Therefore, for every \( X \in \mathfrak{g} \),

\[
0 = \left. \frac{d}{dt} \right|_{t=0} (Ad_{d\exp(tX)}(Y), Ad_{d\exp(tX)}(Z)) = (d(X)Y, Z) + (Y, d(X)Z) \tag{1}
\]

On the other hand, if the above holds, then \( (\cdot, \cdot) \) must be right invariant in a small neighborhood of the identity (i.e., on which \( \exp \) diffeomorphically maps to). To see this, note:

\[
0 = (d(X)Y, Z) + (Y, d(X)Z) = \left. \frac{d}{dt} \right|_{t=0} (dR_{d\exp(tX)}Y, dR_{d\exp(tX)}Z) \\
= \left. \frac{d}{dt} \right|_{t=0} (dR_{d\exp(tX)}Y_{d\exp(sX)}, dR_{d\exp(tX)}Z_{d\exp(sX)}) \\
= \left. \frac{d}{dt} \right|_{s} (dR_{d\exp(tX)}Y, dR_{d\exp(tX)}Z)
\]

i.e., \( (dR_{d\exp(tX)}Y, dR_{d\exp(tX)}Z) \) is constant for all \( t \).

As \( G \) is connected and \( \eta \) is right-invariant in a neighborhood of \( e \), it must be right-invariant everywhere.

(c) Again, as a basis of left-invariant vector fields forms a \( C^\infty(G) \)-basis of \( \mathcal{X} \), we may assume we are working everywhere with left-invariant vector fields only. Note then that \( \nabla \) induced by \( \frac{1}{2}[X, Y] \) is a metric connection; for any \( X, Y, Z \in \mathfrak{g} \):

\[
0 = \nabla_X \eta(Y, Z) = \eta(\nabla_X Y, Z) + \eta(X, \nabla_X Z)
\]
where the left equality holds because $\eta$ left-invariant implies $\eta(Y, Z)$ is a constant and the right equality holds because it translates exactly to (1), up to a factor of $1/2$ (note we only needed connectedness for the other direction of b). So $\nabla$ induced by $\frac{1}{2}[X, Y]$ is metric and torsion invariant (i.e., it is the Lev-Civita connection of $\eta$). It follows that geodesics of $\eta$ are geodesics of $\nabla$ and, by part b of the previous problem, it follows that $\exp(tX)$ is a geodesic for any $X \in \mathfrak{g}$. 