1. The complex classical Lie groups are:
   - $SL(n + 1, \mathbb{C}) = \{ A \in GL(n + 1, \mathbb{C}) : \det A = 1 \}$;
   - $SO(2n + 1, \mathbb{C}) = \{ A \in SL(2n + 1, \mathbb{C}) : A^T A = I \}$;
   - $Sp(n, \mathbb{C}) = \{ A \in GL(2n, \mathbb{C}) : A^T J A = J \}$;
   - $SO(2n, \mathbb{C}) = \{ A \in SL(2n, \mathbb{C}) : A^T A = I \}$;

   where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Show that these are complex Lie groups with Lie algebras:
   - $A_n := \mathfrak{sl}(n + 1, \mathbb{C}) = \{ X \in \mathfrak{gl}(n + 1, \mathbb{C}) : \text{tr} X = 0 \}$;
   - $B_n := \mathfrak{so}(2n + 1, \mathbb{C}) = \{ X \in \mathfrak{gl}(2n + 1, \mathbb{C}) : X^T + X = 0 \}$;
   - $C_n := \mathfrak{sp}(n, \mathbb{C}) = \{ X \in \mathfrak{gl}(2n, \mathbb{C}) : X^T J + JX = 0 \}$;
   - $D_n := \mathfrak{so}(2n, \mathbb{C}) = \{ X \in \mathfrak{gl}(2n, \mathbb{C}) : X^T X = 0 \}$;

2. The compact classical Lie groups are:
   - $SU(n + 1) = \{ A \in SL(n + 1, \mathbb{C}) : A^T A = 1 \}$;
   - $SO(2n + 1) = \{ A \in SL(2n + 1, \mathbb{R}) : A^T A = I \}$;
   - $Sp(n) = \{ A \in U(2n) : A^T J A = J \}$;
   - $SO(2n) = \{ A \in SL(2n, \mathbb{R}) : A^T A = I \}$;

   Show that these are real Lie groups and find their Lie algebras.

3. Show that, up to isomorphism, there are exactly two Lie algebras of dimension 2 over $\mathbb{R}$.

4. Let $G$ be a Lie group and let $\alpha : \mathbb{R} \to G$ and $\beta : \mathbb{R} \to G$ be two smooth curves with $\alpha(0) = \beta(0) = e$. Show that the curve $\gamma : \mathbb{R} \to G$ defined by $\gamma(t) = \alpha(t)\beta(t)$ is smooth and that the tangent vectors at the identity are related by:

   $$\dot{\gamma}(0) = \dot{\alpha}(0) + \dot{\beta}(0).$$
5. Recall that the quaternions $\mathbb{H}$ is the division $\mathbb{R}$-algebra of dimension 4 with basis $\{1, i, j, k\}$ and multiplication defined by:

$$ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik, \quad i^2 = j^2 = k^2 = -1.$$ 

Identify the unit 3-sphere with the quaternions of length 1:

$$S^3 = \{q \in \mathbb{H} : ||q|| = 1\}.$$ 

Show that $S^3$ is a Lie group with Lie algebra isomorphic to $\mathbb{R}^3$ with Lie bracket $[u, v] = u \times v$.

6. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. The center of $G$ is the subgroup:

$$Z(G) = \{g \in G : gh = hg, \forall h \in G\}.$$ 

Assuming that $Z(G)$ is a Lie subgroup of $G$, show that its Lie algebra is the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$, i.e.:

$$\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} : [X, Y] = 0, \forall Y \in \mathfrak{g}\}.$$ 

Can you prove that $Z(G)$ is always a Lie subgroup of $G$?