An invitation to Poisson geometry and its applications

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Contents:

- Poisson brackets and Hamiltonian dynamics
- Poisson manifolds
- Local Poisson geometry
- Global Poisson geometry
- Deformation quantization
Definition
A Poisson bracket on a manifold $M$ is a Lie bracket

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

satisfying the **Leibniz identity**:

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$ 

The pair $(M, \{\cdot, \cdot\})$ is called a Poisson manifold.
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---

**Definition**

A **Poisson map** $\phi : (M_1, \{ \cdot, \cdot \}_1) \to (M_2, \{ \cdot, \cdot \}_2)$ is a smooth map such that pullback is a Lie algebra morphism:

$$\{ f \circ \phi, g \circ \phi \}_2 = \{ f, g \}_1 \circ \phi, \quad \forall f, g \in C^\infty(M_2).$$
Hamiltonian Dynamics

On a Poisson manifold \((M, \{\cdot, \cdot\})\) a function \(h \in C^\infty(M)\) determines a **hamiltonian vector field** \(X_h\) by:

\[
X_h(f) := \{h, f\}, \quad \forall f \in C^\infty(M).
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Basic Properties

▶ \(l\) is a first integral of \(X_h\) if and only if \(\{h, l\} = 0\);

▶ \(h\) is always a first integral of \(X_h\);

▶ If \(l_1\) and \(l_2\) are first integrals of \(X_h\), then \(\{l_1, l_2\}\) is also a first integral of \(X_h\).
Classical Mechanics (Newton’s Equations)

Motion of a particle \( q(t) \in \mathbb{R}^n \) in a potential \( V : \mathbb{R}^n \rightarrow \mathbb{R} \):

\[
m_i \ddot{q}_i(t) = - \frac{\partial V}{\partial q_i} \quad \Leftrightarrow \quad \begin{cases}
\dot{q}_i = \frac{p_i}{m_i} \\
\dot{p}_i = - \frac{\partial V}{\partial q_i}
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\( M = \mathbb{R}^{2n} \) with coordinates \((x_1, \ldots, x_{2n}) = (q_1, \ldots, q_n, p_1, \ldots, p_n)\):

\[
\begin{align*}
\{f_1, f_2\} &= \sum_{i=1}^{n} \left( \frac{\partial f_1}{\partial p_i} \frac{\partial f_2}{\partial q_i} - \frac{\partial f_1}{\partial q_i} \frac{\partial f_2}{\partial p_i} \right) \\
h &= \sum_{i=1}^{n} \frac{p_i^2}{2m_i} + V(q)
\end{align*}
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$$h = \sum_{i=1}^{n} \frac{p_i^2}{2m_i} + V(q)$$

Then Newton’s equations are equivalent to:

$$\dot{x}_a = \{h, x_a\}, \quad (a = 1, \ldots, n)$$
Elasticity (Euler’s Equation)

- Motion of a top in absence of gravity, moving around its center of mass, with moments of inertia $l_1$, $l_2$ and $l_3$:

\[
\begin{align*}
\dot{x}_1 &= \frac{l_2-l_3}{l_2 l_3} x_2 x_3 \\
\dot{x}_2 &= \frac{l_3-l_1}{l_3 l_1} x_3 x_1, \\
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$M = \mathbb{R}^3$ with coordinates $(x_1, x_2, x_3)$:

$$\{f, g\}(x) = (\nabla f(x) \times \nabla g(x)) \cdot x.$$

$$h(x) = \sum_{i=1}^{3} \frac{x_i^2}{2l_i}.$$
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Then Euler’s equations are equivalent to:

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Population Dynamics (Lotka-Volterra equations)

- The dynamics of $n$ biological species $(x_1, \ldots, x_n)$ interacting in a closed ecosystem:

$$\dot{x}_i = \varepsilon_i x_i + \sum_{j=1}^{n} a_{ij} x_i x_j,$$

where $(a_{ij})$ is skew-symmetric and there is a solution $q = (q_1, \ldots, q_n)$.
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$M = \mathbb{R}_+^n$ with coordinates $(x_1, \ldots, x_n)$:

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\{f_1, f_2\}(x) = \sum_{1 \leq i < j \leq n} a_{ij} x_i x_j \frac{\partial f_1}{\partial x_i} \frac{\partial f_2}{\partial x_j}
\]

\[
h(x) = \sum_{i=1}^{n} (q_i \log x_i - x_i)
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Then the Lotka-Volterra equations are equivalent to:

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Problems in Hamiltonian Dynamics

- How does the Poisson geometry constrain the dynamics?
- Is the system stable under perturbation?
- What are symmetries of a system? Reduction using symmetries?
- What is a (completely) integrable system?
- How to build numerical integrators that take into account the Poisson geometry?
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Many open questions beyond the *symplectic case*. 
Poisson tensors

1:1 correspondence:

\[
\left\{ \text{Poisson brackets } \{ \cdot, \cdot \} \right\} \leftrightarrow \left\{ \text{bivector fields } \pi \in \Gamma(\wedge^2 TM) \text{ satisfying } [\pi, \pi] = 0 \right\}
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\pi(df, dg) = \{f, g\}
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\end{cases} \\
&\leftarrow \leftarrow \\
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In a local chart \((U, x^i)\):

\[
\pi|_U = \sum_{i<j} \pi^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \quad \text{where } \pi^{ij} = \{x^i, x^j\}.
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\[
\pi^\# : T^* M \to TM, \alpha \mapsto \pi(\alpha, \cdot),
\]
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\[ \pi^\#: T^*\!M \rightarrow T\!M, \alpha \mapsto \pi(\alpha, \cdot), \]

In this language:

- **Hamiltonian vector field:** \(X_h = \pi^\#(df)\) ("gradient of \(h\)"
- **rank at \(x \in M\):** \(\text{rank}_x \pi = \dim(\text{Im}(\pi^\#))\) (even integer).
Some examples of Poisson manifolds

- **symplectic manifolds**: $(M, \omega)$ where $\omega \in \Omega^2(M)$ is closed and non-degenerate:
  
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Conversely, any Poisson structure $\pi$ with $\text{rank}_x \pi = \dim M$, everywhere, defines a symplectic structure.
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  \[ \{f, g\}(\xi); = \langle [d_\xi f, d_\xi g], \xi \rangle \]
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  Conversely, a Poisson structure on a vector space \(V\) such that the bracket of linear functions is linear, takes this form: \(V = g^*\).
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- **Oriented 3-manifolds:** $(M^3, \mu)$ where $\mu \in \Omega^3(M)$ is a volume form. Every $F \in C^\infty(M)$ determines a Poisson structure:

  \[ \{ f, g \}_F := \mu^{-1}(df, dg, dF) \]
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- **b-symplectic structures:** A symplectic form with a log-type singularity along a divisor $Z \subset M$, determines a *smooth* Poisson structure. In local coordinates:

$$\omega = \frac{1}{x} dx \wedge dy + \sum_{i=1}^{n-1} dq_i \wedge dp_i \iff \pi = x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \sum_{i=1}^{n-1} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$$

- **Poisson-Lie groups:** A Lie group $G$ with a Poisson structure $\pi$ such that the multiplication is a Poisson map:

$$m: (G \times G, \pi \oplus \pi) \to (G, \pi), (g, h) \mapsto gh.$$

These are semi-classical limits of quantum groups (examples can be obtained from solutions of CYBE).

- **Moduli spaces of flat connections:** The moduli space $M$ of principal $G$-bundles with a flat connection over a surface $\Sigma$ with boundary:

$$M = \text{Hom}(\pi_1(\Sigma), G) / G,$$ has a natural Poisson structure (symplectic if $\partial \Sigma = \emptyset$).
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Symplectic Foliation

For any two hamiltonian functions $h_1$ and $h_2$:

$$[X_{h_1}, X_{h_2}] = X_{\{h_1, h_2\}}$$

Define an **equivalence relation** on $M$ by declaring two points equivalent if they can be joined by trajectories of hamiltonian vector fields.
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**Theorem (Weinstein, 1983)**

The equivalence classes form a *singular foliation* of $(M, \{\cdot, \cdot\})$ by *symplectic submanifolds*. 
Examples of symplectic foliations

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  \mathfrak{su}^*(2)
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Examples of symplectic foliations

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\[\text{su}^*(2)\]  \[\text{sl}^*(2, \mathbb{R})\]  \[b\]
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- **Oriented 3-manifolds:** leaves of \((M^3, \mu, F)\) are contained in the level sets of \(F : M^3 \to \mathbb{R}\).
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Local Poisson Geometry

A point $x_0 \in M$ where $\pi$ vanishes is called a **singular point** (so $\{x_0\}$ is a 0-dim symplectic leave).

**Definition**

The **isotropy Lie algebra** of a singular point $x_0$ is:

$$g_{x_0} := T^* M \quad \text{with} \quad [d_{x_0} f, d_{x_0} g] := d_{x_0} \{f, g\}.$$ 

The dual space $T_x M$ with its linear Poisson structure is called the **linear approximation** to $(M, \pi)$ at $x_0$. 
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In local coordinates centered at $x_0$:

$$\{x^i, x^j\}(x) = \{x^i, x^j\}(x_0) + \sum_k \frac{\partial \{x^i, x^j\}}{\partial x^k}(x_0) x^k + o(2).$$

$$= \sum_k c^i_j x^k + o(2).$$
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$$= \sum_k c^{i j}_k x^k + o(2).$$

**Linearization Problem:** Can one choose coordinates around $x_0$ where $\pi$ is linear (no higher order terms)?
**Theorem (Conn, 1985)**

Let $x_0$ be a singular point of $(M, \pi)$. If $\mathfrak{g}_{x_0}$ is a compact semisimple Lie algebra then $\pi$ can be linearized around $x_0$: there are local coordinates $(x^1, \ldots, x^m)$ centered at $x_0$ where the Poisson bracket is linear:

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Remarks:

▶ The original proof used a Nash-Moser fast convergence method, requiring some hard analysis. A (more soft) geometric proof was obtained in 2011 by M. Crainic & RLF.
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Remarks:

- The original proof used a Nash-Moser fast convergence method, requiring some hard analysis. A (more soft) geometric proof was obtain in 2011 by M. Crainic & RLF.

- For other types of singularities one does not know a complete set of invariants.
Global Poisson Geometry - Stability

Stability of leaves: In general, one does not expect symplectic leaves to persist under perturbations of $\pi$: 
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![Diagram of a rolled-up cylinder](image)
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Theorem (M. Crainic & RLF (2010))

Let $L$ be a compact symplectic leaf of $(M, \pi)$ and assume that $H^2_{\pi}(M, L) = 0$.

Then $L$ is stable: every nearby Poisson structure has a family of nearby diffeomorphic leaves smoothly parametrized by $H^1_{\pi}(M, L)$. 

$H^\bullet_{\pi}(M, L)$ is the relative Poisson cohomology, the cohomology of the complex of multivector fields along $L$:

$$X^\bullet(M, L) = \Gamma(\wedge^\bullet T^*M),$$

d$\pi = [\pi, \cdot] : X^\bullet(M, L) \to X^{\bullet+1}(M, L)$.

(this is an elliptic complex). There is also a version for strong stability where “diffeomorphic” is replaced by “symplectomorphic.” The proofs involve some ideas on deforming linear complexes to non-linear complexes, that can be traced back to unpublished work of R. Hamilton on deformations of foliations.
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Global Poisson geometry - Symplectic groupoid

A Poisson bracket makes \((C^\infty(M), \{\cdot, \cdot\})\) into a Lie algebra.

▶ **Question:** Is there a Lie group “integrating” \((M, \{\cdot, \cdot\})\)?

Such a \(\infty\)-dim Lie group, if it exists, should play a fundamental role in global Poisson geometry. Amazingly, the answer is even better:

▶ **Answer:** (M. Karasev; A. Weinstein) There is a group-like object, a symplectic groupoid, associated with every Poisson manifold \((M, \{\cdot, \cdot\})\).

But there are no free meals...

▶ **Addenda:** (M.Crainic & RLF) This object always exists as a topological groupoid, is finite dimensional, but may fail to be smooth. The precise obstructions to smoothness are known.
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$X$ – topological space; look at paths $\gamma : [0, 1] \rightarrow X$
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▶ product:

• $\tau(1)$
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• $\gamma(0)$
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identity:

$$u : X \hookrightarrow \Pi_1(X)$$
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★ inverse:

$\iota : G \longrightarrow G$
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- The space \( \Pi_1(X) \) has a natural topology and the source, target, multiplication and inverse are all continuous maps: \( \Pi_1(X) \xrightarrow{\sim} X \) is an example of a topological groupoid.
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- The space \( \Pi_1(X) \) has a natural topology and the source, target, multiplication and inverse are all continuous maps: \( \Pi_1(X) \rightrightarrows X \) is an example of a topological groupoid.

- If \( X = M \) is a manifold, the space \( \Pi_1(M) \) is a manifold and the source, target, multiplication and inverse are all smooth maps: then \( \Pi_1(M) \rightrightarrows M \) is an example of a Lie groupoid.
\((M, \pi)\) – Poisson manifold; look at cotangent paths:

\[ a : [0, 1] \rightarrow T^* M \]

For any Poisson manifold \((M, \pi)\), there is a topological groupoid \(\Sigma(M) \Rightarrow M\) "integrating" it. \(\Sigma(M) = P(T^* M) // G\) is a symplectic quotient (A. Cattaneo & G. Felder).
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$\Sigma(M) = P(T^* M) // G$ is a symplectic quotient (A. Cattaneo & G. Felder).
$\Sigma(M) \Rightarrow M$ and $\Pi_1(M) \Rightarrow M$ differ substantially:

$\Pi_1(M)$ is always smooth while $\Sigma(M)$ may fail to be smooth;

$\Pi_1(M)$ has one orbit (if $M$ connected) while orbits of $\Sigma(M)$ are the symplectic leaves of $(M, \pi)$;

The homotopy groups $\pi_1(M, x) = \{\text{loops in } M \text{ based at } x\}$ homotopy are discrete while the Poisson homotopy groups $\Sigma(M, x) = \{\text{cotangent loops in } M \text{ based at } x\}$ cotangent homotopy are Lie groups (if smooth).
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Theorem (Crainic & RLF, 2004)

Let \((M, \pi)\) be a Poisson manifold and fix a symplectic leaf \(L\). There is a group morphism

\[
\partial_x : \pi_2(L, x) \to \nu_x^*(L)
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controlling integrability: \(\Sigma(M)\) is smooth if and only if the groups \(\text{Im}(\partial_x)\) are uniformly discrete.
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Theorem (Crainic & Marcut (2012))

Let $(M, \pi)$ be a Poisson manifold. If $\Sigma(M, x)$ is smooth and the source map is proper, then a neighborhood of any symplectic leaf $L$ is Poisson diffeomorphic to the first order model of $\pi$ around $L$.

- There is an explicit local model, which depends on some choices.
- This result can be strengthened by replacing $\Sigma(M)$ by other symplectic groupoids integrating $(M, \pi)$.
- This result can be generalized by replacing the symplectic leaf $L$ by more general Poisson submanifolds.
- Several proofs are available. The most geometric uses a new notion of simplicial metric on the nerve of a groupoid, which has many potential applications (del Hoyo and RLF (2016)).
Definition

A **star product** is an associative product $\star_{\hbar}$ on $C^\infty(M)[[\hbar]]$ deforming the usual product:

$$f \star_{\hbar} g = \sum_{n=0}^{\infty} B_n(f, g) \hbar^n, \quad \text{where } B_0(f, g) = fg.$$
Deformation quantization

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where $B_0(f, g) = fg$.

▶ We assume that $\star_\hbar$ is natural, meaning that each $B_k$ is a bidifferential operator of order $\leq k$. 

$\triangleright$ We have Shr"odinger's Equation:

$$\frac{df}{dt} = \frac{1}{\hbar} [h, f] \star_\hbar \hbar.$$
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- Given $h \in C^\infty(M)$ we have Shrödinger’s Equation:

$$\frac{df}{dt} = \frac{1}{\hbar} [h, f]_{\star_\hbar}$$
Existence of deformation quantizations

Theorem (Kontsevich (2002))

Given a Poisson manifold \((M, \pi)\) there exists a star product \(\star_{\hbar}\) inducing \(\pi\).
Existence of deformation quantizations

Theorem (Kontsevich (2002))

*Given a Poisson manifold* \((M, \pi)\) *there exists a star product* \(\star_\hbar\)
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- This theorem is a consequence of a much more general result, Kontsevich’s Formality Theorem, which asserts the existence of a certain \(L_\infty\)-isomorphism between two DGLA.
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Kontsevich gives an explicit formula for \(\star_\hbar\).
Existence of deformation quantizations

**Theorem (Kontsevich (2002))**

*Given a Poisson manifold \((M, \pi)\) there exists a star product \(\ast_\hbar\) inducing \(\pi\).*

- This theorem is a consequence of a much more general result, Kontsevich’s Formality Theorem, which asserts the existence of a certain \(L_\infty\)-isomorphism between two DGLA.
- Kontsevich gives an explicit formula for \(\ast_\hbar\).
- Kontsevich’s Formality also gives a *classification* of all star products \(\ast_\hbar\) inducing \(\pi\).
Non-formal deformation quantization

Kontsevich’s Theorem gives existence of *formal* star products $\star_\hbar$. What about *non-formal* star products?

Conjecture (RLF, 2018)
If there exists a non-formal star product $\star_\hbar$ inducing $\pi$, then $(M, \pi)$ must be integrable by a symplectic groupoid $G \Rightarrow M$.

Together with Alejandro Cabrera (UFRJ), we have the following strategy to prove this conjecture:

Step 1 From a non-formal star product $\star_\hbar$ construct a *local* symplectic groupoid $G \Rightarrow M$ integrating $(M, \pi)$;

Step 2 Associativity of $\star_\hbar$ implies that $G \Rightarrow M$ satisfies $n$-associativity for all $n \in \mathbb{N}$, i.e., it is *globally* associative.

Step 3 Use result of RLF & Michiels (2018): if $G \Rightarrow M$ is globally associative then it extends to a global symplectic groupoid.

Note: This works in the formal case, producing a *formal* symplectic groupoid (Karabegov, Cattaneo & Felder, Contreras)
Non-formal deformation quantization

Kontsevich’s Theorem gives existence of formal star products $\star \hbar$. What about non-formal star products?

**Conjecture (RLF, 2018)**

*If there exists a non-formal star product $\star \hbar$ inducing $\pi$, then $(M, \pi)$ must be integrable by a symplectic groupoid.*

Together with Alejandro Cabrera (UFRJ), we have the following strategy to prove this conjecture:

**Step 1** From a non-formal star product $\star \hbar$ construct a local symplectic groupoid $G \Rightarrow M$ integrating $(M, \pi)$;

**Step 2** Associativity of $\star \hbar$ implies that $G \Rightarrow M$ satisfies $n$-associativity for all $n \in \mathbb{N}$, i.e., it is globally associative.

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Many other directions in Poisson geometry

▶ $b$-symplectic manifolds: Guillemin, Miranda & Pires; Gualtieri, Pelayo & Ratiu; Marcut & Osorno-Torres . . .

▶ Generalized complex geometry: Hitchin; Bursztyn, Calvacanti & Gualtieri, Baley; . . .

▶ Poisson-Lie groups and Poisson homogeneous spaces: Drinfeld; Semenov-Tian-Shansky; Lu & Evens; Yakimov; Kosmann-Schwarzbach; Reshetikhin . . .

▶ Moduli spaces and twisted-Poisson structures: Alekseev & Meinrenken; Boalch; Li-Bland & Severa; . . .

▶ Cluster algebras: Fomin & Zelevinsky; Gekhtman, Shapiro & Vainshtein; . . .

▶ Poisson manifolds of compact type: Crainic, RLF, Martinez-Torrres, Zung; . . .

▶ . . .
there is still a lot of very tasty *Poisson* to be fished!!!

http://poissongeometry.org