8) Tubular Neighborhood Theorems

One can use the previous results to construct "local models" for
symplectic manifolds around submanifolds.

Thus (Lagrangian Tubular Neighborhood)

Let \((M,\omega)\) be a symplectic manifold and \(L \subset M\) a compact
Lagrangian submanifold. There are neighborhoods \(L \subset N_0 \subset T^*L\)
and \(L \subset N_1 \subset M\) and a symplectomorphism \(\phi : (N_0, \omega_{can}) \to (N_1, \omega|_N)\)
such that:

\[
\begin{array}{ccc}
(T^*L, \omega_{can}) & \xrightarrow{\phi} & N_1 \subset (M, \omega) \\
(\text{zero section}) i_0 & \circlearrowleft & i_1 \ (\text{inclusion})
\end{array}
\]

Proof:

Apply the theorem to \(L \subset (M,\omega)\) and \(0 \subset (T^*L, \omega_{can})\)
and \(\phi = i_0 : L \to O_L\) The zero section.

A codimension 1 submanifold (= hypersurface) of \((M,\omega)\) is
a coisotropic submanifold. We have:

Proposition:

let \(i : C \subset (M, \omega)\) be a compact, orientable, hypersurface. Then there exists
neighborhood \(C \subset N \subset M\) which is symplectomorphic
to \(C \times \mathbb{R} \subset \mathbb{R}^n\) equipped with the symplectic form:

\[
\omega_c = \rho^*_c (i^*\omega) + d(\pm \rho^*_c \alpha),
\]
where the \(\rho^*_c(C)\) is any 1-form such that \(\alpha(\nu) \neq 0, \forall \nu \in \mathfrak{e}_c(i^*\omega)\).
Proof:
\[ d \omega_c = p_c^*(v^d \omega) + d^2 \left( t p_c^* \alpha \right) = 0 \]
\[ \omega_c |_{C \times I_{05}} = (p_c^* (i^* \omega) + t \, dp_c^* \alpha + dt \wedge p_c^* \alpha) |_{C \times I_{05}} \]
\[ = (p_c^* (i^* \omega) + dt \wedge p_c^* \alpha) |_{C \times I_{05}} \]

\[ \Rightarrow \omega_c \text{ is symplectic for } \epsilon < 1. \]

Hence:
\[ C \subset (C \times I_{1}) \subset \omega \text{ is coisotropic} \]
\[ (M, \omega) \text{ coisotropic.} \]

So we can apply coisotropic neighborhood theorem. \( \Box \)

Rmk: Since \( M \) is orientable (it is symplectic!) we have:
\[ C \cap N \text{ is orientable } \iff \mathcal{U}(C) \text{ is orientable (i.e., } C \text{ is co-orientable}) \]
\[ \iff \exists \, x \in \mathfrak{X}(N), \quad T_{C}N = TC \oplus \langle x \rangle \]

Then the 1-form \( \alpha \in \Omega^1(C) \) given by:
\[ \alpha = i^x (l_x \omega) \]

Satisfies:
\[ \alpha (v) \neq 0, \quad \forall v \in \ker i^x \omega \]

Conversely, if such a form \( \alpha \) exists, then let \( \tilde{\alpha} \in \Omega^1(N) \) be an extension to \( M \): \( \alpha = i^* \tilde{\alpha} \). If \( x \in \mathfrak{X}(M) \) is the vector field
\[ l_x \omega = \tilde{\alpha} \]

Then:
\[ T_{C}N = TC \oplus \langle x \rangle \]

so \( C \) is orientable. \( \Box \)
Rmk: Conical Cones Tubular Neighbourhoods Then:

Given \( C \subset (M,\omega) \) conical, there is a "local model", but depends on choices (any different choices lead to local models that are symplectomorphic):

1. \( K\nu(i^*\omega) = (TC)\omega \cap TC \) is a bundle over \( C \)
2. \( \dim (K\nu(i^*\omega)) = \dim (TC)\omega = \dim C + \operatorname{codim} C = \dim N \)

Thus exists a symplectic form \( \omega_c \) in a neighborhood of zero section of \((TC)\omega \) such that \( \iota_0^* \omega_c = i^*\omega \)

To construct \( \omega_c \) need additional data...

In proposition above \( K\nu(i^*\omega) \cong C \times \mathbb{R} \), because \( C \) is orientable, and extra data is \( \rho \).

Conical Cones Tubular Neighbourhoods Then:

If \( C \subset (M,\omega) \) is conical, then a neighborhood of \( C \) in \( M \) is symplectomorphic to a neighborhood of the zero section in \((K\nu(i^*\omega), \omega_c)\).

<table>
<thead>
<tr>
<th>Type</th>
<th>(M,\omega)</th>
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<tbody>
<tr>
<td>Symplectic submanifold S</td>
<td>Need to know symplectic u.v. ((TS)\omega)</td>
</tr>
<tr>
<td>Lagrangian submanifold L</td>
<td>Determined: ((TL \oplus TL, \mathcal{O}_{\text{loc}}))</td>
</tr>
<tr>
<td>Coisotropic submanifold C</td>
<td>Determined: ((K\nu(i^*\omega), \mathcal{O}_C))</td>
</tr>
<tr>
<td>Isotropic submanifold I</td>
<td>Need to know: ((TI)\omega / TI)</td>
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