6) Moser path method

Suppose \( \Lambda_t \in \Omega^2(M) \) is a smooth family of symplectic forms with an exact derivative:

\[
\frac{d}{dt} \omega_t = d\xi_t
\]

Then we can produce an isotopy \( p_t \in \text{Diff}(M) \) such that

\[
p_t^* \omega_t = \omega_0
\]

(unless some completeness assumption). Indeed, as in the proof, if we differentiate:

\[
0 = \frac{d}{dt} p_t^* \omega_t = p_t^* \left( \mathcal{L}_{X_t} \omega_t + \frac{d\omega_t}{dt} \right)
\]

where \( X_t \) is the time-dependent vector field:

\[
X_t(x) = \frac{d}{dt} p_t(y) \quad \text{at} \quad y = p_t^{-1}(x)
\]

and it must satisfy:

\[
\mathcal{L}_{X_t} \omega_t + \frac{d\omega_t}{dt} = \mathcal{L}_{X_t} \omega_t + d\xi_t = 0
\]

\[\Leftrightarrow \quad \mathcal{L}_{X_t} \omega_t = -d\xi_t
\]

Since \( \omega_t \) is symplectic, we can solve the last equation.

Hence, if \( X_t \) has flow \( p_t := \varphi_t^{\ast} \) defines up to time \( 1 \), then (*) holds, so \( \omega_2 \) and \( \omega_0 \) are symplectomophic.
Theorem (Moser-Global Version)

Let $M$ be a closed manifold. If $\omega, \omega_0 \in \Omega^2(n)$ are symplectic forms that can be connected by a smooth family of symplectic forms $\omega_t$ with fixed cohomology class $[\omega_t] = c$, then $\omega_0$ and $\omega$ are symplectomorphic.

Proof:
Because $M$ is compact, we can apply Moser's path method, because

$$[\omega_t] = c \iff \exists \alpha_t \in \Omega^2(n) : d\omega_t = d\alpha_t$$

(For this, one can use, e.g., Hodge Theory, e.g., Poincaré-Lemma for compactly supported cohomology and Mayer-Vietoris.)

So for $M$ closed:

$$S_c = \frac{1}{2} \text{ symplectic forms } \omega \text{ with } [\omega] = c \in H^2(n)$$

All symplectic forms in same path-connected component are symplectomorphic.

Remark: D. McDuff gave example of compact manifold $M$ and symplectic forms $\omega_0, \omega$, with $[\omega_0] = [\omega]$ which are not symplectomorphic (they are not in same path-connected component!).
Then (Moser - Relative version)

Let $M^n$ be a manifold, $Q \subset M$ a compact submanifold, $\omega_0, \omega_1 \in \Omega^2(M)$ closed 2-forms such that:

- For all $q \in Q$, $\omega_0|_q = \omega_1|_q$ are non-degenerate.

Then there exist open neighborhoods $N_0$ and $N_1$ of $Q$ and a diffeo $\phi : N_0 \to N_1$ such that:

$\phi|_Q = \text{id}$, $\phi^* \omega_1 = \omega_0$.

Note: If $Q = \{p\}$, $\Rightarrow$ Dae Uoon's theorem.

Proof: We apply Moser's Theorem to the path

$\omega_t = t \omega_1 + (1-t) \omega_0$.

If we can prove that there exists $\alpha \in \Omega^2(N_0)$ defined on some open neighborhood $Q \subset N_0 \subset M$ such that:

- $\alpha|_Q = 0$, $\forall q \in Q$
- $\frac{d}{dt} \alpha = \frac{d}{dt} \omega_t = \omega_1 - \omega_0$

Then shrinking $N_0$, if necessary, we have:

- $\omega_t$ is non-degenerate in $N_0$
- $X_t$ given by $L_{X_t} \omega_t = -\alpha$, vanishes in $Q$, and $X_t$ has flow $\phi^t$, defined up to $t = 1$ in $N_0$.

$\Rightarrow \phi = \phi_1$. 

Why is there a tubular neighborhood!

Proposition: Let $Q \subset M$ be a closed submanifold and $Q \subset U \subset M$ a tubular neighborhood. Then:

$$H^k(U) \xrightarrow{i^*} H^k(Q)$$

is an isomorphism. Moreover, if we $C^d(U)$ is closed and $i^* \omega = 0$, then there exists $\alpha \in C^d(U)$ with $\alpha|_Q = 0$ such that $\omega = \partial\alpha$.

Detour - Tubular neighborhood:

Thm: Let $Q \subset M$ be an embedded submanifold. There is an embedding $\phi: U(Q) \rightarrow M$ such that the following diagram commutes:

$$\begin{CD}
U(Q) @>\phi>> M \\
@V\iota_0 VV @VVi V \\
Q @>>\iota<
\end{CD}$$

The open set $U = \phi(U(Q))$ is called a tubular neighborhood of $Q$. 

\[ \begin{align*}
\phi & : U(Q) \rightarrow M \\
\iota_0 & : Q \rightarrow U(Q) \\
\iota & : Q \rightarrow M \\
\pi & : U(Q) \rightarrow Q
\end{align*} \]
Locally Proof:

1. Choose metric $g$ on $M$.

2. Identify:

$$U(\Theta) \cong (T\Theta)^\perp \subset T\Theta \cdot M$$

3. Given $v_q \in U(\Theta)$, let

$$\exp(v_q) = \gamma_{v_q}(1), \quad \text{with initial condition } \gamma_{v_q}(0) = v_q$$

4. There exist $\varepsilon : \Theta \to \mathbb{R}^+$ smooth such that

$$\exp : \left\{ v_q \in U(\Theta) : \|v_q\| < \varepsilon(q) \right\} \to M$$

is an embedding. If $\Theta$ is compact, we can take $\varepsilon$ constant.

5. The map $c_\varepsilon : U(\Theta) \to U(\Theta)$ given by

$$c_\varepsilon(v_q) = \varepsilon(q) \frac{\|v_q\|}{1 + \|v_q\|^2} v_q$$

is a diffeomorphism with image in $V^\varepsilon$.

6. The composition $\phi := \exp \circ c_\varepsilon$ is the desired tubular neighborhood map.

The tubular neighborhood $\Theta \subset U \subset M$ comes with a fibration $\pi : U \to \Theta$ which is a deformation retract; if $i : \Theta \to M$, then

$$\pi \circ i = \text{id}_\Theta, \quad i \circ \pi = \text{id}_U$$

So we have:

$$i^* : H^*(U) \to H^*(\Theta)$$ is isomorphism.
To get a statement at level of differential forms one uses homotopy operads:
\[ p_t : U \rightarrow U, \quad (q,v) \mapsto (q,tv) \]

Then:
- \[ p_1 = id \]
- \[ p_0 = i \circ \pi \]
- \[ p_t|_{\partial U} = id_{\partial U} \]

\[ \Rightarrow p_t \text{ is a strong deformation retract} \]

We have a homotopy operad \( H : \Omega^d(U) \rightarrow \Omega^d(U) : \)
\[ T - (i \circ \pi)^* = dH + Hd \quad (\star) \]

Gives by:
\[ H(\omega) = \int_0^1 p_t^* (i_{V_t} \omega) \, dt \quad \text{with} \quad \frac{d}{dt} V_t(p) = \frac{d}{dt} p_t(p) \]

**Remark:** \( V_t = \frac{t}{t} (\text{Euler v.f.}) \) if \( t \neq 0 \), \( V_0 = 0 \) \( \Rightarrow \) not smooth at \( t=0 \). But \( H(\omega) \) is still smooth.

Thus if \( d\omega = 0 \) and \( \omega|_{\partial U} = 0 \), \( \forall q \in G \), we have:
- \( \omega = dH(\omega) \)
- \( H(\omega)_q = 0 \) \( \Rightarrow \) \( V_t(q) = 0 \), \( \forall q \in G \)

so the proposition follows.

**Verification of (\star):**
\[ dH(\omega) + H(d\omega) = \int_0^1 p_t^* (d i_{V_t} \omega + i_{V_t} d\omega) \, dt = \int_0^1 (p_t^* d i_{V_t} \omega) \, dt \]
\[ = \int_0^1 \frac{d}{dt} p_t^* \omega \, dt = p_1^* \omega - p_0^* \omega = \omega - (i \circ \pi)^* \omega \]