3) Submanifolds of symplectic manifolds

Recall

- A \textbf{submanifold} of \( M \) is a pair \((N, i)\) where \( N \) is a manifold and \( i : N \subset M \) is an injective immersion (i.e., \( di_x : T_N \to T_M \) is injective for all \( x \in N \)).

- An \textbf{embedded submanifold} of \( M \) is a submanifold \((N, i)\) where \( i : N \to i(N) \) is a homeomorphism.

- A \textbf{closed submanifold} of \( M \) is an embedded submanifold such that \( i(N) \subset M \) is closed.

- A \textbf{subset} \( N \subset M \) has at most one smooth structure such that \( i : N \subset M \) (inclusion) is an embedding.

- A \textbf{subset} \( N \subset M \) can have many smooth structures such that \( i : N \subset M \) (inclusion) is an injective immersion.

\textbf{Examples:}

- \textbf{Submanifolds:}
  \[
  \begin{align*}
  N &= \mathbb{R}^2 \\
  \mathbb{R}^2 &\hookrightarrow \mathbb{R}^3
  \end{align*}
  \]

- \textbf{Not Embedded:}
  \[
  \begin{align*}
  \mathbb{R} &\hookrightarrow \mathbb{R}^2 \\
  \mathbb{R} &\not\hookrightarrow \mathbb{R}^2
  \end{align*}
  \]
Embedded submanifolds

\[ M = \mathbb{R}^2 \]

\[ M = S^2 \] Not closed

\[ N = \mathbb{R} \]

Closed embedded submanifolds

\[ N = \mathbb{R}^2 \]

\[ M = S^2 \]

\[ N = \mathbb{R} \]

\[ N = S^1 \]

Exercise:

An embedding \( i : N \to M \) is closed iff it is proper

(Note: A map is proper if the inverse image of any compact set is compact.)

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In the book "submanifold" = "closed submanifold" [Z]

Der: Let \((M,\omega)\) be a symplectic manifold and \( i : N \to M \)

a submanifold. We say that:

(i) \( N \) is isotropic if \( T_N \cap T_M \) is isotropic, \( \forall x \in N \)

(ii) \( " \) coisotropic \( " \) \( " \) coisotropic, \( " \)

(iii) \( " \) symplectic \( " \) \( " \) symplectic \( " \)

(iv) \( " \) Lagrangian \( " \) \( " \) Lagrangian \( " \)

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**Rinks:**
1. $N$ is isotropic $\iff i^*\omega = 0$
2. $N$ is involution $\iff i^*\omega = 0 \& \text{dim } N = \frac{1}{2} \text{dim } M$
3. $N$ is Lagrangian $\iff (N, i^*\omega)$ is symplectic.

**Fact:** If $T : M \to N$ is an involution ($T^2 = \text{id}$) then the fixed point set:

$$\text{Fix}(T) = \{ x \in N : T(x) = x \}$$

has connected components which are closed embedded submanifolds.

**Exercise:** Let $T : (N,\omega) \to (N,\omega)$ be an involution. Then

(i) If $T$ is symplectic ($T^*\omega = \omega$), then the connected components of $\text{Fix}(T)$ are symplectic submanifolds.

(ii) If $T$ is anti-symplectic ($T^*\omega = -\omega$), then the connected components of $\text{Fix}(T)$ are Lagrangian submanifolds.

**Examples:**
1. If $N \subset (M,\omega)$ is a symplectic submanifold then $N$ is isotropic.
2. If $N \subset (M,\omega)$ is a coisotropic submanifold then $N$ is coisotropic.
Example: \( M = T^* X, \omega_{can} \)

\[ \text{deficit}(x) \mapsto \alpha : X \rightarrow T^* X, \text{ is a (closed embedded) submanifold} \]

\[ x \mapsto \alpha_x \quad \omega \text{ clvm } = \frac{1}{2} \text{ clvm } T^* X \]

Recall that:

\[ \alpha^* \omega_{can} = d \alpha \]

So \( \text{graph}(\alpha) \subset T^* X \) is lagrangian iff \( \alpha \) is a closed form.

Note: This example fits in the exercise above. Given any \( \alpha \in \Omega^1(X) \), define \( T : T^* X \rightarrow T^* X \) by:

\[ T(\xi_x) = 2\alpha(x) - \xi_x \]

This is a symplectomorphism iff \( d\alpha = 0 \) and we have:

\[ F_{\alpha}(\xi) = \frac{1}{2} \alpha(x), \forall \xi \in T_x X. \]

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Are there other lagrangian submanifolds of \( (T^* X, \omega_{can}) \)?

Yes, e.g., a fiber \( T^* x \subset T^* X \).

Def: let \( Y \subset X \) be a submanifold. The normal bundle is the subbundle of the cotangent bundle:

\[ U(Y) = T_Y X / T_Y \]

(i.e. \( U(Y)_y = T^* X / T^*_Y \))

The conormal bundle is the dual vector bundle \( U^*(Y) \rightarrow Y \)

\[ U^*(Y) = \{ \alpha : U(Y) \rightarrow \mathbb{R} \text{ linear} \} \]

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There is a natural identification of $\mathcal{U}(Y)$ with the subbundle $\mathcal{U}(Y)^0 \subset T^*X$:

$$\mathcal{U}(Y)^0 := \{ \alpha \in T^*_o X : \alpha \big|_{T_o Y} = 0 \}$$

Namely, we have the bundle isomorphism:

$$\mathcal{U}(Y)^0 \xrightarrow{\sim} \mathcal{U}^*(Y)$$

$$d \mapsto ([v], \alpha(v))$$

(This is the bundle version of the statement for vector spaces:

$W \subset V$ subspace $\Rightarrow (V/W)^* \cong W^*$)

There are short exact sequences of vector bundles:

$$0 \rightarrow T Y \rightarrow T Y X \rightarrow \mathcal{U}(Y) \rightarrow 0$$

$$0 \rightarrow (T Y)^0 \rightarrow T^*_Y X \rightarrow (T Y)^* \rightarrow 0$$

**Examples:**

1. If $Y = \{x_0 \}$ $\subset X$ (0-dim submanifold) then:

$$T Y = 0 \subset T X \quad \Rightarrow \quad \mathcal{U}^*(Y) = \{0\}_o = T^*_{x_0} X$$

2. If $\phi : X \rightarrow Z$ is a submersion, then

$$Y = \phi(\bar{z}_0) \quad \text{is a submanifold of } X$$

$$T Y = \ker d \phi \quad \Rightarrow \quad \mathcal{U}^*(Y) = (\ker d \phi)^0 = (d \phi)^*(T^*_0 Z)$$

**Exercise:** Show that $\mathcal{U}^*(Y) \cong \phi^*(T^*_0 Z) | Y$
Proposition: For any submanifold $Y \subset X$, the conormal bundle $U^*(Y) \subset (T^*X, \omega)$ is a Lagrangian submanifold.

Proof:
Since $\dim U^*(Y) = \frac{1}{2} \dim (T^*X)$, we need to check that $i^* \omega = 0$, where $i : U^*(Y) \to T^*X$ is the inclusion. Given $y_0 \in Y$, we check that $(i^* \omega)_{y_0} = 0$, $\forall x \in U^*(Y)_{y_0}$.

Choose coordinates $(U, x^1, \ldots, x^n)$ for $X$ centered at $y_0$, such that:

$$C = \text{Connected Component of } Y \ni y_0$$

Let $(x^1, \ldots, x^n, p_1, \ldots, p_n)$ be the induced coord on $T^*U$ so that:

$$\omega_{can} |_U = \sum_{i=1}^{n} d x^i \wedge dp_i;$$

Then:

$$T(Y \cap U) = \left\langle \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\rangle, \forall x \in C$$

Connected component of $U^*(Y) \cap T^*U$ containing $U^*(Y)_{y_0}$

$$\Rightarrow (i^* \omega) = \sum_{i=1}^{n} i^* dx^i \wedge dp_i + \sum_{i=1}^{n} i^* x^i \wedge dp_i = 0$$