15) Kähler Manifolds

Definition: A Kähler manifold is a triple \((M, \omega, J, g)\) where:

(i) \(\omega\) is a symplectic form;
(ii) \(J\) is a \(\omega\)-compatible, integrable, almost complex structure;
(iii) \(g(\cdot, \cdot) = \omega(\cdot, J\cdot)\) is the associated Riemannian metric.

By the Newlander-Nirenberg Thm, a Kähler manifold is a complex manifold.

Conversely, given a complex manifold \(M\), a real symplectic form \(\omega \in \Omega^2(M)\) defines a Kähler structure in the unique almost complex \(\mathbb{C}\)-structure \(J\) is \(\omega\)-compatible, i.e., if
\[
g(\cdot, \cdot) = \omega(\cdot, J\cdot)
\]
is a Riemannian metric.

Proposition: If \(M\) is a complex manifold and \((\mathbb{C}, z^1, \ldots, z^n)\) are holomorphic coordinates on \(z^i = x^i + y^i\), Then:
\[
T_{1,0} M = \left\{ \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n} \right\} \quad \frac{\partial}{\partial z^i} = i \left( \frac{\partial}{\partial x^i} - \frac{\partial}{\partial y^i} \right)
\]
\[
T_{0,1} M = \left\{ \frac{\partial}{\partial \bar{z}^1}, \ldots, \frac{\partial}{\partial \bar{z}^n} \right\} \quad \frac{\partial}{\partial \bar{z}^i} = i \left( \frac{\partial}{\partial \bar{x}^i} + \frac{\partial}{\partial \bar{y}^i} \right)
\]
\[
T^{1,0} M = \left\{ dz^1, \ldots, dz^n \right\} \quad dz^i = dx^i + dy^i
\]
\[
T^{0,1} M = \left\{ d\bar{z}^1, \ldots, d\bar{z}^n \right\} \quad d\bar{z}^i = dx^i - dy^i
\]
Proof:
We have that:
\[ \mathcal{O}(\frac{\partial}{\partial \bar{z}^i}) = \frac{\partial}{\partial \bar{z}^i}, \quad \mathcal{O}(\frac{\partial}{\partial z^i}) = -\frac{\partial}{\partial z^i}. \]

So:
\[ \mathcal{O}(\frac{\partial}{\partial \bar{z}^i}) = \frac{1}{2} \left( \frac{\partial}{\partial \bar{z}^i} + \bar{\eta} \frac{\partial}{\partial z^i} \right) = \sqrt{\eta} \frac{\partial}{\partial z^i} \rightarrow \frac{\partial}{\partial z^i} \in T_{\eta^1} \mathcal{N} \]

Thus, \( \dim \mathcal{O}(T_{\eta^1} \mathcal{N}) = m \), the result follows.

The other equalities are similar.

So:
\[ \Omega^{1,0}(\mathcal{U}) = \left\{ \sum_{i=1} \Sigma b_i d\bar{z}^i : b_i \in \mathcal{C}^\infty(\mathcal{U}, \mathcal{E}) \right\} \]
\[ \Omega^{0,1}(\mathcal{U}) = \left\{ \sum_{i=1} b_i d\bar{z}^i : b_i \in \mathcal{C}^\infty(\mathcal{U}, \mathcal{E}) \right\} \]
\[ \Omega^{1,1}(\mathcal{U}) = \left\{ \sum_{i\neq j} a_{ij} d\bar{z}^i d\bar{z}^j : a_{ij} \in \mathcal{C}^\infty(\mathcal{U}, \mathcal{E}) \right\} \]
\[ \Omega^{1,0}(\mathcal{U}) = \left\{ \sum_{i\neq j} a_{ij} d\bar{z}^i d\bar{z}^j : a_{ij} \in \mathcal{C}^\infty(\mathcal{U}, \mathcal{E}) \right\} \]
\[ \Omega^{0,2}(\mathcal{U}) = \left\{ \sum_{i\neq j} a_{ij} d\bar{z}^i d\bar{z}^j : a_{ij} \in \mathcal{C}^\infty(\mathcal{U}, \mathcal{E}) \right\} \]

etc.

Do not confuse \((k,0)\)-forms as holomorphic forms!!

A holomorphic form locally takes the form:
\[ \sum a_{i_1...i_k} d\bar{z}^{i_1} ... d\bar{z}^{i_k} \]
where \(a_{i_1...i_k}\) are holomorphic functions: \( \overline{\partial} a_{i_1...i_k} = 0 \).
**Proposition 2:** For a complex manifold $M$ one has:

$$d = \partial + \bar{\partial}$$

and moreover:

$$d^2 = \partial \bar{\partial} + \bar{\partial} \partial = \partial^2 = 0$$

**Proof:**

Since $\mathbf{T} \otimes \mathbb{C}$ is im local holomorphic chart has basis:

$$dz^1, \ldots, dz^n, d\bar{z}^1, \ldots, d\bar{z}^n$$ (exact theory!)

We can write a $(K, i)$-form, as:

$$\beta = \sum_{i_k < \ldots < i_1} \beta_{i_1 \ldots i_n} dz^{i_1} \wedge \ldots \wedge d\bar{z}^{i_n}$$

Hence:

$$d\beta = \sum_{i_k < \ldots < i_1} d\beta_{i_1 \ldots i_n} dz^{i_1} \wedge \ldots \wedge d\bar{z}^{i_n}$$

$$= \sum_{i_k < \ldots < i_1} \left( \partial \beta_{i_1 \ldots i_n} + \bar{\partial} \beta_{i_1 \ldots i_n} \right) dz^{i_1} \wedge \ldots \wedge d\bar{z}^{i_n}$$

$$= \sum_{i_k < \ldots < i_1} \partial \beta_{i_1 \ldots i_n} dz^{i_1} \wedge \ldots \wedge d\bar{z}^{i_n} +$$

$$\sum_{i_k < \ldots < i_1} \bar{\partial} \beta_{i_1 \ldots i_n} dz^{i_1} \wedge \ldots \wedge d\bar{z}^{i_n}$$

$$\Rightarrow d^2 \beta = \partial \bar{\partial} \beta + \bar{\partial} \partial \beta$$

Also:

$$0 = d^2 \beta = \partial \bar{\partial} \beta + (\partial \bar{\partial} + \bar{\partial} \partial) \beta + \bar{\partial} \partial \beta \Rightarrow \begin{cases} \partial^2 \beta = 0 \\ \bar{\partial}^2 \beta = 0 \end{cases}$$
**Remark:** For functions be $C^\infty(U; C)$ we have:

$$db = \sum \left( \frac{\partial b}{\partial x^i} \, dx^i + \frac{\partial b}{\partial y^i} \, dy^i \right)$$

$$= \sum \left[ \left( \frac{\partial b}{\partial x^i} - \frac{\partial b}{\partial y^i} \right) (dx^i + idy^i) + \left( \frac{\partial b}{\partial x^i} + \frac{\partial b}{\partial y^i} \right) (dy^i + idx^i) \right]$$

$$= \sum \frac{\partial b}{\partial \bar{z}^i} \, d\bar{z}^i + \sum \frac{\partial b}{\partial z^i} \, dz^i$$

Similarly, for $(k,l)$-forms:

$$\beta = \sum \beta_{i_1 \cdots i_k \bar{j}_1 \cdots \bar{j}_l} \, dz^{i_1} \wedge \cdots \wedge dz^{i_k} \wedge d\bar{z}^{\bar{j}_1} \wedge \cdots \wedge d\bar{z}^{\bar{j}_l}$$

$$\Rightarrow db = \sum \sum \left( \frac{\partial \beta_{i_1 \cdots i_k \bar{j}_1 \cdots \bar{j}_l}}{\partial x^{i_1}} \, dx^{i_1} + \cdots + \frac{\partial \beta_{i_1 \cdots i_k \bar{j}_1 \cdots \bar{j}_l}}{\partial y^{i_k}} \, dy^{i_k} \right) \beta_{i_1 \cdots i_k \bar{j}_1 \cdots \bar{j}_l}$$

$$+ \sum \sum \left( \frac{\partial \beta_{i_1 \cdots i_k \bar{j}_1 \cdots \bar{j}_l}}{\partial x^{\bar{j}_1}} \, dx^{\bar{j}_1} + \cdots + \frac{\partial \beta_{i_1 \cdots i_k \bar{j}_1 \cdots \bar{j}_l}}{\partial y^{\bar{j}_l}} \, dy^{\bar{j}_l} \right) \beta_{i_1 \cdots i_k \bar{j}_1 \cdots \bar{j}_l}$$

**Remark 2:** The Dolbeaut cohomology of a complex manifold is defined by:

$$H^{k,l}_{\text{Dolbeaut}}(M) := \frac{\text{Ker}(\bar{\partial} : \Omega^{k,l}(U) \to \Omega^{k+1,l}(U))}{\text{Im}(\bar{\partial} : \Omega^{k-1,l}(U) \to \Omega^{k,l}(U))}$$

The Dolbeaut theorem states that it is isomorphic to the sheaf cohomology of the sheaf $\mathcal{O} \to \Omega^{k,l}(U)$. 

\[ \therefore \]
Example: $M = \mathbb{C}^n$ with global holomorphic coordinates $z^i = x^i + y^i$. The canonical symplectic form is:

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i = \frac{i}{2} \sum_{i=1}^{n} dz^i \wedge d\bar{z}^i \in \Omega^{2,0}(M)$$

Let $(M, \omega, \xi, \theta)$ be a Kähler manifold. Then:

$$\omega \in \Omega^2(M; \mathbb{C}) = \Omega^{2,0}(M) \oplus \Omega^{1,1}(M) \oplus \Omega^{0,2}(M)$$

In local holomorphic coordinates:

$$\omega = \sum_{\alpha \beta} a_{\alpha \beta} dz^\alpha \wedge d\bar{z}^\beta + \sum_{\delta \gamma} b_{\delta \gamma} dz^\delta \wedge d\bar{z}^\gamma + \sum_{\delta \epsilon} c_{\delta \epsilon} d\bar{z}^\delta \wedge dz^\epsilon$$

$$\partial_i (\frac{\partial}{\partial z^i}) = \frac{1}{i} \frac{\partial}{\partial z^i}, \quad \partial_i (\frac{\partial}{\partial \bar{z}^i}) = -\frac{1}{i} \frac{\partial}{\partial \bar{z}^i}$$

$$\partial^i (dz^i) = \frac{1}{i} dz^i \quad \partial^i d\bar{z}^i = -\frac{1}{i} d\bar{z}^i$$

Now:

$$\omega(\partial u, \partial z) = \omega(u, z)$$

$$\therefore \partial^i \omega = 0 \iff$$

$$\partial^i \omega = -\sum_{\alpha \beta} a_{\alpha \beta} dz^\alpha \wedge d\bar{z}^\beta + \sum_{\delta \gamma} b_{\delta \gamma} dz^\delta \wedge d\bar{z}^\gamma - \sum_{\delta \epsilon} c_{\delta \epsilon} d\bar{z}^\delta \wedge dz^\epsilon = 0$$

$$\omega = \sum_{\alpha \beta} a_{\alpha \beta} dz^\alpha \wedge d\bar{z}^\beta + \sum_{\delta \gamma} b_{\delta \gamma} dz^\delta \wedge d\bar{z}^\gamma + \sum_{\delta \epsilon} c_{\delta \epsilon} d\bar{z}^\delta \wedge dz^\epsilon$$

$$\Rightarrow a_{\alpha \beta} = c_{\alpha \beta} = 0 \quad \Rightarrow \omega = \sum_{\delta \gamma} b_{\delta \gamma} dz^\delta \wedge d\bar{z}^\gamma \in \Omega^{1,1}(M)$$

Since $d\omega = 0$ we also have:

$$\bar{\omega} = \bar{\omega} = 0 \Rightarrow \sum \bar{b}_{\alpha \beta} \in H^{1,1}_{\text{dol}}(M)$$
Let us write:
\[ \omega = \frac{\sqrt{\gamma}}{2} \sum_{i, x} h_{i x} \, dz^i \wedge d\bar{z}^x \]

- **Simu \( \omega \) is a real form:**
  \[ \overline{\omega} = \omega \iff - \frac{\sqrt{\gamma}}{2} \sum_{i, x} \overline{h_{i x}} \, d\overline{z}^i \wedge dz^x = \frac{\sqrt{\gamma}}{2} \sum_{i, x} h_{i x} \, dz^i \wedge d\overline{z}^x \]

  \[ \Rightarrow h_{i x} = \overline{h_{i x}} \iff \begin{cases} \text{Hermitean matrix} \\
  \text{matrix} \end{cases} \]

- **Simu \( \omega \) is non-degenerate:**
  \[ \omega^m \neq 0 \iff m! \left( \frac{\sqrt{\gamma}}{2} \right)^m \det(h_{i j}) \, dz^i \wedge d\overline{z}^j ... dz^n \wedge d\overline{z}^n \neq 0 \]

  \[ \Rightarrow \det(h_{i j}) \neq 0 \]

- Finally, we have positivity:
  \[ \omega(Su, u) > 0 \text{ if } u \neq 0 \iff H = (h_{i j}) \text{ is positive matrix} \]

**Corollary:** A 2-form \( \omega \in \Omega^2(M, \mathbb{C}) \) on a complex manifold is Kähler if it is a 1-form, \( \partial \)-closed, \( \overline{\partial} \)-closed, and in any local holomorphic coordinates:
\[ \omega = \frac{\sqrt{\gamma}}{2} \sum_{i, \bar{j}} h_{i \bar{j}} \, dz^i \wedge d\overline{z}^\bar{j} \]

Then \( H = (h_{i \bar{j}}) \) is positive-definite, Hermitian matrix.
Theorem
Let \( \omega \in \Omega^{1,1}(M) \) be a closed 1-form on a complex manifold.
Every \( \omega \in M \) has a neighborhood \( U \) and a function \( \rho \in C^\infty(U, \mathbb{R}) \) such that:
\[
\omega|_U = \frac{i}{2} \partial \bar{\partial} \rho
\]

Proof:
We know the Poincare Lemma for \( \partial \neq \bar{\partial} \):
\[
d \omega = 0 \Rightarrow \partial \omega = 0 \Rightarrow \omega = \partial \alpha, \ \alpha \in \Omega^{0,1}(U)
\]
Note that \( \bar{\partial} \omega \in \Omega^{0,2}(U) \) and since \( \partial \partial + \bar{\partial} \partial = 0 \),
\[
\theta \bar{\partial} \alpha = -\partial \alpha = \bar{\partial} \omega = 0 \Rightarrow \bar{\partial} \alpha = \partial \bar{\partial} \bar{\partial} \omega = 0
\]
Now Poincare' Lemma for \( \bar{\partial} \) gives:
\[
d = \bar{\partial} \rho
\]
Now:
\[
\omega = \bar{\omega} \Rightarrow \rho \in \Omega^{1,0}(U) \Rightarrow \omega = \frac{i}{2} \partial \bar{\partial} \rho, \ \rho \in C^\infty(U, \mathbb{R}).
\]

Given Kähler manifold \((M, \omega, J, g)\) a function \( \rho \in C^\infty(U) \) such that
\[
\omega|_U = \frac{i}{2} \partial \bar{\partial} \rho
\]
is called a (local) Kähler potential. The function \( \rho \) satisfies
\[
\left[ \frac{\partial^2 \rho}{\partial \zeta_j \partial \bar{\zeta}_k} \right]
\]
is positive definite
and is called strictly plurisubharmonic (spsh).
Conversely, given a complex manifold \( M \) a smooth function \( \rho \in \mathcal{C}^\infty(M) \) defines a Kähler form or global Kähler potential:

\[
\omega = 2\pi i \partial \bar{\partial} \rho
\]

**Example:**

\( M = \mathbb{C}^n \) has a global Kähler potential the smooth function:

\[
\rho(x, y) = \sum_{i=1}^{n} (x_i^2 + y_i^2) = \sum_{i=1}^{n} x_i^2 = \| x \|^2
\]

Many more examples can be obtained from The Following:

**Proposition:**

Let \((M, \omega, J, g)\) be a Kähler manifold and \( \mathbb{C}^n \subset M \) a complex submanifold. Then \( (N, i^*\omega, J, J_N, g|_N) \) is Kähler. If \( \rho \in \mathcal{C}^\infty(N) \) is a global Kähler potential for \( M \), then \( \rho \mid_\mathbb{C}^n \) is a global Kähler potential for \( N \).

**Proof:**

Use The result about almost complex submanifold of \((M, \omega, J)\) are symplectic submanifolds.

**Example:**

- Every complex submanifold of \( \mathbb{C}^n \) is Kähler, so for example, every smooth affine variety (the zero locus of a family of polynomials) is a Kähler manifold.
Examples:

Let $G \in \mathbb{C}^{n+1} \setminus \{0\}$ by: $\lambda \cdot (z_0, \ldots, z_n) = (\lambda z_0, \ldots, \lambda z_n)$

This action is free and proper, by holomorphic maps, so the quotient

$$\mathbb{C}P^n := \mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}$$

is a complex manifold.

We can choose holomorphic charts as follows:

$$U_i = \{ [z_0 : \ldots : z_n] : z_i \neq 0 \}$$

$$\varphi_i : U_i \to \mathbb{C}^n, \quad \varphi_i([z_0 : \ldots : z_n]) = \left( \frac{z_0}{z_i}, \frac{z_1}{z_i}, \ldots, \frac{z_n}{z_i} \right)$$

The transition functions are

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j) \quad (i \neq j)$$

$$\mathbb{C}^n \setminus \{ \omega_i \neq 0 \} \to \mathbb{C}^n \setminus \{ \omega_i \neq 0 \}$$

$$(\omega_1, \ldots, \omega_n) \mapsto \left( \frac{\omega_1}{\omega_i}, \ldots, \frac{\omega_i}{\omega_i}, \frac{1}{\omega_i}, \frac{\omega_{i+1}}{\omega_i}, \ldots, \frac{\omega_n}{\omega_i} \right)$$

Now:

$$\rho(\omega_1, \ldots, \omega_n) = \log(1 + ||\omega||^2)$$

is a smooth function, so we have a Kähler form on $\mathbb{C}^n$:

$$\omega_{FS} = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \rho = \sqrt{-1} \partial \left( \sum_{i=1}^{n} \frac{\omega_i}{||\omega||^2 + 1} \; d\bar{\omega}_i \right)$$

$$= \frac{\sqrt{-1}}{||\omega||^2 + 1} \left( \sum_{i=1}^{n} d\omega_i \wedge d\bar{\omega}_i - \sum_{i=1}^{n} \frac{\omega_i \bar{\omega}_i}{||\omega||^2 + 1} \; d\omega_i \wedge d\bar{\omega}_i \right)$$

This is known as the Fubini-Study Kähler form.
Exercise:

Check that under a change of coordinates for $\mathbb{CP}^n$ one has:

$$(\psi_0 \circ \psi_1)^* \omega_{FS} = \omega_{FS}$$

It follows that $\omega_{FS}$ induces a Kähler form on $\mathbb{CP}^n$ known as the Fubini-Study form.

Conclusion: Every complex submanifold of $\mathbb{CP}^n$ is Kähler. So every smooth projective variety is Kähler.

More examples of Kähler manifolds:

1) Compact Riemann surfaces: an area form is symplectic and any compatible almost complex structure is integrable (due to 2)

2) Complex tori: $\mathbb{C}^n / \Lambda$ where $\Lambda \subset \mathbb{C}^n$ a full rank lattice (i.e., $\Lambda$ is a discrete subgroup of $(\mathbb{R}^n, +)$: $\omega = \frac{1}{2} \sum d\lambda_i da_i$

$: T\mathbb{C}^n \to T\mathbb{C}^n$ both descend to $\mathbb{C}^n / \Lambda$.

3) Stein manifolds: Kähler manifolds $(M, \omega, J, g)$ admitting a global proper Kähler potential $\phi: M \to \mathbb{R}$ (proper $\Rightarrow$ compact $\Rightarrow \phi^{-1}(K)$ compact). They are precisely those properly embedded analytic submanifolds of $\mathbb{C}^n$. 
**Hodge Theorem**

For a compact Kähler manifold:

\[ H^d_{\text{dolh}(M; \mathbb{C})} = \bigoplus_{k \leq d} H^{k,\bar{k}}_{\text{dolh}}(M) \]

with \( H_{\text{dolh}}^{k,\bar{k}}(M) \subseteq H_{\text{dolh}}^{k,\bar{k}}(M) \).

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\[ b^d = \dim_{\mathbb{R}} H^d_{\text{dolh}}(M; \mathbb{R}) = \dim_{\mathbb{C}} H^d_{\text{dolh}}(M; \mathbb{C}) \] (Betti numbers)

\[ h^{k,\bar{k}} = \dim_{\mathbb{C}} H^{k,\bar{k}}_{\text{dolh}}(M) \] (Hodge numbers)

**Hodge Thm** For a compact Kähler manifold:

\[ b^d = \sum_{k \leq d} h^{k,\bar{k}}, \quad h^k = h^{k,\bar{k}} \]

**Corollary:** For a compact Kähler manifold the odd Betti numbers are even:

\[ b^{2d+1} = 2 \sum_{e=0}^d h^{e,2d+1-e} \]

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Recall that for a compact symplectic manifold

the even Betti numbers are positive:

**Proof:** \( H^{2k}(M) = 0 \) so \( \omega^k = d\omega \Rightarrow \int_M \omega^n = \int_M d(\omega \wedge \omega^{n-k}) = 0 \)
**Corollary:**

For a compact Kähler manifold $h^{0,0} \neq 0$.

**Proof:**

Note that $\omega \in \Omega^{0,0}(M) \Rightarrow \omega \in \Omega^{0,0}(M)$

Also: $d\omega = 0 \Rightarrow d\omega^e + \bar{\partial}\omega^e = 0 \Rightarrow \bar{\partial}\omega^e = 0

It follows that

$$H^{0,0}(M) \ni [\omega] \leftrightarrow [\omega] \in H^{0,0}(M)$$

under the Hodge decomposition. So we must have $H^{0,0}(M) \neq \{0\}$.

**Kodaira-Thorston Example:**

Let $\Gamma \subset \text{Aff}(\mathbb{R}^4)$ be (discrete) group generated by

\[
\begin{align*}
(\alpha, \beta, \gamma, \delta) & \xrightarrow{\gamma_1} (\alpha + 1, \beta, \gamma, \delta) \\
(\alpha, \beta, \gamma, \delta) & \xrightarrow{\gamma_2} (\alpha, \beta + 1, \gamma, \delta) \\
(\alpha, \beta, \gamma, \delta) & \xrightarrow{\gamma_3} (\alpha, \beta, \gamma + 1, \delta) \\
(\alpha, \beta, \gamma, \delta) & \xrightarrow{\gamma_4} (\alpha, \beta, \gamma, \delta + 1)
\end{align*}
\]

This action is proper and free so defines a manifold:

$$M := \mathbb{R}^4 / \Gamma$$

Since the action of $\Gamma$ preserves $\omega = dx_1dy_1 + dx_2dy_2$, we have a symplectic form on $M$.
**Exercise:** Check that the projection

\[ \pi : \mathbb{R}^n \to \mathbb{R}^2, (x_1, x_2, x_3, x_4) \to (y_1, y_2) \]

induces a submersion:

\[ \pi : M \to \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2 \]

and the fibers of this map are Tori, so \( M \) is a torus fibration over \( \mathbb{T}^2 \).

Now observe that:

\[
\begin{align*}
M &= \mathbb{R}^n/\mathbb{Z}^n \\
\mathbb{R}^n &\text{ 1-connected}
\end{align*}
\]

So that:

\[
H^i(M,\mathbb{Z}) \xrightarrow{\pi_*} \mathbb{Z} = \mathbb{Z}, \quad \text{has rank 3}
\]

Because the commutators of Guernouane are all trivial except for

\[
(\delta x, \delta y) = \delta z. \quad \text{Hence:}
\]

\( b^1 = 3 \) is not even \( \Rightarrow M \) has no Kahler structure.

**Remark:** Kodaira has proved that \( M \) does not admit a complex structure!

**Theorem (Gromov, Tischler)**

A compact symplectic manifold \((M,\omega)\) with integral cohomology class \( [\omega] \) admits a symplectic embedding \( i : (M,\omega) \to \mathbb{CP}^n \) for some \( n \geq n \).

**Remark:** This is a symplectic analogue of the Nash isometric embedding theorem for Riemannian manifolds.