14) Almost Complex Manifolds

Def: An almost complex structure on a manifold $M$ is a smooth section $J \in \Gamma(TM \oplus TM)$ such that $J^2 = -\text{id}$. So we have a complex structure $J_x : T_x M \to T_x M$ tangent smoothly at $x$.

Recall that a complex manifold $M$ is given by a manifold with an atlas $\{(U_x, \phi_x)\}$ where $\phi_x : U_x \to \mathbb{C}^n$ and the transition functions are holomorphic maps:

$$\phi_p \circ \phi_x^{-1} : \phi_x(U_x \cap U_p) \to \phi_p(U_x \cap U_p)$$

We call $(U_x, \phi_x)$ holomorphic coordinates: $\phi_x = (z^1, \ldots, z^n)$, $z^i = x^i + ih^i$. Given a complex manifold, we have an almost complex structure $J$ defined in holomorphic coordinates $z^i = x^i + h^i$ by

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}. \quad (\star)$$

Defn. An almost complex structure $J$ on a manifold $M$ is said to be integrable if $M$ has a holomorphic atlas such that $J$ is given by $(\star)$. Not every almost complex structure is integrable.
\textbf{Defn:} Given a manifold \( M \) and a \((1,1)\)-tensor field \( A : TM \rightarrow TM \), its \underline{Nijenhuis torsion} is the \((2,1)\)-tensor:

\[
N_A(x,y) := A\left( [A(x),y] + [x,A(y)] - A([x,y]) \right) - [A(x),A(y)]
\]

If \( J \) is an integrable complex structure, then computing \( N_J \) in complex coordinates \( (x^a, J(x)) \) we see immediately that:

\[
N_J = 0
\]

\textbf{Thm (Newlander-Nirenberg)}

An almost complex structure \( J : TM \rightarrow TM \) is integrable iff \( N_J = 0 \).

The proof of this result requires some hard analysis.

\textbf{Exercise}

Show that if \( \dim M = 2 \) then every almost complex structure \( J : TM \rightarrow TM \) is integrable.

Integrability is too strong! We give up on integrability and study symplectic manifolds \((M, \omega)\) with a compatible almost complex structure \( J : TM \rightarrow TM \), so that:

\[
J(\cdot, \cdot) = \omega(\cdot, J \cdot) \text{ is Riemannian metric}
\]
Given $(M,\omega)$ symplectic manifold, we have a fiber bundle:

$$J(M,\omega) \to M \text{ with } \mathcal{J}_\omega := \{ J : \omega \text{-compatible complex st. } \}$$

Sections of this bundle are precisely the $\omega$-compatible almost complex structures.

Thus

The fiber bundle $J(M,\omega) \to M$ has contractible fibers. In particular, the space of $\omega$-compatible almost complex structures is non-empty and contractible.

Proof:

Fix $\omega \in \mathcal{J}_\omega$. We need to show that there is a homotopy

$$h_t : \mathcal{J}_\omega \to \mathcal{J}_\omega$$

starting at $h_0 = \text{id}$ and finishing at $h_1$ the constant map with value some $J_0 \in \mathcal{J}_\omega$.

For that observe that if $\tilde{G}_1$ and $\tilde{G}_0$ are any 2 inner products, then

$$G_t := t \tilde{G}_1 + (1-t) \tilde{G}_0$$

is also an inner product (exercise!). Hence, fix $\tilde{J}_0$, a $\omega$-compatible complex st. and set:

$$G_0 := \omega_x(-,\tilde{J}_0)$$

Now, given a $\omega_x$-compatible $J$, we have the inner product

$$G_J := \omega_x(-,J)$$
So we can consider the family of such potentials

$$ G_t = t G_\omega + (1-t) G_0 $$

and these determine $\omega_t$-compatible CR-structures $J_t$, joining $J$ to $J_0$. So we can define:

$$ h_t(J) := J_t $$

Remarks:

Note that we have not used that $d\omega = 0$. So:

- Result works for almost symplectic manifolds $(M, \omega)$
  i.e., $\omega$ is only non-degenerate.
- Obstruction to existence of almost symplectic structure
  is the same as existence of an almost complex structure.
- Result works more generally for symplectic vector
  bundles $(E, \omega) \to M$, $\omega_0 : E \times E \to \mathbb{R}$, non-deg.

Proposition 1

Let $J$ be an almost complex structure on $M$, which compatible for
two symplectic structures $\omega_0, \omega_1 \in \Omega^2(M)$. Then there exists a smooth
family of symplectic structures $\omega_t \in \Omega^2(M)$ joining $\omega_0$ to $\omega_1$.

Proof:

Let $\omega_t = t \omega_1 + (1-t) \omega_0$

This is clearly closed. We have:
Given an almost complex manifold \((M, J)\) and an almost complex submanifold is a submanifold \(N \subseteq M\) such that \(J(TN) \subset TN\).

**Proposition 2**

Let \((M, \omega)\) be a symplectic submanifold and let \(J\) be a \(\omega\)-compatible almost complex structure. If \(N \subset M\) is an almost complex submanifold, then \(N\) is a symplectic submanifold.

**Proof:**

We need to check that \(i^*\omega\) is symplectic, where \(i: N \hookrightarrow M\).

It is clearly closed. To check that it is non-degenerate:

\[
g_N(\cdot, \cdot) = \omega(\cdot, J \cdot)\]

\(\Rightarrow\) \(\omega\) restricts to define \(g_N\) on \(N\)

\(\Rightarrow\) \(\omega\) restricts to complex structure \(J_N\) on \(N\)

\(\Rightarrow\) \(i^*\omega(\cdot, \cdot) = g_N(J_N \cdot, \cdot)\)

\(\Rightarrow\) \(i^*\omega(\cdot, \cdot) = g_N(J_N \cdot, \cdot)\)

is non-degenerate.

\[\square\]

We will see later examples of (compact) symplectic manifolds (hence almost complex) which do not admit a complex structure, and conversely.
It may be hard to decide if a manifold has a complex structure: e.g., $S^5$ has almost complex structure, it is still not known if it has a complex structure (Atiyah claims it doesn't, but his moment is incomplete).

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**J-Holomorphic & J-anti-holomorphic tangent/cotangent bundles**

If $V$ is an $R$-vector space, its complexification is the $C$-vector space $V \otimes C$ with scalar multiplication:

$$\lambda \cdot (v \otimes z) = v \otimes (\lambda z)$$

$$\dim_R V = m \implies \dim_C (V \otimes C) = m.$$ 

For any manifold $M$, the complexified tangent bundle $TM \otimes C$ has fibers the complexifications $T_x M \otimes C$.

Let $(M, J)$ be an almost complex manifold. Then we can extend $J$ to $TM \otimes C$ by:

$$J (v \otimes z) := J(v) \otimes z$$

Since $J^2 = -id$ and $(TM \otimes C)_x$ is a $C$-vector space, we have splittings into $\pm i$-eigen spaces:

$i$-eigen space: $T_{x0} M := \{ v \in TM \otimes C : J(v) = i v \}$

$$= \{ v \otimes 1 - J(v) \otimes i : v \in TM \}$$

($J$-holomorphic tangent space)
-1- Eigenspace: \( T_{0,1} \mathbb{N} := \{ \omega \in T^* \mathbb{C} \otimes \mathbb{C} : \mathfrak{d}(\omega) = -i \omega \} \)
\[ = \{ \omega \otimes 1 + i(\omega \cdot i) : \omega \in TN \} \]

(\( J \)-anti-holomorphic \( TM \), \( TM \) cat space)

We have:
\[
TM \otimes \mathbb{C} = T_{1,0}M \oplus T_{0,1} \mathbb{N}
\]

The projections \( \pi_{1,0} \) and \( \pi_{0,1} \) restrict to \( TM \subset TM \otimes \mathbb{C} \)

- Real bundle isomorphisms: \( \pi_{1,0} : TN \rightarrow T_{1,0} \mathbb{N} \) such that
\[
\pi_{1,0} \circ \mathfrak{d} = i \cdot \pi_{1,0}, \quad \pi_{1,0}(\omega) = \frac{1}{2}(\omega \otimes 1 - i(\omega \cdot i))
\]
\[
\Rightarrow (TM, \mathfrak{d}) \simeq T_{1,0}M \quad (\text{as } \mathbb{C}-\text{vector bundles})
\]

- Real bundle isomorphism: \( \pi_{0,1} : TM \rightarrow T_{0,1} \mathbb{M} \) such that
\[
\pi_{0,1} \circ \mathfrak{d} = -i \cdot \pi_{0,1}, \quad \pi_{0,1}(\omega) = \frac{1}{2}(\omega \otimes 1 + i(\omega \cdot i))
\]
\[
\Rightarrow (TM, \mathfrak{d}) \simeq T_{0,1} \mathbb{M} \quad (\text{as } \mathbb{C}-\text{vector bundles})
\]

Similarly, for the cotangent bundle:

\[
T^* M \otimes \mathbb{C} \simeq T_{1,0}^* M \oplus T_{0,1}^* M
\]

above:
\[
T_{1,0}^* M = \{ \eta \in T^* M \otimes \mathbb{C} : \mathfrak{d}^* \eta = +i \eta \} \]
\[
= \{ \omega \otimes 1 - (\omega \cdot i) \otimes i : \omega \in TM \}
\]
\[
T_{0,1}^* M = \{ \eta \in T^* M \otimes \mathbb{C} : \mathfrak{d}^* \eta = -i \eta \} \]
\[
= \{ \omega \otimes 1 + (\omega \cdot i) \otimes i : \omega \in TM \}
\]
We also have projections:
\[ \pi^{10} : T^{*}N \otimes \mathbb{C} \rightarrow T^{10}N, \quad \eta \mapsto \frac{1}{2} (\eta - i \bar{\eta}) \]
\[ \pi^{01} : T^{*}M \otimes \mathbb{C} \rightarrow T^{01}N, \quad \eta \mapsto \frac{1}{2} (\eta + i \bar{\eta}) \]

**Differential Forms:**
\[ \Omega^k(N; \mathbb{C}) = \mathcal{P}(\Lambda^k(T^*N \otimes \mathbb{C})) - \mathbb{C}\text{-valued forms on } M \]

This gets decomposed:
\[ \Lambda^k(T^*M \otimes \mathbb{C}) = \bigoplus_{a+b=k} (\Lambda^a(T^*N) \otimes \Lambda^b(T^0N)) \]
\[ \otimes \bigoplus_{a+b=k} T^{a,b}M \]

**Differential Forms of Type** \((a,b)\):
\[ \Omega^{a,b}(N; \mathbb{C}) = \mathcal{P}(T^{a,b}N) \]
\[ \Omega^k(N; \mathbb{C}) = \bigoplus_{a+b=k} \Omega^{a,b} \]

We denote the associate projections by
\[ \pi^{a,b} : \Omega^k(N; \mathbb{C}) \rightarrow \Omega^{a,b} \]

(Since \(k = a+b\), this is not ambiguous).

**Defn:**
\[ \partial := \pi^{a+b,0} \circ d : \Omega^{a,b} \rightarrow \Omega^{a-1,b} \]
\[ \overline{\partial} := \pi^{a,0+b} \circ d : \Omega^{a,b} \rightarrow \Omega^{a,b+1} \]

\[ d \beta = \sum_{a+b = k+1} \pi^{a,b} d \beta = \pi^{0,k+1} d \beta + \cdots + \partial \beta + \overline{\partial} \beta + \cdots + \pi^{k+1,0} d \beta \]
In general: \( df = \partial f + \bar{\partial} f \)

**Defn:** A function \( f: M \to \mathbb{C} \) is called \( J \)-holomorphic if
\[
df \circ J = i df
\]
and is called \textbf{anti-holomorphic} if
\[
df \circ J = -i df
\]

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**Exercise:**
1. \( f \) is \( J \)-holomorphic \( \iff \) \( \bar{\partial} f = 0 \)
2. \( f \) is \( J \)-anti-holomorphic \( \iff \) \( \partial f = 0 \)

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**Rmk:** In general, an almost complex manifold \((M, J)\) has very few holomorphic functions. But it has many \( J \)-holomorphic curves, i.e., maps \( f: \mathbb{C} \to M \) such that
\[
df \cdot i = 0 \circ df
\]
These \( J \)-holomorphic curves play an important role in symplectic geometry.

Gromov-Witten invariants \( \equiv \) counts \# \( J \)-holomorphic curves