12) Symplectic vs Contact Geometry

(Mention Homework set #3; Due Oct. 21)

$(\Lambda,\omega)$ - symplectic manifold

$i: N^m \to \Lambda^n$ - codimension 1 submanifold

$X \in \mathcal{X}(M)$ - vector field defined on some tubular neighborhood of $N$.

(i) $\mathcal{L}_X \omega = \omega <\omega (\varphi^t_\omega)\omega = e^t \omega$ (Liouville vector field)

(ii) $X + N : T\Lambda M = T\Lambda N \oplus \langle X \rangle$

Then:

$\alpha := i^x(l_x \omega) \in \Omega^1(M)$

$\Rightarrow d\alpha = i^x(d(l_x \omega)) = i^x(l_x d\omega) = i^x \omega$

$\Rightarrow \alpha \wedge (\wedge(d\alpha))^{-1} = i^x(l_x \omega \wedge \omega^{-1})$ non-vanishing!

$\Rightarrow \alpha \in \Omega^1(N)$ is a contact form. We say that $N$ is of contact type

Conversely, any contact form is obtained in this way:

Theorem (symplectization)

Let $\alpha \in \Omega^1(N)$ be a contact form. Then $M = N \times \mathbb{R}$ carries the symplectic form:

$\omega = d\left(e^t \left|_{N} \alpha \right\right)$

Remark: The vector field $X = \frac{\partial}{\partial t}$ is transverse to $N \times \{0\} \subset N \times \mathbb{R}$ and:

$\mathcal{L}_X \omega = \omega, \ \alpha = i^x(l_x \omega)$
Proof: \( \omega \) is clearly closed. So all one needs to check is non- degeneracy:
\[
\omega = e^t \left( dt \wedge p^x dx + p^y dw \right)
\]
\[
\Rightarrow \omega^m = e^t \left( dt \wedge p^x dx \wedge (da \wedge (dx^y)) \right) \text{ man-degenerate.}
\]

Properties of The Symplectization:

- \( L \subset (\mathbb{N}, \lambda) \) is a Legendrian submanifold iff \( L \times \mathbb{R} (\mathbb{M}, \omega) \) is Lagrangian.
- \( \psi : \mathbb{N} \rightarrow \mathbb{N} \) is contactomorphism with \( \psi^* \omega = e^\lambda \omega \) iff the map \( \hat{\psi} : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{N} \times \mathbb{R} \)
  \[
  \hat{\psi}(\alpha, t) = (\psi(\alpha), t - \beta(\alpha))
  \]
  is a symplectomorphism.

If \((\mathbb{M}, \omega)\) is symplectic and \( N \subset (\mathbb{M}, \omega) \) is an hypersurface of contact type, then it has a "positive side" into each any Liouville vector field points (exercise).

Definition:

A contact manifold \((\mathbb{N}, \xi = \ker \alpha)\) is called symplectically fillable if there exists a symplectic manifold \((\mathbb{M}, \omega)\) where \( N \) embeds as an hypersurface of contact type and \( N = \partial W \), where \( W \subset M \) is a compact submanifold which lies on the negative side of \( N \).
In dimension 3:

- A 2-disk $D(c(N, S))$ is called _overtwisted_ if $N$ is one of the foliations:

\[
\begin{align*}
\text{Degenerate} & \quad \text{Non-degenerate}
\end{align*}
\]

A contact structure $(N^3, \xi)$ is called _overtwisted_ if it admits an overtwisted disk. Otherwise, it is called a _tight_ contact structure.

**Exercise:** Show that $(\mathbb{R}^3, \omega_{std})$ is overtwisted.

---

**Eliashberg:** Given a compact 3-manifold, every oriented plane field is homotopic via oriented plane fields to an overtwisted contact structure.

(Generalise to any odd dimensions by Borman, Eliashberg and Murphy for appropriate notion of "overtwisted").

**Eliashberg-Gromov:** If a compact contact 3-manifold is fillable, then it is tight.

**Etzur-Rouan:** There exists a compact 3-manifold that does not support any tight contact structure.