Math 520 - Symplectic Geometry

- MWF- 9.00-9.50 AH 447
- Office Hours: M & Th: 1.30-2.30 pm
- Grading Policy: expect everyone will have an "A",
  - Homework every 2 weeks
  - 3-5 pages paper on some topic (in latex)
- Textbooks:
  - A. Cannas da Silva: Lectures on Symplectic Geometry
  - D. McDuff & D. Salamon: Introduction to Sympl. topology

Syllabus:
- Linear Symplectic Geometry
- Symplectic Manifolds
- Symplectomorphisms
- Contact Manifolds
- Almost complex structures
- Hamiltonian Group actions
- Toposes (?)

What is a symplectic manifold?
**Riemannian Manifold:**

\[ g_p (u, v) = \text{inner product in each tangent space } T_p M \]

- Symmetric:
  \[ g_p (u, v) = g_p (v, u) \]

- Non-degenerate:
  \[ g_p (u, v) = 0 \implies v = 0 \]

- Positive definite:
  \[ g_p (u, u) > 0 \text{ if } u \neq 0 \]

**Local invariants (curvature):**

(Both local & global aspects)

**Symplectic Manifold:**

\[ \omega_p (u, v) \equiv \text{"skew-symmetric product"} \]

- Symmetric:
  \[ \omega_p (u, v) = -\omega_p (v, u) \]

- Non-degenerate:
  \[ \omega_p (u, v) = 0 \implies v = 0 \]

- Integrability condition:
  \[ \omega \in \Omega^2 (M) \]
  \[ d \omega = 0 \]

Darboux's Thm

\( \Rightarrow \) No local invariants

(Topological aspects)
Defn: A **symplectic form** on a manifold $M$ is a 2-form $\omega \in \Omega^2(M)$ s.t.:

1. $\omega$ is non-degenerate;
2. $\omega$ is closed: $d\omega = 0$.

The pair $(M, \omega)$ is called a **symplectic manifold**.

A map between symplectic manifolds $\phi: (M, \omega_1) \to (M_2, \omega_2)$ is called a **symplectic map** if $\phi^* \omega_2 = \omega_1$. ($\Rightarrow \dim M = \dim M_2$)

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I. Linear Symplectic Geometry

$V$ - vector space (finite dim)

$\Omega: V \times V \to \mathbb{R}$ biliinear & skew-symmetric

**Thm (canonical basis)**

There is a basis $u_1, \ldots, u_n, e_1, \ldots, e_m, f_1, \ldots, f_m$ such that:

1. $\Omega(u_i, u_j) = 0$
2. $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$
3. $\Omega(e_i, f_j) = \delta_{ij}$

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$\{u_1, e_1, f_1\}$ - canonical basis $e$ (but not unique!)

$$\Omega(u,v) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{I} \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$
Proof: Define:

\[ Ku \Omega = \{ u \in V : \Omega (u, v) = 0, \forall v \in V \} \quad \text{(subspace)} \]

and choose bases \( \{ u_1, \ldots, u_k \} \) for \( Ku \Omega \) and complementary space \( W \) to \( Ku \Omega \):

\[ V = Ku \Omega \oplus W \]

Choose \( e \in W \). There exists \( v \in W \) s.t. \( \Omega (e, v) = a \neq 0 \). So we set \( f = \frac{1}{a} v \) and we have \( \Omega (e, f) = 1 \).

Thus:

\[ W_1 = \langle e, f \rangle \subseteq W \]

And we define:

\[ W_1^{\Omega^2} = \{ u \in W : \Omega^2 (u, v) = 0, \forall v \in W_1 \} \]

We check that:

\[ W = W_1 \oplus W_1^{\Omega^2} \]

1. \( W_1 \cap W_1^{\Omega^2} = \{ 0 \} \):

\[ u = a e_i + b f_i \in W_1^{\Omega^2} \]

\[ \Rightarrow \begin{cases} 0 = \Omega^2 (u, e_i) = a \Omega^2 (e_i, e_i) + b \Omega^2 (f_i, e_i) = -b \\ 0 = \Omega^2 (u, f_i) = a \Omega^2 (e_i, f_i) + b \Omega^2 (f_i, f_i) = a \end{cases} \]

\[ \Rightarrow u = 0. \]

2. \( W_1 = V_1 + W_1^{\Omega^2} \): Let \( u \in W_1 \). If \( c = \Omega (u, e_i) \neq d = \Omega (u, f_i) \)

we find:

\[ u = \frac{d e_i - c f_i}{m} + \frac{u - d e_i + c f_i}{m} \]

where \( m = \Omega (u, u) \neq 0 \).
Now we can repeat the argument for $W_i\otimes^2$, choosing $e_i, f_i \in W_i$ with $\otimes^2(e_i, f_i) = 1$, set $W_i = \langle e_i, f_i \rangle$. Note that $\otimes^2(e_i, e_i) = \otimes^2(f_i, f_i) = \delta_{i0}$.

Since $\dim V < +\infty$, this must stop:

$$V = \text{Ker} \otimes^2 \oplus W_1 \oplus W_2 \oplus \cdots \oplus W_m$$

$$\langle e_i, f_i \rangle \quad \langle e_i, f_i \rangle \quad \langle e_i, f_i \rangle$$

($\otimes^2$-orthogonal decomposition) \[ \square \]

*Remark:* This is the analogue of the Gram-Schmidt orthogonalisation.

We obtain an orthonormal set $(V, \otimes^2)$:

$$\dim (\text{Ker} \otimes^2) = \dim V - \dim (\text{Ker} \otimes^2) = \text{rank } \otimes^2$$

Even number

If $\phi : (V, \otimes^2) \to (W, \otimes^2)$ is a linear isomorphism such that $\otimes^2 \otimes^2 w = \otimes^2 \otimes^2 v \iff \otimes^2 \otimes^2 \phi(w), \phi(v) = \otimes^2 \otimes^2 (w, v)$ for all $w, v \in V$.

Then $\text{rank } \otimes^2 w = \text{rank } \otimes^2 v$.

Given $(V, \otimes^2)$ we obtain a linear map

$$\otimes^2 : V \to V^*$$

$$u \mapsto (v \mapsto \otimes^2(u, v))$$

Notice that:

$$\text{Ker } \otimes^2 = \text{Ker } \otimes^2$$
Definition:
- \( \Omega : V \times V \to \mathbb{R} \) is called a \textbf{linear symplectic form} if \( \text{Ker} \, \Omega = \{0\} \Longleftrightarrow \Omega^* : V \to V^* \) is an isomorphism.

The pair \((V, \Omega)\) is called \textbf{a symplectic vector space}.

A \underline{linear symplectomorphism} is a \underline{linear isomorphism} \( \phi : (V, \Omega) \to (W, \Omega_0) \) such that \( \phi^* \Omega_0 = \Omega \).

\[ \]

Corollary:
A symplectic vector space \((V, \Omega)\) has a \underline{basis} \( \{e_1, e_2, \ldots, e_n\} \) satisfying:
\[ \Omega(e_i, f_j) = \delta_{ij} \]

So \((V, \Omega)\) is symplectomorphic to \((\mathbb{R}^{2n}, \Omega_0)\)

where \( \Omega_0 \) is the canonical \underline{linear symplectic form}:
\[ \Omega_0((u_1, \ldots, u_{2n}),(v_1, \ldots, v_{2n})) = \sum_{i=1}^{n} (u_i v_{i+n} - v_i u_{i+n}) \]

\[ \]

Proof: \( \phi : V \to \mathbb{R}^{2n} \) is the unique linear map such that
\[ \phi(e_i) = (1, \ldots, 0, 0), \phi(e_n) = (0, \ldots, 1, 0, \ldots, 0) \]
\[ \phi(f_i) = (0, \ldots, 0, 1, \ldots, 0), \phi(f_n) = (0, \ldots, 0, 1) \]

All symplectic vector spaces of the same dimensions look alike: \((\mathbb{R}^{2n}, \Omega_0)\).
Note: This is not true for a non-degenerate symplectic bilinear form \( B : V \times V \to \mathbb{R} \):

\[
B = \begin{bmatrix}
+1 & 0 \\
0 & -1 \\
\end{bmatrix}
\]

The condition of non-degeneracy can be expressed in several distinct forms:

**Proposition:** Let \( \Omega : V \times V \to V \) be a skew-symplectic bilinear form. The following are equivalent:

(i) \( \Omega \) is non-degenerate

(ii) The map \( \Omega^*: V \to V^* \), \( u \mapsto \Omega(\cdot, u) \) is an isomorphism

(iii) \( \dim V = 2n \) and \( \Omega^n = \Omega_{2n-1} \ldots \Omega_1 \Omega = 0 \).

**Proof:** Use symplectic basis.

The linear symplectomorphisms of \((V, \Omega)\) form a group. How does it look like?
\[ S_p(V, \Omega) = \{ \phi : V \rightarrow V : \Omega(\phi(u), \phi(v)) = \Omega(u, v) \} \]

We can assume that \((V, \Omega) \cong (\mathbb{R}^{2n}, \Omega_0)\). Then,

\[ S_p(m, \mathbb{R}) = \{ \phi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} : \Omega_0(\phi(u), \phi(v)) = \Omega_0(u, v) \} \]

If we write:

\[ \phi = \begin{bmatrix} A & B \\ c & D \end{bmatrix} \quad \Omega_0 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \]

Then,

\[ (***) \Rightarrow \begin{bmatrix} A & B \\ c & D \end{bmatrix}^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ c & D \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \]

\[ \Leftrightarrow \begin{cases} A^T c = C^T A \\ B^T D = D^T B \quad (***) \\ A^T D - C^T B = I \end{cases} \]

\[ S_p(m, \mathbb{R}) = \left\{ \begin{bmatrix} A & B \\ c & D \end{bmatrix} \text{ such that (***) holds} \right\} \]

Let \((V, \Omega_2)\) be a symplectic vector space.
Let \(W \subset V\) be a subspace.

\[ W, \Omega_2 \quad \mid \quad \forall u, v \in W : \Omega_2(u, v) = 0, \forall v \in W \]

P. B. Jespersen.
When $R_{c1123}$ is bounders one can choose to place the origin so that all symmetries $T_p s_3$ are linear maps $n_{TCI}g_{AI}$.

This explains why in our examples we could represent symmetries by matrices $A_{AIIAnos}_{dctA_t1}$.

Exercise: Check that in the examples $A_{AIIAnos}_{dctA_t1}$.

Rotation 2 Permutations Symmetries of $R_{c1}$ form a Group under composition.

We will see another important example $I_2^3^4 B_{G}$.

A permute them if $1_{2_{3_{4_{0_{0_{8_{0_{3_{I_{4_{3}}}}}}}}}}}$.

Given a Finite set $F$, a punctuation $S$ is a bijection $1_T F F$.

Recall bijection; it exists inverse $I_Y X$ if it is both injective $x = x x e x$ and surjective $y = y y e y$.

The $n_{TCI}g_{AI}$ of a finite $F \in R_{c1}$ is a bijection.

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**Definition:**

- $W \subset (V, \Omega)$ is called a **symplectic subspace** if $W \cap W^\perp \neq 0$.

- $W \subset (V, \Omega)$ is called a **lanczos subspace** if $W = W^\perp$.

- $W \subset (V, \Omega)$ is both isochephe and coisochephe ($\dim W \leq \frac{1}{2} \dim V$).

**Example:** $(R^m, \Omega_0)$:

- $\{ (v_1, \ldots, v_k, 0, \ldots, 0) : v_i \in \mathbb{R}^3 \}$ - isochephe if $k \leq m$.

- $\{ (0, \ldots, 0, v_k, \ldots, v_1) : v_i \in \mathbb{R}^3 \}$ - coisochephe if $k \geq m$.

- $\{ (v_1, \ldots, v_k, 0, \ldots, 0, w_1, \ldots, w_n, 0, \ldots, 0) \}$ - symplectic.