A PROOF OF DE RHAM'S THEOREM

JAMES WRATTEN

Abstract. This is an expository paper on de Rham’s Theorem.

1. Introduction

De Rham cohomology is one of the basic cohomology theories which obey the Eilenberg-Steenrod axioms. Also used frequently are simplicial, singular, sheaf, cellular, and Čech cohomology. These cohomology theories are defined on different subcategories of the category of topological spaces. Singular cohomology is, in some sense, the most general cohomology theory in that it is defined on all topological spaces and we say that this is a cohomology theory in the category of topological spaces. Simplicial (respectively: cellular, de Rham) cohomology is a cohomology theory in the category of simplicial complexes (respectively: CW-complexes, smooth manifolds).

An important result is that all cohomology theories obeying the Eilenberg-Steenrod axioms agree on finite CW-complexes. This last statement holds as one can show that the Eilenberg-Steenrod axioms (ie. homotopy, excision, dimension, additivity, and exactness) alone determine the cohomology for finite CW-complexes. If we drop the dimension criteria then we get what is called an extraordinary homology theory, an example of which is K-theory.

An important point to make is that the Eilenberg-Steenrod axioms only determine the cohomology of finite CW-complexes. In particular different cohomology theories can disagree on spaces which are not homeomorphic to a CW-complex. A canonical example of this is the closed topologist’s sine curve, in which case the Čech and singular cohomologies differ. It turns out that the zeroeth Čech cohomology is $H^0(X) = \mathbb{Z}$, if $X$ is the closed topologists sine curve, while the zeroeth singular cohomology is $H^0(X) = \mathbb{Z}^2$.

The aim of this paper is to present a proof of de Rham’s theorem via simplicial methods. We will follow the treatment in [1] and [2] the closest. In simplest terms, de Rham’s theorem says that the de Rham cohomology is a topological invariant. De Rham’s Theorem is quite similar, in flavor, to some other theorems one might see in an algebraic topology course. For example, if we have a CW-complex, $X$, then we know that its cellular cohomology is isomorphic to its singular cohomology. This means that no matter what CW-structure I put on a topological space, assuming that it admits a CW structure, its cellular homology will be the same. Likewise, de Rham’s theorem says that the de Rham cohomology is independent of the smooth structure we put on a topological manifold. Furthermore, it agrees with the singular cohomology, where both are defined, namely on smooth manifolds. This means that

Date: April 30th, 2014.
the homologies of the exotic spheres are identical to the homologies of the standard sphere.

The general idea here is that we want to compute the singular cohomology of some topological space. Suppose that our topological space admits an additional structure (e.g., a smooth structure giving us a smooth manifold or a CW-structure yielding a CW-complex). The hope is that we find a cohomology theory in this new category which is simpler to compute. This would be worthless if our result depended on the particular structure we imposed, as this would not tell us the cohomology of the underlying topological space. The type of theorem discussed above tells us that the result does not depend on the structure imposed thus allowing us to use the new cohomology theory to compute the singular cohomology.

**Background**

Throughout the paper we will use $M$ to denote a smooth manifold, $\Omega^k(M)$ to denote the space of $k$-forms on $M$, and $Z^k_{dR}(M)$, respectively $B^k_{dR}(M)$, to denote the closed, respectively exact, $k$-forms on $M$. As usual $d : \Omega^k(M) \to \Omega^{k+1}(M)$ will be the exterior derivative. Then we define the de Rham cohomology space of order $k$ by $H^k_{dR}(M) \equiv Z^k_{dR}(M)/B^k_{dR}(M)$, which is a vector space over $\mathbb{R}$. As $M$ is a topological space we are able to use singular homology as well. We denote the standard $k$-simplex by $\Delta^k = \{ \sum_{i=0}^{k} t_i e_i : \sum_{i=0}^{k} t_i = 1, t_i \geq 0 \} \subset \mathbb{R}^k$ where $e_i$ is the $i$-th basis vector in $\mathbb{R}^k$ for $1 \leq i \leq k$ and $e_0$ is the origin. Then a singular $k$-simplex is simply a continuous map $\sigma : \Delta^k \to M$. A singular $k$-chain is a formal $\mathbb{R}$-linear combination of singular $k$-simplices. Then, we let $S_k(M, \mathbb{R})$ be the free $\mathbb{R}$-vector space generated by all singular $k$-simplices. We use $\partial : S_k(M; \mathbb{R}) \to S_{k-1}(M; \mathbb{R})$ as the boundary map giving us the complex $(S_\bullet(M; \mathbb{R}), \partial)$. Finally, we define the singular homology of $M$ with real coefficients by $H_k(M, \mathbb{R}) \equiv Z_k(M, \mathbb{R})/B_k(M, \mathbb{R})$ where $Z_k(M, \mathbb{R})$ is the real vector space of all cycles and $B_k(M, \mathbb{R})$ is the real vector space of all boundaries. We readily see that the complex $(S_\bullet(M; \mathbb{R}), \partial)$ has the subcomplex $(S_\infty(M; \mathbb{R}), \partial)$, which is formed by only considering the smooth singular $k$-chains, whose homology is denoted by $H^\infty_k(M; \mathbb{R})$. We also have the dual complex of $(S_\bullet(M; \mathbb{R}), \partial)$, namely $(S_\bullet(M; \mathbb{R}), d)$, as well as the dual complex of $(S_\infty(M; \mathbb{R}), \partial)$, namely $(S_\infty(M; \mathbb{R}), d)$ were $d$ is the adjoint of the map $\partial$. From these dual complexes we find the simplicial cohomology.

2. Approximation

As mentioned above we have the $(S_\infty_\bullet(M; \mathbb{R}), \partial)$ is a subcomplex of $(S_\bullet(M; \mathbb{R}), \partial)$ via the inclusion map. Our first important result is that they agree on homology.

**Theorem 2.1.** The inclusion map from $S_\infty_k(M; \mathbb{R})$ to $S_k(M; \mathbb{R})$ induces an isomorphism in homology, namely $H^\infty_k(M; \mathbb{R}) \cong H_k(M; \mathbb{R})$ for all $k \in \mathbb{N}$. Similarly, $H^{\infty,k}(M; \mathbb{R}) \cong H^{k}(M; \mathbb{R})$ for all $k \in \mathbb{N}$.

The proof of this theorem is quite technical and relies on the following lemma’s.

**Lemma 2.2.** For any smooth manifold, $M$, and any singular $k$-simplex $\sigma : \Delta^k \to M$ there exists a homotopy, $H_\sigma : \Delta^k \times I \to M$, from $\sigma$ to a smooth $k$-simplex, $\bar{\sigma}$ with the following properties.

1. Given that $\sigma$ is already smooth the homotopy is the identity.
(2) For each face map \( F_{i,k} : \Delta^{k-1} \to \Delta^k \) we have \( H_{\sigma \circ F_{i,k}}(x,t) = H_{\sigma}(F_{i,k}(x),t) \) for all \((x,t) \in \Delta^{k-1} \times I\).

**Proof.** Sketch

We proceed by induction on the dimension of \( \sigma \). The base case is when we have \( \sigma : \Delta^0 \to M \). This is trivially smooth so we define \( H_{\sigma}(x,t) = \sigma(x) \). Now, we assume the inductive hypothesis that for each \( k \)-simplex with \( k < k_0 \in \mathbb{Z}^+ \) the above theorem holds. Let, \( \sigma \) be a \( k_0 \)-simplex in \( M \). In the case that \( \sigma \) is smooth we define \( H_{\sigma} \) to be the identity. Otherwise we consider the set \( S := (\Delta^{k_0} \times \{0\}) \cup (\partial \Delta^{k_0} \times I) \subset \Delta^{k_0} \times I \). We define the map \( H_0 : S \to M \) by

\[
H_0(x,t) = \begin{cases} 
\sigma(x) & : x \in \Delta^{k_0}, t = 0 \\
\sigma \circ F_{i,p}(F_{i,p}^{-1}(x),t) & : x \in \partial \Delta^{k_0}, t \in I 
\end{cases}
\]

On overlap these functions agree implying that \( H_0 \) is continuous. Then we extend \( H_0 \) to all of \( \Delta^{k_0} \times I \) by using a retraction from \( \Delta^{k_0} \times I \) onto \( S \), namely \( R : \Delta^{k_0} \times I \to S \). We then let \( H(x,t) := H_0(R(x,t)) \). Using the Whitney approximation theorem allows us to alter \( H \) so that it is a homotopy from \( \sigma \) to a smooth \( k_0 \)-simplex. The remaining details can be found in [1].

**Proof.** Theorem 2.1

First we define a \( \mathbb{R} \)-vector space homomorphism \( s : S_k(M; \mathbb{R}) \to S_k^\infty(M; \mathbb{R}) \) which we will later show is an isomorphism with inverse the inclusion. We define, \( i_0, i_1 : \Delta^k \to \Delta^k \times I \) by \( i_0(x) = (x,0), i_1(x) = (x,1) \). These are trivially smooth maps. Then we define \( s \) by \( \sigma \to H_{\sigma} \circ i_1 \) and extending linearly to chains. We observe that \( s \) is a chain map. Hence, \( s \) induces a map on homology denoted by \( s_* : H_k(M; \mathbb{R}) \to H_k^\infty(M; \mathbb{R}) \). We want to show that \( s_* \) is an isomorphism with inverse \( i_* \), induced via the inclusion map. From the preceding lemma we have that \( s \circ i \) is the identity on \( H_k^\infty(M; \mathbb{R}) \) which implies that \( (s \circ i)_* = s_* \circ i_* \) is the identity on \( H_k^\infty(M) \). The other direction requires a homotopy operator \( h : S_k(M; \mathbb{R}) \to S_{k+1}(M; \mathbb{R}) \) for which \( \partial \circ h + h \circ \partial = i \circ s - Id_{S_k(M; \mathbb{R})} \). We define \( h \) by \( \sigma \mapsto \sum_{i=0}^k (-1)^i H_{\sigma} \circ G_{i,k} \) where \( G_{i,k} : \Delta^{k+1} \to \Delta^k \times I \) is the \((k+1)\)-simplex with vertices \((e_0,0), \ldots, (e_i,0), (e_i',1), \ldots, (e_k,1)\).

**3. Integration Map and Proof of De Rham’s Theorem**

Now, we will introduce a way to integrate differential \( k \)-forms over smooth \( k \)-simplices. We define the integral of any \( \omega \in \Omega^k(M) \) over a smooth \( k \)-simplex \( \sigma : \Delta^k \to M \) by \( \int_{\sigma} \omega = \int_{\Delta^k} \sigma^* \omega \) and extend linearly for any smooth singular \( k \)-chain. Note : \( \int_{\Delta^k} \sigma^* \omega \) is just the Riemann integral of the function \( f dx^1 \cdots dx^n \) over \( \Delta^k \subset \mathbb{R}^k \) assuming that \( \sigma^* w = f dx^1 \wedge \cdots \wedge dx^n \).

We need the following analog of Stokes Theorem

**Theorem 3.1.** Let \( w \in \Omega^{k-1}(M) \) and \( c \) be a smooth singular \( k \)-chain. Then, \( \int_c dw = \int_{\partial c} w \).

This allows us to find a chain map \( I : (\Omega^\bullet M, d) \to (S^{\infty \bullet}(M; \mathbb{R}), d) \) defined by \( I(w)(\sigma) = \int_{\sigma} w \) for \( w \in \Omega^k(M) \) and \( \sigma \in S^\infty_k(M; \mathbb{R}) \). The fact that this is a chain map is a trivial application of the above version of Stokes Theorem. This induces a map in cohomology, \( I_* : H^k_{dR}(M) \to H^{\infty,k}(M; \mathbb{R}) \cong H^k(M; \mathbb{R}) \). The DeRham theorem states that this map is actually as isomorphism.
Lemma 3.2. Let $F : M \to N$. Then,

\[
\begin{array}{ccc}
H^k_{dR}(N) & \xrightarrow{F^*} & H^k_{dR}(M) \\
\downarrow{I_*} & & \downarrow{I_*} \\
H^k(N; \mathbb{R}) & \xrightarrow{f^*} & H^k(M; \mathbb{R})
\end{array}
\]

Let $U, V$ be open subsets covering $M$. Let $f, g$ be the connecting homomorphisms for the Mayer-Vietoris sequences. Then,

\[
\begin{array}{ccc}
H^{k-1}_{dR}(U \cap V) & \xrightarrow{f} & H^k_{dR}(M) \\
\downarrow{I_*} & & \downarrow{I_*} \\
H^{k-1}(U \cap V; \mathbb{R}) & \xrightarrow{g} & H^k(M; \mathbb{R})
\end{array}
\]

Definition 3.3. We will say that a manifold is de Rham, if $I$ is an isomorphism for each $k \in \mathbb{N}$.

Definition 3.4. An open cover $\{U_i\}$ is called a de Rham cover if each $U_i$ is a de Rham manifold and each finite intersection of $U_i$’s is also de Rham. An open cover which is also a basis for the topology is called a de Rham basis for $M$.

Lemma 3.5. Every convex open set of $\mathbb{R}^n$ is de Rham.

Proof. Let $U$ be some open subset of $\mathbb{R}^n$. We know, from Poincare’s lemma, that $H^0_{dR}(U)$ is trivial unless $k = 0$ in which case $H^0_{dR}(U) \cong \mathbb{R}$. We also know the same holds for simplicial cohomology. Then we consider, $I_*[w][\sigma] = \int_\sigma w = \int_{\Delta^0} \sigma^* f$ since $H^0_{dR}(U)$ consists of the equivalence class of the function with constant vaule one while $H^0(U; \mathbb{R})$ consists of the equivalence class of the (smooth) map $\sigma : \Delta^0 \to U$. Hence, we get that $I_*[w][\sigma] = \int_\sigma w = (f \circ \sigma)(0) = 1$. So, I is indeed an isomorphism as both spaces are one dimensional real vector spaces and the map is non-trivial. □

Lemma 3.6. If $M$ has a finite de Rham cover, then $M$ is de Rham.

Proof. We will also prove this result via induction on the number of open sets is the finite de Rham cover. If our de Rham cover consists of one open set then by definition our manifold is de Rham. Now, suppose that our de Rham cover consists of two open sets $U_1, U_2$. Using Lemma 2.4 we get the following commutative diagram
This is a commutative diagram and from the hypothesis all downward arrows except the middle one are isomorphisms. Hence, the five lemma tells us that the middle map is indeed an isomorphism giving us the desired result. Now, we make the inductive hypothesis that the theorem holds for de Rham covers of size at most \( k \) where \( k \in \mathbb{Z}^+ \). Suppose that we have a de Rham cover of \( M \) with \( k + 1 \) open sets, \( U_i \) for \( 1 \leq i \leq k + 1 \) Then we have that \( M = A \cup B \) where \( A = U_1 \cup \cdots \cup U_k \) and \( B = U_{k+1} \). Then we use the result for covers of size two to finish the proof of the lemma.

\begin{lemma}
If \( M \) has a de Rham basis then \( M \) is de Rham.
\end{lemma}

\begin{proof}
Omitted, uses exhaustion functions. Refer to to [1] for full proof.
\end{proof}

\begin{lemma}
Any open subset of \( \mathbb{R}^n \) is de Rham.
\end{lemma}

\begin{proof}
We know that \( U \) has a basis of Euclidean balls, which is clearly a de Rham basis. So the preceding lemma gives the result.
\end{proof}

\begin{theorem}
Every smooth manifold is de Rham.
\end{theorem}

\begin{proof}
Every smooth manifold has a basis of of domains diffeomorphic to open subsets of \( \mathbb{R}^n \), namely the domains of particular smooth coordinate charts. This basis is de Rham, hence any smooth manifold is de Rham completing the proof of de Rham’s Theorem
\end{proof}

\begin{references}
1. Lee, John , Introduction to Smooth Manifolds
2. Fernandes, Rui, Differential Geometry Course Notes
\end{references}