1 Acknowledgments

This work heavily borrows from Applications of Lie Groups to Differential Equations by Peter J Olver. The following is essentially a summary of of sections 2.1-2.4 of Olver’s text.

2 Introduction

Here we will provide an overview of the method of finding symmetries of partial differential equations. First we will discuss some basic facts regarding local groups of transformations, symmetries of algebraic equations and group actions on functions. We will then proceed to a discussion of how to interpret the task of solving a PDE in terms of jet spaces, and the meaning of group actions in this setting. Finally, we will illustrate the method of finding symmetries by outlining an example.

Complete proofs for every claim will not be provided. Instead we will give general outlines and the important relevant facts.

3 Preliminaries

Because our interest is in localized symmetries of equations, we will need to make clear the notion of a local group of transformations.

Definition. Let $M$ be a smooth manifold. A Local Group of Transformations consists of a lie group $G$, a connected open set $U$ and a function $\Phi : U \to M$ such that

$$\{0\} \times M \subset U \subset G \times M$$

- If $(g,x) \in U$ and $(h,\Phi(g,x)) \in U$, then $\Phi(h,\Phi(g,x)) = \Phi(hg,x)$
- For all $x$ we have $\Phi(e,x) = x$
- If $(g,x) \in U$, then $(g^{-1},\Phi(g,x)) \in U$
For the sake of notation we will generally refer to a local group of transformations on $M$ as $G = (G, \mathcal{U})$.

**Example.** Let $M$ be the open upper half plane of $\mathbb{C}$. Let $G = SO(2)$. Then we can take $\mathcal{U} = \{(\psi, re^{i\theta} : 0 < \psi + \theta < \pi\}$. Then the canonical action of $G$ on $M$ gives a local group of transformations. Notice that $\theta \cdot x$ is not going to be defined in general.

**Theorem 1.** Suppose $G$ is a local group of transformations acting on $M$. Let $g$ be the Lie Algebra of $G$. Then there is a Lie Algebra homomorphism $\Phi : g \rightarrow \mathfrak{X}(M)$, such that
\[
\exp t\mathbf{x} \cdot p = \exp(t\Phi(x))(p), \text{ whenever } (\exp t\mathbf{x}, p) \in \mathcal{U}.
\]
Here we use the convention of describing $g$ as right invariant vector fields.

**Proof.** The proof of this proposition is essentially identical to the standard case where $\mathcal{U} = G \times M$. \hfill \Box

**Notation.** For a Lie group $G$, we will typically identify an element $v \in g = \mathfrak{L}(G)$ with its corresponding element $\Phi(v)$. Hence, if $f \in C^\infty(M)$ then if we write $v(f)$ for $v \in g$ then what we actually mean is $\Phi(v)(f)$.

### 3.1 Symmetries of Algebraic Equations

**Definition.** Suppose we have a smooth system of equations $\{f_i\}_{i=1}^n \subset C^\infty(M)$. Suppose $G$ is a local group of transformations on $M$. Then we say that $\{f_i\}$ is $G$ invariant if for any solution $x$ we have that $g \cdot x$ is also a solution.

**Example.** Let $M = \mathbb{R}^2$, and $F(x,y) = \frac{x}{y}$ and $G = (\mathbb{R}, \mathbb{R} \times M)$ where $\lambda \cdot (x,y) = (\lambda x, \lambda y)$

**Proposition 1.** Suppose $G$ is a local group of transformations acting on $M$. Suppose that $\{f_i\}_{i=1}^n \subset C^\infty(M)$ is a system of continuous functions of maximal rank. Then $\{f_i\}$ is $G$ invariant if $v(F) = 0$ everywhere on the solution set of $F$.

**Proof.** The proof of this fact is somewhat involve and utilizes the construction of functional independence and group invariants, which we do not discuss here. \hfill \Box

### 3.2 Groups Acting on Functions

Here $f : X \rightarrow \mathbb{R}^n$ will refer to a function of $m$ independent variables and $n$ dependent variables (i.e. $X \subset \mathbb{R}^m$). Suppose $G$ is a local group of transformations acting on $X \times U \subset \mathbb{R}^{m+n}$. We will denote the graph of $f$ by $\Gamma_f$.

**Lemma 1.** Suppose we have $f$ and $G$ as above. Then for any $x \in X$ there is a connected open set $e \in O \subset G$ and a connected open set $x \in \Omega$ such that there exists a unique smooth function $F : O \times \Omega \rightarrow \mathbb{R}^n$ such that $F(e,x) = f(x)$ and if we let $g \cdot f = F(g, -)$ then $\Gamma_{g \cdot f} \subset g \Gamma_f$. We call $g \cdot f$ the transform of $f$ by $g$. 

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Proof. This is shown by just showing that acting on graph of a small enough restriction of \( f \) by a small enough element of \( g \) still gives the graph of a function. Uniqueness follows from the fact that a function can be identified with its graph.

This lemma shows that we can locally transform a function around any point in its domain, if \( G \) acts locally on its graph.

**Notation.** Suppose we have \( f, G, \) and \( \Omega \) as above. For \( u = f(x) \) we will write the transformed variables as \( g \cdot (x, u) = (\Xi_g(x, u), \Phi_g(x, u)) = (\tilde{x}, \tilde{u}) \). Further, we will denote the transformed function \( g \cdot f \) by \( \tilde{f}(\tilde{x}) \).

Choosing \( \Omega \) does not often present much obstruction, it usually occurs naturally in the process of finding the formula for a transformed function, as the following proposition shows.

**Proposition 2.** Using the notation as above, we can write \( \tilde{f}(\tilde{x}) \) more explicitly as

\[
g \cdot f = [\Phi_g \circ (\text{Id} \times f)] \circ [\Xi_g \circ (\text{Id} \times f)]^{-1}
\]

It is clear that the ability to invert \( \Xi_g \circ (\text{Id} \times f) \) around a point \( x \) is the only obstruction for our choice of \( \Omega \).

**Example.** Take a linear function on the plane \( f(x) = ax + b \). Consider the group of rotations acting on the graph of \( f \). Computation, using the above formulas as a guide gives that

\[
\tilde{f}(\tilde{x}) = \frac{\sin \theta + a \cos \theta}{\cos \theta - a \sin \theta} \tilde{x} + \frac{b}{\cos \theta - a \sin \theta}
\]

Which will be defined for \( \cot \theta \neq a \).

**Definition.** Suppose we have a system of partial differential equations \( \Delta \) defined for functions \( f : X \to \mathbb{R}^n \) and local group of transformations \( G \) defined on \( X \times \mathbb{R}^n \). Then we say \( G \) is a symmetry of \( \Delta \) if for every solution \( f \) and group element \( g \) such that \( g \cdot f \) is defined on some domain \( \Omega \), then \( g \cdot f \) is a solution to \( \Delta \) on \( \Omega \).

### 4 Jets and Prolongation

**Notation.** For a natural number \( n \in \mathbb{N} \) and function \( f \) of \( m \) variables, let \( (\partial_{(n)}f) \) denote the \( q^{(m+n-1)} \)-tuple of combinations of \( n \)th order derivatives of \( f \).

**Definition.** Suppose \( f \in C^\infty(\mathbb{R}^m) \), then define the \( n \)th prolongation of \( f \) to be a smooth function \( \text{pr}^n f : \mathbb{R}^m \to \mathbb{R}^k \) given by

\[
\text{pr}^n f(x) = (f(x), \partial_{(1)} f, \partial_{(2)} f, \ldots, \partial_{(n)} f)
\]

We call \( \mathbb{R}^m \times \mathbb{R}^k \) the \( n \)th order *Jet Space* of \( \mathbb{R}^m \).
We can define the Jet Space invariants for a general manifold by taking the space $C^\infty(M)$ and applying the equivalence relation $f \sim g$ if their $n$th order Taylor polynomials are equal in every chart. We will denote the order $n$ jet space of a manifold $M$ by $J^n(M)$.

It is clear from this definition that it is natural to think of a system of partial differential equations order $n$ as being given by finitely many smooth functions $\{P_i\} \subset C^\infty(J^n)$. Solving a system of partial differential equations then amounts to finding a function $f$ such that $P_i(x, \text{pr}^n f(x)) = 0$ for every $i$.

### 4.1 Prolongation of Group Actions

**Proposition 3.** Suppose $G$ is a local group of transformations. Suppose that we have functions $f_1, f_2$ and an action $g \cdot f_1$ and $g \cdot f_2$ defined on some domain, then $\text{pr}^n f_1(x) = \text{pr}^n f_2(x)$ if and only if $\text{pr}^n f_1(\tilde{x}) = \text{pr}^n f_2(\tilde{x})$.

Hence $G$ induces a well defined local action on $J^n(M)$.

**Proof.** This is primarily a result of our formula for $g \cdot f$ and the chain rule combined with the definition of prolongation.

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**Definition.** Suppose we have a local group of transformations $G$ on $M$. Then we say that the order $n$ prolongation of $G$, which we will denote as $\text{pr}^n G$, is the local group of transformations on on $J^n(M)$ induced by $G$.

It turns out that if $G$ is a local group of transformations on $M$ with underlying group $G$ that $\text{pr}^n G$ also has underlying group $G$. In particular, if $G$ is a 1 parameter local group of transformations, then $\text{pr}^n G$ is a 1 parameter local group of transformations. Hence, we can make the following definition.

**Definition.** Let $X$ be a vector field on $M$. Let $\mathcal{G}$ be the local group of transformations induced by the flow of $X$. Then we define $\text{pr}^n X$ to be the vector field on $J^n(M)$ given by the infinitesimal generator of $\text{pr}^n \mathcal{G}$.

**Example.** Let $G$ be given by rotations of the plane $\mathbb{R}^2$. Then $J^1(\mathbb{R})$ can be written in the coordinates $x, u, u_x$. Since we know the form of rotated linear functions from above, we can explicitly compute that for rotations by angle $\theta$ the local action of $\text{pr}^1 G$ is given by

$$\theta \cdot (x, u, u_x) = (x \cos \theta - u \sin \theta, x \sin \theta + u \cos \theta, \frac{\sin \theta + u_x \cos \theta}{\cos \theta - u_x \sin \theta}).$$

If $v$ is the infinitesimal generator of $G$ then the infinitesimal generator of $\text{pr}^1 G$ is given by taking derivatives of the above formula and setting equal to zero to get

$$\text{pr}^1 v = -u \partial_x + x \partial_u + (1 + u_x^2) \partial_{u_x}.$$
4.2 Symmetries of PDEs

**Definition.** Suppose we have a system $\Delta$ of partial differential equations given by a finite collection $\{P_i\}_{i=1}^n \subset C^\infty(J^n(M))$. We say that $\Delta$ is of maximal rank if the corresponding function $P \in C^\infty(J^n(M),\mathbb{R}^k)$ is maximal rank on the subvariety $V_\Delta = \{x \in J^n(M) : P(x) = 0\}$.

In practice the maximal rank condition is not often a significant obstacle, since if $V_\Delta$ is sufficiently well behaved one can find an "equivalent" system $\Delta'$ which is of maximal rank.

**Theorem 2.** Suppose $G$ is a local group of transformations and $\Delta$ is a system of PDEs of maximal rank and order $n$. If for every infinitesimal generator $v$

$$\text{pr}^n v(D_i)(x) = 0 \text{ whenever } x \in V_\Delta,$$

then $G$ is a symmetry group of the equation.

**Proof.** This is, mostly, an immediate consequence of Theorem 1.

Under some additional assumptions about the behavior of $\Delta$, we can show that this condition is also necessary. To make full use of this theorem, we will give a general formula for the prolongation of a vector field. First, a definition.

**Definition.** Let $P \in C^\infty(J^k(M))$, then the total derivative with respect to $x^i$ is the unique function $D_i P \in C^\infty(J^{k+1}(M))$ such that for any smooth function $f$,

$$D_i P(x, \text{pr}^n f(x)) = \frac{\partial}{\partial x^i} (P(x, \text{pr}^n f(x))).$$

For a $k$-tuple, $J = (j_1, ..., j_k)$, the $J$-th total derivative will refer to

$$D_J = D_{j_1} \cdot \cdot \cdot D_{j_k}.$$

A consequence of the definition of total derivative is that the order of $J$ will not matter.

**Theorem 3.** Let $v = \sum_{i=1}^p \xi_i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \phi_\alpha(x,u) \frac{\partial}{\partial u^{\alpha}}$ be a vector field on $X \times U$. The $n$-th prolongation of $v$ is given by

$$\text{pr}^n v = v + \sum_{\alpha=1}^q \sum_J \phi^J_\alpha(x, \text{pr}^n u) \frac{\partial}{\partial u^J_\alpha}.$$

Where $J = (j_1, ..., j_k)$ ranges over unordered $n$-tuples of $1 \leq j_i \leq p$ with $1 \leq k \leq n$. The coefficients $\phi^J_\alpha$ are given by

$$\phi^J_\alpha(x, \text{pr}^n u) = D_J \left( \phi_\alpha - \sum_{i=1}^p \xi_i u^i_\alpha \right) + \sum_{i=1}^p \xi^i u^J_{j,i}.$$

Where $u^\alpha_i = \partial u^\alpha/\partial x^i$ and $u^J_{j,i} = \partial u^J_\alpha/\partial x^i$.
Example. Let \( p = 2 \) and \( q = 1 \) in the prolongation formula, then for

\[
v = \xi(x, t, u) \partial_x + \tau(x, t, u) \partial_t + \phi(x, t, u) \partial_u,
\]

we have

\[
\phi^t = \phi^t - \xi_t u_x + (\phi_u - \tau_t) u_t - \xi_u u_x u_t - \tau_u u_t^2
\]

and, even worse,

\[
\phi^{xx} = \phi_{xx} + (2\phi_{xx} - \xi_{xx}) u_x - \tau_{xx} u_t + (\phi_{uu} - 2\xi_{xx}) u_x^2 - 2\tau_{xx} u_x u_t,
\]

\[
- \xi_{uu} u_x^3 - \tau_{uu} u_x^2 u_t + (\phi_u - 2\xi_x) u_{xx} - 2\tau_{u} u_{x} - 3\xi_{u} u_{xx}
\]

\[
- \tau_{u} u_{x} u_{xx} - 2\tau_{u} u_{x} u_{xt}.
\]

These are just two of the components of the prolonged vector field. While the prolongation formula seems complex, its advantage is that it can be expressed very concretely. These equations are both just polynomials in derivatives of \( u \).

We will now demonstrate an example of using the prolongation formula to find the symmetries of a differential equation.

5 Example: The Heat Equation

Consider the heat equation.

\[
u_t = u_{xx}
\]

In this case, the criterion from the above theorem is \( pr^2(u_t - u_{xx}) = 0 \) which is the same as \( \phi'_{xx}(x, t, pr^2 u) = \phi'_{xx}(x, t, pr^2 u) \), whenever \( u_t = u_{xx} \). We have computed these coefficients above, and setting them equal gives a system of ten (individually very simple) differential equations which we write below.

\[
\begin{align*}
0 &= 2\tau_u \\
0 &= -2\tau_x \\
-\tau_u &= -\tau_u \\
0 &= \tau_{uu} \\
-\xi_u &= -2\tau_{xx} - 3\xi_u \\
\phi_u - \tau_t &= -\tau_{xx} + \phi_u - 2\xi \\\n0 &= -\xi_{uu} \\
0 &= \phi_{uu} - 2\xi_{xx} \\
-\xi_t &= 2\phi_{xx} - \xi_{xx} \\
\phi_1 &= \phi_{xx}
\end{align*}
\]

Solving this system is not too difficult and some work shows that,
\[ \xi = c_1 + c_4 x + 2c_5 t + 4c_6 xt \]
\[ \tau = c_2 + 2c_4 t + 4c_6 t^2 \]
\[ \phi = (c_3 - c_5 x - 2c_6 t - c_6 x^2)u + \alpha(x,t). \]

For constants \( c_i \) and a solution to the heat equation \( \alpha \). By fixing various constants we get that the infinitesimal symmetries of the heat equation are spanned by

\[ v_1 = \partial_x \]
\[ v_2 = \partial_t \]
\[ v_3 = u\partial_u \]
\[ v_4 = x\partial_x + 2t\partial_t \]
\[ v_5 = 2t\partial_x - xu\partial_u \]
\[ v_6 = 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u \]
\[ v_\alpha = \alpha(x,t)\partial_u \]

Each of these vector fields corresponds to some symmetry of the heat equation. Some with more obvious geometric interpretations than others. Interestingly, integrating \( v_6 \) and doing some reparameterization actually gives the fundamental solution of the heat equation!

The study of symmetries goes much deeper than these basic theorems and examples, but the author hopes that this brief summary demonstrates the vast potential for both theoretical and practical applications of the subject.