Darboux’s theorem and symplectic geometry

Liang, Feng

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Abstract

Symplectic geometry is a very important branch of differential geometry, it is a special case of poisson geometry, and could also give some geometric view of classical mechanics. In this project, we will give some basic concepts and properties of Symplectic manifold; moreover, we will also introduce the very fundamental property of Symplectic geometry- Darboux’s theorem, which shows that any Symplectic manifold is locally isomorphic to some \( \mathbb{R}^{2n} \) with its standard symplectic form.

1 Basic concepts

A **Symplectic manifold** is a smooth manifold \( M \) of even dimension \( 2n \) with a symplectic structure: a nondegenerate closed 2-form \( \omega \in \Omega^2(M) \)

Notice such \( \omega \) satisfies that:

- \( \omega(u, v) = -\omega(v, u) \)
- if \( \omega(u, v) = 0 \) for \( \forall v \), then \( u = 0 \)

We call a pair \( (V, \omega) \) of a finite dimensional real space with a bilinear form \( \omega: V \times V \to \mathbb{R} \) a **Symplectic vector space** if the \( \omega \) satisfies the two conditions above. By such definition, \( (T_pM, \omega_q) \) is a Symplectic vector space.

Moreover, as \( \omega \) closed, it defines a homotopy class \( a = [\omega] \in H^2(M; \mathbb{R}) \)

Give some examples:

**Example 1.1**
Sympletic manifold $\mathbb{R}^{2n}$ with standard symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

Notice: $\omega_0 \wedge \cdots \wedge \omega_0$ is a volume form.

- Every Riemann surface with its area form is a symplectic manifold
- The product of two symplectic manifold $M_1 \times M_2$ is a symplectic manifold with the symplectic form $\omega_1 \oplus \omega_2$

**Remark**

$\omega$ gives a isomorphism: $TM \to T^*M : X \mapsto \iota(X)\omega = \omega(X, \cdot)$

Under this isomorphism, we can consider more structures on a Sympletic manifold.

A *symplectomorphism* of a symplectic manifold $(M, \omega)$ is a diffeomorphism $\psi \in \text{Diff}(M)$ which preserves the symplectic form

$$\omega = \psi^* \omega$$

denote the group of symplectomorphisms by $\text{Symp}(M, \omega)$, or just $\text{Symp}(M)$.

A vector field $X \in \chi(M)$ is called *symplectic* if $\iota(X)\omega$ is closed.

**Example 1.2**

- $\psi : \mathbb{R}^4 \to \mathbb{R}^2; (x, y) \mapsto (x + j, A_j y + k), A_j = \left( \begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix} \right), (j, k) \in \mathbb{Z}^2$.

  It preserves the standard syplectic form.

In the course of differential manifold, we have seen that a vector field on a smooth manifold $M$ can give a flow on it which is a family of diffeomorphisms in $\text{Diff}(M)$; conversely, we can get a vector field by differentiating a family of diffeomorphisms. Indeed, we can also get a similar correspondence between symplectomorphisms and symplectic vector fields. It follows the next proposition

**Proposition 1.1** Let $M$ be a closed manifold. If $t \mapsto \psi_t \in \text{Diff}(M)$ is a smooth family of diffeomorphisms generated by a family of vector fields $X_t \in \chi(M)$ via

$$\frac{d}{dt} \psi_t = X_t \circ \psi_t, \quad \psi_0 = \text{id},$$
then \( \psi_t \in \text{Symp} (M, \omega) \) for every \( t \) if and only if \( X_t \in \chi(M, \omega) \) for every \( t \). Moreover, if \( X, Y \in \chi(M, \omega) \) then \([X, Y] \in \chi(M, \omega)\) and
\[
\iota([X, Y]) \omega = dH, \text{where } H = \omega(X, Y).
\]

Proof: (1)
\[
\frac{d}{dt} \psi_t^* \omega = \psi_t^* (\iota(X_t) d\omega + d(\iota(X_t) \omega)) = \psi_t^* d(\iota(X_t) \omega),
\]
and
\[
\mathcal{L}_X \omega = \iota(X) d\omega + d(\iota(X) \omega)
\]

(2)
\[
\iota([X, Y]) \omega = \frac{d}{dt} \bigg|_{t=0} \iota(\psi_t^* X) \omega = \mathcal{L}_Y (\iota(X) \omega) = d\iota(Y) \iota(X) \omega = d(\omega(X, Y)).
\]

As we mentioned in the very beginning, to give a well relation with the classical mechanics, we also want to study the object related to Energy, or more generally, Hamiltonian.

For a Symplectic manifold \((M, \omega)\), consider a smooth function \( H : M \to \mathbb{R} \), we can determine a vector field by:
\[
\iota(X) \omega = dH
\]
We call such vector field the **Hamiltonian vector field** associated to the **Hamiltonian function** \( H \). Also, define the associated Hamiltonian flow \( \phi_H^t \in \text{Diff}(M) \), which satisfies:
\[
\frac{d}{dt} \phi_H^t = X_H \circ \phi_H^t, \quad \phi_H^0 = id
\]

**Remark**
By noticing the fact:
\[
dH(X_H) = (\iota(X_H) \omega)(X_H) = \omega(X_H, X_H) = 0
\]
can see that Hamiltonian vector field is along to the level set of Hamiltonian function \( H \).

**Example 1.3**
\( H : \mathbb{S}^2 \to \mathbb{R}; (x_1, x_2, x_3) \mapsto x_3 \), \( \phi_H^t \) is the rotation along \( x_3 \)-axis with constant speed, \( X_H \) is simply \( \frac{\partial}{\partial \theta} \)
As expected, there are some properties for Hamiltonian things.

**Proposition 1.2** 
\((M, \omega)\) symplectic manifold.

i Hamiltonian flow \(\phi^t_H \in \text{Symp}(M, \omega)\), tangent to the level surfaces of \(H\).

ii \(H : M \to \mathbb{R}\) and \(\psi \in \text{Symp}(M, \omega)\), then \(X_{H \circ \psi} = \psi \ast X_H\).

iii \([X_F, X_G] = X_{\{F,G\}}\), Poisson bracket defined as \(F, G = \omega(X_F, X_H) = dF(X_H)\)

**Proof:** (i) directly from last Proposition 3.2 and closedness from definitions. (ii) comes from:
\[
i(X_{H \circ \psi}) \omega = d(H \circ \psi) = \psi \ast dH = \psi i(X_H) \omega = \psi i(\psi \ast X_H) \omega.
\]

(iii) By (i), \(\phi^t_G \in \text{Symp}(M, \omega)\), then
\[
[X_F, X_G] = -\frac{d}{dt} \bigg|_{t=0} (\phi^t_F) \ast X_G = -\frac{d}{dt} \bigg|_{t=0} X_{G \circ \phi^t_F}.
\]

Hence
\[
i([X_F, Y_G]) \omega = -\frac{d}{dt} \bigg|_{t=0} d(G \circ \phi^t_F) = -d \frac{d}{dt} \bigg|_{t=0} G \circ \phi^t_F = -d(dG(X_F)) = -d\{G, F\}
\]
\[
= d\{F, G\} = X_{\{F,G\}}
\]

After knowing some basic properties of Hamiltonian things. We want to extend some more concepts in differential geometry, give the definition of **Hamiltonian isotopies**.

In topology or differential geometry, Isotopy is a topology concepts, which is a family of embeddings connecting 2 continuous embeddings in a smooth manifold or a topological space. Then, relate it with Symplectic manifold.

\(M\) Symplectic, consider smooth map \([0, 1] \times M \to M : (t, q) \mapsto \psi_t(q)\) with \(\psi_t \in \text{Symp}(M, \omega)\) and \(\psi_0 = \text{id}\). Call such a family of symplectomorphism **symplectic isotopies**.

Given \(\psi_t\), we can find its related vector fields \(X_t\) by \(\frac{d}{dt} \psi_t = X_t \circ \psi_t, X_t\) symplectic by previous proposition, shows \(i(X_t) \omega\) closed. Furthermore, if it is a boundary with \(i(X_t) \omega = dH\), we call
such $H$, time-dependent Hamiltonian, call such $\psi_t$ Hamiltonian isotopy. For $\psi \in \text{Symp}(M,\omega)$ with a Hamiltonian isotopy $\psi_t$ connecting it and identity, i.e. $\psi_0 = \text{id}$, $\psi_1 = \psi$, we call such $\psi$ is Hamiltonian. Denote the group of Hamiltoniansymplectomorphisms by $\text{Ham}(M,\omega)$, it is normal in $\text{Symp}(M)$.

Example 1.4

- unknot is not isotopic to Trefoil knot.

Before proving the Darboux’s theorem, we want to give a cotangent bundle $T^*L$ a symplectic structure to finish the basic concept part.

For a smooth $M$, let $x = (x_1, ..., x_n): U \to \mathbb{R}^n$ the local coordinate chart on $L$, as $T_q^*L$ has the basis $x_i$, $\forall v^* \in T_q^*L$, it is in a unique form $\lambda_{can} := \sum_{i=1}^{n} y_i dx_i$, this gives the canonical 1-form, by taking the differential, we get the canonical 2-form $\omega_{can} = -d\lambda_{can} = dx \wedge dy$.

By this construction, we see that indeed, canonical 1-form at $(x, v^*)$ is just $v^*$ itself in some sense. Moreover, not hard to check that $(T^*L, \omega_{can})$ is a symplectic manifold, which means that we find a symplectic structure for that cotangent bundle.

There is also a proposition without proof here follows:

Proposition 1.3

$\lambda_{can}$ is uniquely characterized by the property: $\sigma^* \lambda_{can} = \sigma$, $\forall$ 1-form $\sigma$.

2 Darboux’s theorem

Finally, we come to a very important and fundamental results in many fields-Darboux’s theorem. It was found by Jean Gaston Darboux when he solved the Pfaff problem. Here,
we will talk about it in the most known form for symplectic geometry.

**Darboux’s theorem**

Every symplectic form $\omega$ on $M$ is locally diffeomorphic to the standard form $\omega_0$ on $\mathbb{R}^{2n}$

This theorem tells us any symplectic manifold in same dimension looks same locally.

Next, we give a proof by applying the *Moser’s argument*.

**Lemma 1/ Moser’s argument**

For every family of symplectic forms $\omega_t \in \Omega^2(M)$ which are exact:

$$\frac{d}{dt} \omega_t = d\sigma_t$$

then, $\exists$ a family of diffeomorphisms $\psi_t \in \text{Diff}(M)$ such that:

$$\psi_t^* \omega_t = \omega_0$$

Proof: We want to think $\psi_t$ as a flow of some $X_t$, via $\frac{d}{dt} \psi_t = X_t \circ \psi_t, \quad \psi_0 = id$. It satisfies the condition for the manifold, still need to solve the equation to satisfies the condition for the symplectic form on it, differentiate both side of

$$\psi_t^* \omega_t = \omega_0$$

, get:

$$0 = \frac{d}{dt} \psi_t^* \omega_t = \psi_t^* (\frac{d}{dt} \omega_t + \iota(X_t)d\omega_t + dt(X_t)\omega_t) = \psi_t^* (d\sigma_t + dt(X_t)\omega_t)$$

Solve: $\psi_t^* (d\sigma_t + dt(X_t)\omega_t)$, as $\omega_t$ nondegenerate, get such $X_t$.

Move on to last Lemma for the Proof of Darboux theorem.

**Lemma 2** Let $M$ be a $2n$-dimensional smooth manifold and $Q \subset M$ be a compact submanifold. Suppose that $\omega_0, \omega_1 \in \Omega^2(M)$ are equal and non-degenerate on $T_qM$. Then there exist open neighborhoods $N_0$ and $N_1$ of $Q$ and a diffeomorphism $\psi : N_0 \to N_1$ such that

$$\psi |_Q = id, \quad \psi \ast \omega_1 = \omega_0.$$ 

This lemma is very strong. If we take $Q$ as a point, this lemma tells us if two symplectic form are identical or isomorphic at some point, they are identical or isomorphic on their
germ. Also, we can see any symplectic form are isomorphic on $M$ at a point, we can think locally $M$ as $\mathbb{R}^{2n}$. After this isomorphism, we are in $\mathbb{R}^{2n}$. As we just point out, it then gives a isomorphism to the standard form at a point, then, by lemma, we get an isomorphism to $\omega_0$ on the germ of this point. Composite all germs, we get the Darboux’s theorem.

Proof of lemma 2:

**Proof:** By Moser: enough to show $\exists$ some 1-form $\sigma \in \Omega^1(\mathcal{N}_0)$ such that

$$\sigma |_{T_QM} = 0, \quad d\sigma = \omega_1 - \omega_0.$$  

With $\omega_t = \omega_0 + t(\omega_1 - \omega_0) = \omega_0 + t d\sigma$ satisfies the condition of Moser, on $\mathcal{N}_0$.

Shrinking $\mathcal{N}_0$ to ensure that $\omega_t$ is non-degenerate in $\mathcal{N}_0$ for every $t$ as Moser requires. Shrinking $\mathcal{N}_0$ again to ensure the solution for Moser exist on the required time interval $0 \leq t \leq 1$.

Prove what we claim at the beginning of the proof:

Consider the restriction of exponential map to the normal bundle $TQ^\perp$ of the submanifold $Q$ with respect to any Riemannian metric on $M$:

$$\exp: TQ^\perp \to M.$$  

Consider the neighbourhood of the zero section:

$$U_\varepsilon = \{(q,v) \in TM \mid q \in Q, v \in T_qQ^\perp, |v| < \varepsilon\}.$$  

Then get a diffeomorphism: $\mathcal{N}_0 = \exp(U_\varepsilon)$ for $\varepsilon > 0$ sufficiently small.

Define $\phi_t: \mathcal{N}_0 \to \mathcal{N}_0$ for $0 \leq t \leq 1$ by

$$\phi_t(\exp(q,v)) = \exp(q,tv).$$  

$\phi_t$ is a diffeomorphism for $t > 0$ with $\phi_0(\mathcal{N}_0) \subset Q$, $\phi_1 = \text{id}$, and $\phi_t |_Q = \text{id}$. This implies $\phi_0 * \tau = 0$, $\phi_1 * \tau = \tau$, where $\tau = \omega_1 - \omega_0$.

Then, can define the vector field

$$X_t = (\frac{d}{dt}\phi_t) \circ \phi_t^{-1}$$  

for $t > 0$. $X_t$ singular at $t = 0$. However, we obtain

$$\frac{d}{dt} \phi_t * \tau = \phi_t * \mathcal{L}_{X_t}\tau = d(\phi_t * \iota(X_t)\tau) = d\sigma_t,$$

but such family of $\sigma_t(q,v) = \tau(\phi_t(q); \frac{d}{dt}\phi_t(q), d\phi_t(q)v)$ smooth at $t = 0$ and vanishes on $Q$.  

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Finally, we get:

\[ \tau = \phi_1 \ast \tau - \phi_0 \ast \tau = \int_0^1 \frac{d}{dt} \phi_t \ast \tau dt = d\sigma, \quad \sigma = \int_0^1 \sigma_t dt. \]

This finish the proof.

This paper follows part of the reference text Chapter 3, gives a introduction to the symplectic geometry.

Darboux’s theorem is very fundamental and also very important. We can not expect the math world without it nowadays. Hope we can find more such theorems in the future. The way to the future is not flat but we will go further. Thank you.
References