Abstract. This is an expository paper for the purposes of the course Differentiable Manifolds II, which took place in Spring 2014 at UIUC. After reviewing some basic facts on group actions, we include a modern version of the proof of Bochner’s Linearization Theorem from the book of Duistermaat and Kolk [5].

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1. Introduction

The investigations by Cartan [3] of automorphisms and general transformations of a domain in several complex variables into itself by means of complex analytic functions led to the following result: If a compact group of automorphisms has a fixed point then in suitably chosen local coordinates around the fixed point, all transformations are linear. Bochner [1] then proved a similar result in the smooth case:

**Theorem:** Let $G \times M \to M$ be a smooth action of a compact Lie group $G$ on a smooth manifold $M$, which has a fixed point. Then in suitably chosen local coordinates around the fixed point the action is linear.

Palais and Smale then suggested extending the above result to the non-compact case. In this direction, Hermann [7] proved that if $G$ is a semisimple algebra of vector fields on a manifold which have a common zero point, then all the vector fields in $G$ can be simultaneously linearized by a formal power series change of coordinates.

In the case when the group $G$ is connected, finding a linear system of coordinates for the Lie group action, is the same as finding a linear system of coordinates for the associated Lie algebra action. However, since a Lie algebra action does not always integrate to a Lie group action, the linearization problem for Lie algebra actions is harder. To make this more precise, considering an action $\rho : g \to \mathfrak{X}(M)$ of a Lie algebra $g$ on a manifold $M$ with a fixed point $x_0$, we are asking whether there...
are coordinates \((x^1, \ldots, x^m)\) around the fixed point \(x_0\) such that for all \(X \in \mathfrak{g}\), the vector fields \(\rho(X)\) are linear in these coordinates.

The problem of linearization of Lie algebra actions turns out to have remarkable similarities with the problem of linearization of Poisson brackets, which is of central interest in the area of Poisson Geometry. For more details on the linearization of Poisson brackets and how this is connected to the case of Lie algebras see the paper of Fernandes and Monnier [6], as well as the book of Dufour and Zung [4].

In the preceding paragraphs we described one direction in which the linearization of compact Lie group actions can be extended. In what follows, we are including a proof of the Bochner Linearization Theorem using an averaging principle argument.

2. Group actions

**Definition 2.1.** Let \(M\) be a set. An action of a group \(G\) on \(M\) is a homomorphism \(A: G \to GL(M)\). Writing for \(g \in G, x \in M\)

\[(A(g))(x) = A(g,x)\]

we can also describe the action of \(G\) on \(M\) as a mapping \(A: G \times M \to M\) such that

\[A(gh,x) = A(g,A(h,x))\]

for \(g,h \in G, x \in M\).

In particular, if \(G\) is a Lie group, \(M\) is a smooth manifold and the map \(A\) is smooth, we say that we have a smooth action. In this case, each \(A(g): M \to M\) is a diffeomorphism of \(M\), so one also says that \(G\) is a transformation group of \(M\).

If \(G\) is a topological group and \(M\) a topological space, the action is said to be continuous if the mapping \(A: G \times M \to M\) is continuous.

Moreover, in case \(M\) is a vector space \((V, +, \cdot)\) over a field \(k\), the action \(A\) will be called linear if it is compatible with the linear structure of \(V\), i.e. the following hold:

\(i)\) for every \(v_1, v_2 \in V\) and \(g \in G\):

\[A(g,v_1 + v_2) = A(g,v_1) + A(g,v_2)\]

\(i)\) for every \(\lambda \in k, g \in G\) and \(v \in V\):

\[A(g,\lambda v) = \lambda A(g,v)\]

A smooth linear action corresponds precisely to a representation \(\rho: G \to GL(V)\) (for further information, see Lee [8], p. 209).

**Definition 2.2.** For each \(x \in M\), the orbit through \(x\) for the action \(A\) is defined to be the set

\[A(G(x)) = G \cdot x = \{A(g)(x) | g \in G\}\]

**Lemma 2.3.** The relation \(y \in G \cdot x\) is an equivalence relation in \(M\).

**Proof.** If 1 denotes the identity element of \(G\), then \(A(1)(x) = id(x) = x\), so \(x \in G \cdot x\) showing that the relation is reflexive. Moreover, if \(y \in G \cdot x\) then \(y = A(g)(x)\) for some \(g \in G\). But then \(x = A(g^{-1})(y) = A(g^{-1})(g)\) hence \(x \in G \cdot y\), showing that the relation is symmetric. Finally, if \(y \in G \cdot x\) and \(z \in G \cdot y\) then \(y = g \cdot x\) and \(z = h \cdot y\), for some \(g, h \in G\). Thus \(z = h \cdot (g \cdot x) = (hg) \cdot x\), or \(z \in G \cdot x\), providing transitivity. \(\square\)

Therefore the space \(M\) is partitioned into orbits; the set of orbits is called the quotient \(G \backslash M\) of \(M\) under the action of \(G\) on \(M\). Moreover, the mapping \(\pi: M \to G \backslash M\) with \(x \mapsto G \cdot x\) is called the canonical projection.

The action is said to be transitive if there is only one orbit, i.e. if there is an \(x \in M\) such that for each \(y \in M\), \(y = g \cdot x\) for some \(g \in G\). In this case, \(M\) is called a homogeneous space.
The quotient topology on the orbit space $G \setminus M$ is defined so that a set $V$ is open in $G \setminus M$ if and only if the $G$-invariant subset $\pi^{-1}(V)$ is open in $M$. In general, the quotient topology need not be Hausdorff, i.e. it is not necessarily true that: \{ $G \cdot x$ \} is closed in $G \setminus M$ if and only if $G \cdot x$ is closed in $M$.

**Definition 2.4.** A subset $N$ of $M$ is said to be invariant under the action $A$ or $G$-invariant, if $g \cdot y \in N$ whenever $g \in N$, $y \in G$; in other words $A(G \times N) \subset N$. Note that $N$ is invariant if and only if $N$ is a union of orbits.

**Definition 2.5.** For each $x \in M$, we see that $G_x = \{ g \in G | g \cdot x = x \}$ is a subgroup of $G$, called the isotropy group of the action at the point $x$, or the stabilizer of $x$ under the action. Whenever $G_x = \{1\}$, the action is said to be free at $x$. Moreover, an action is said to be free, if it is free at every $x \in M$.

### 3. Bochner Linearization Theorem

We are now getting to the proof of the main theorem as described in the book of Duistermaat and Kolk [5], p. 96. An important tool that we are using is the averaging principle, for which further details are included in the appendix. After completing this proof, we will see that the theorem actually says that near a fixed point, the action of the compact group $G$ can be identified with the linear action of a closed Lie subgroup of the orthogonal group acting on a ball around the origin in an Euclidean space.

For this version of the proof, we are considering $M$ to be a finite-dimensional real-analytic manifold (i.e. transition maps are expressible as convergent power series in a neighborhood of each point) and $G$ to be a compact topological group acting continuously on $M$ by means of $C^k$ transformations, where $1 \leq k \leq \omega$ (here $C^\omega$ denotes the real-analytic mappings).

**Theorem 3.1. Bochner’s Linearization Theorem.** Let $A$ be a continuous homomorphism from a compact group $G$ to $\text{Diff}^k(M)$, with $k \geq 1$ and let $x_0 \in M$, with $A(g)(x_0) = x_0$, for all $g \in G$. Then there exists a $G$-invariant open neighborhood $U$ of $x_0$ in $M$ and a $C^k$ diffeomorphism $\chi$ from $U$ onto an open neighborhood $V$ of 0 in $T_{x_0}M$, such that:

$$\chi(x_0) = 0, \quad T_{x_0} \chi = I : T_{x_0}M \rightarrow T_{x_0}M$$

and:

$$\chi(A(g)(x)) = T_{x_0}A(g)\chi(x), \quad \text{for } g \in G, x \in U$$

**Proof.** We construct first the $G$-invariant open neighborhood of $x_0$:

For every open neighborhood $U'$ of $x_0$ in $M$, there exists a $G$-invariant open neighborhood $U$ of $x_0$ in $M$ that is contained in $U'$. Indeed, because $A : G \times M \rightarrow M$ with $(g, x) \mapsto A(g)(x)$ is continuous at $(g, x_0)$ and $A(g, x_0) = x_0$, there exist for each $g \in G$ open neighborhoods $W(g)$ of $g$ in $G$, and $U'(g)$ of $x_0$ in $M$, respectively, such that $A(W(g) \times U'(g)) \subset U'$. Because $G$ is compact, there is a finite subset $F$ of $G$ such that $\bigcup_{g \in F} W(g) = G$ and it follows that $A(G \times U'') \subset U'$, where $U'' = \bigcap_{g \in F} U(g)$ is an open neighborhood of $x_0$ in $M$. Now $U = A(G \times U'') = \bigcup_{g \in G} A(g)(U'')$ is the desired $G$-invariant neighborhood of $x_0$ in $M$. 

Let $\tilde{\chi}$ be a $C^k$ mapping from an open neighborhood $\tilde{U}$ of $x_0$ in $M$, to $T_{x_0}M$, such that $\tilde{\chi}(x_0) = 0$ and $T_{x_0}\tilde{\chi} = Id_{T_{x_0}M}$. The existence of such $\tilde{\chi}$ is obvious. Restricting now this $\tilde{\chi}$ suitably, we can arrange that $\tilde{U}$ is $G$-invariant.

Now, the mapping:

$$(g, x) \mapsto T_{x_0}A(g) \circ \tilde{\chi} \circ A(g)^{-1}$$

is a continuous representation $\pi$ of $G$ in the space of $C^k$ mappings $\chi : \tilde{U} \to T_{x_0}M$ such that $\chi(x_0) = 0$, which is a complete locally convex topological vector space. Therefore, we can apply the Averaging Principle for this space to form:

$$\bar{\chi} = \int_G T_{x_0}A(g) \circ \tilde{\chi} \circ A(g)^{-1} dg$$

or more explicitly

$$\bar{\chi}(x) = \int_G T_{x_0}A(g) \tilde{\chi}(A(g)^{-1}(x)) dg$$

for $x \in \tilde{U}$. But now, since $av(\pi)(\bar{\chi}) = \tilde{\chi}$ (see in the appendix), we get that:

$$T_{x_0}A(g) \circ \tilde{\chi} \circ A(g)^{-1} = \bar{\chi}$$

for $g \in G$, i.e. $\bar{\chi}$ satisfies on $\tilde{U}$ the relation described in the theorem.

Moreover,

$$T_{x_0} \left( T_{x_0}A(g) \circ \tilde{\chi} \circ A(g)^{-1} \right) = T_{x_0}A(g) \circ T_{x_0}\tilde{\chi} \circ T_{x_0}A(g)^{-1} = Id$$

for all $g \in G$, so $T_{x_0}\bar{\chi} = Id_{T_{x_0}M}$. By the inverse mapping theorem there is an open neighborhood $U'$ of $x_0$ in $M$ such that $\bar{\chi}|_{U'}$ is a $C^k$ diffeomorphism from $U'$ onto an open neighborhood of 0 in $T_{x_0}M$. Restricting further $\bar{\chi}$ to a $G$-invariant open neighborhood of $x_0$ in $U'$ we get the desired $\chi$.

**Remark 3.2.** The result of the theorem indeed describes the linearization of the group action in the following sense:

We will first need to introduce some further terminology.

**Definition 3.3.** Let $X, Y$ be topological spaces and $A : G \times X \to X$, $B : G \times Y \to Y$ actions of a group $G$ on $X$ and $Y$ respectively. We say that a mapping $\Phi : X \to Y$ intertwines the action $A$ with $B$ or is $G$-equivariant $X \to Y$, if the following diagram is commutative

$$
\begin{array}{ccc}
X & \stackrel{\Phi}{\longrightarrow} & Y \\
A(g) \downarrow & \circ & \downarrow B(g) \\
X & \stackrel{\Phi}{\longrightarrow} & Y
\end{array}
$$

in other words, if $\Phi \circ A(g) = B(g) \circ \Phi, \; \forall g \in G$.

In particular, assume $G$ is a Lie group and both $A, B$ are $C^k$ actions on the manifolds $X, Y$ respectively. Then $\Phi$ is called an equivalence of $C^k$ actions, if $\Phi : X \to Y$ is a $C^k$ diffeomorphism intertwining $A$ with $B$. In that case, the actions $A$ and $B$ are said to be $C^k$-equivalent.

Continuing Remark 3.2, we now see that in the above sense, Theorem 3.1 says that the action of the compact Lie group $G$ on the real-analytic manifold $M$, restricted to a suitable $G$-invariant open neighborhood of the fixed point $x_0$, is equivalent to the linear tangent action of $G$ on $T_{x_0}M$, restricted to an open neighborhood of 0 in $T_{x_0}M$. 


Indeed, we have seen that the following diagram is commutative for every $g \in G$
\[
\begin{array}{ccc}
U & \xrightarrow{A} & V \\
\downarrow A(g) & \circlearrowleft & \downarrow T_{x_0}A(g) \\
U & \xrightarrow{\chi} & V
\end{array}
\]

Remark 3.4. If $h$ is an arbitrary inner product on $T_{x_0}M$ then $\bar{h} = \int_G T_{x_0}A(g)^*hdg$ is an inner product on $T_{x_0}M$ that is invariant under the tangent action of $G$ on $T_{x_0}M$. In other words, $G' = \{T_{x_0}A(g) | g \in G\}$ is a compact and hence closed Lie subgroup of the orthogonal group of the Euclidean space $E = (T_{x_0}M, \bar{h})$. Let $B$ be an open ball around 0 in $E$ that is contained in $V$, with $V$ as in the preceding theorem. Then $B$ is $G'$-invariant, therefore $\chi^{-1}(B)$ is a $G$-invariant open neighborhood of $x_0$ in $U$ on which $G$ acts as described in the sentence preceding the Theorem.

Appendix A. The Averaging principle

In this section we will describe the averaging principle for continuous functions $f$ on the compact Lie group $G$ with values in a complete, locally convex, topological vector space $V$. Using the uniform continuity of $f$ on the compact space $G$, one can find for each continuous seminorm $\nu$ on $V$ and each $\varepsilon > 0$, a finite partition of unity with nonnegative continuous functions $h_j$ on $G$, such that $\nu(f(x) - f(y)) < \varepsilon$, if $x, y \in G$ are contained in the support of the same $h_j$. Choosing $x_j \in \text{supp}h_j$ we see that the "Riemann sums":
\[
\sum_j \int_G h_j(x) dx f(x_j)
\]
in $V$ form a Cauchy net, converging to an element of $V$, which by definition is the integral $\int_G f(x) dx$. Clearly, for every continuous linear form $\mu$ on $V$, we have
\[
\mu \left( \int_G f(x) dx \right) = \int_G \mu(f(x)) dx, \text{ for } \mu \in V'
\]
Of course, if $V$ is finite-dimensional, then these definitions are much more elementary and the next proposition just expresses that the averaging is defined coordinate-wise.

If $\pi$ is a representation of $G$ in the complete, locally convex, topological vector space $V$, then one can define for every $f \in C(G)$ the linear mapping $\pi(f) : V \to V$ by
\[
\pi(f)(v) := \int_G f(x) \pi(x)(v) dx, \text{ for } v \in V
\]
A particular case occurs when $f = 1$, the function on $G$ which is constant, equal to 1. In this case, the corresponding operator:
\[
\text{av}(\pi) : V \to V \text{ with } v \mapsto \int_G \pi(x)(v) dx
\]
is called the average of the representation $\pi$. 
For each \( v \in V \) the approximating sums:

\[
\sum_j \int_G h_j (x) f (x_j) \tau (x_j) (v)
\]

form a bounded subset of \( V \). Thus, if \( V \) is a barreled space, we can use the Banach-Steinhaus theorem to conclude that \( \tau (f) : V \to V \) is continuous. See Bourbaki [2], Ch. III, §3, No.6.

**Proposition A.1. Averaging principle.** Let \( G \) be a compact group, and \( \tau \) a representation of \( G \) in the complete, locally convex, topological vector space \( V \). Then \( \text{av}(\tau) \) is a linear projection from \( V \) onto the space \( V^{\tau(G)} := \{ v \in V | \tau (x) (v) = v \} \) of fixed points for the action \( \tau \) of \( G \) on \( V \).

**Proof.** For any \( v \in V, g \in G \) we have

\[
\tau (g) \circ \text{av}(\tau) (v) = \tau (g) \left( \int_G \tau (x) (v) \, dx \right) = \int_G \tau (g) \circ \tau (x) (v) \, dx
\]

\[
= \int_G \tau (gx) (v) \, dx = \int_G \tau (g) (v) \, dx = \text{av}(\tau) (v)
\]

where we have used the continuity of \( \tau (g) \) in the second equality and the left invariance of averaging in the fourth one.

Therefore \( \text{av}(\tau) (V) \subset V^{\tau(G)} \).

On the other hand, if \( v \in V^{\tau(G)} \) then we get:

\[
\text{av}(\tau) (v) = \int_G \tau (x) (v) \, dx = \int_G v \, dx = v
\]

using that \( \int_G dx = 1 \); so \( \text{av}(\tau) \) acts on \( V^{\tau(G)} \) as the identity. \( \square \)

**Remark A.2.** In the proof of Bochner’s Linearization Theorem we have used the Averaging principle in the real version.

**References**