Symmetric Spaces
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May 5, 2014

Notations

\( M \) Riemannian manifold.
\( N \) normal neighborhood of the origin in \( T_pM \).
\( N_p \) normal neighbourhood of \( p \), \( N_p = \text{exp} N_0 \).
\( s_p \) geodesic symmetry with respect to \( p \).
\( f^\Phi \) \( d\Phi f = f \circ \Phi \).
\( X^\Phi \) \( d\Phi X \).
\( K(S) \) sectional curvature of \( M \) at \( p \) along the section \( S \).
\( D^r_s \) set of tensor fields of type \((r, s)\).
\( I(M) \) the set of all isometries on \( M \).

1 Affine Locally Symmetric Space, Isometry Group and Others

Definition 1 (normal neighborhood) A neighborhood \( N_p \) of \( p \) in \( M \) is called a normal neighbourhood if \( N_p = \text{exp} N_0 \), where \( N_0 \) is a normal neighborhood of the origin in \( T_pM \), i.e. satisfying: (1) \( \text{exp} \) is a diffeomorphism of \( N_0 \) onto an open neighborhood \( N_p \); (2) if \( X \in N_0 \), \( 0 \leq t \leq 1 \), then \( tX \in N_p \) (star shaped).

Definition 2 (geodesic symmetry) \( \forall q \in N_p \), consider the geodesic \( t \to \gamma(t) \subset N_p \) passing through \( p \) and \( q \) s.t. \( \gamma(0) = p \), \( \gamma(1) = q \). Then the mapping \( q \to q^\prime = \gamma(-1) \) of \( N_p \) onto itself is called geodesic symmetry w.r.t \( p \), denoted by \( s_p \).

Remark: \( s_p \) is a diffeomorphism of \( N_p \) onto itself and \( (ds_p)_p = -I \).

Definition 3 (Affine locally symmetric) \( M \) is called affine locally symmetric if each point \( m \in M \) has an open neighborhood \( N_m \) on which the geodesic symmetry \( s_m \) is an affine transformation. i.e. \( \nabla_X(Y) = (\nabla_{X^s_m}(Y^{s_m}))^{s_m^{-1}} \), \( \forall X, Y \in \mathfrak{X}(M) \).

Definition 4 (Pseudo-Riemannian structure) Let \( M \) be a \( C^\infty \)-manifold. A pseudo-Riemannian structure on \( M \) is a symmetric nondegenerate (as bilinear form at each \( p \in M \)) tensor field \( g \) of type \((0, 2)\).

Remark: A pseudo-Riemannian manifold is a connected \( C^\infty \)-manifold with
a pseudo-Riemannian structure. \( g \) is called a Riemannian structure iff \( g_p \) is positive definite \( \forall p \in M \).

**Definition 5 (Isometry)** Let \( M \) and \( N \) be two \( C^\infty \) manifolds with pseudo-Riemannian structures \( g \) and \( h \), respectively. Let \( \phi \) be a mapping of \( M \) into \( N \).

Then \( \phi \) is called an isometry if \( \phi \) is a diffeomorphism of \( M \) onto \( N \) and \( \phi^* h = g \). \( \phi \) is called a local isometry if for each \( p \in M \) there exist open neighborhoods \( U \) of \( p \) and \( V \) of \( \phi(p) \) s.t. \( \phi \) is an isometry of \( U \) onto \( V \).

**Definition 6 (Riemannian locally symmetric space)** \( M \) is called a Riemannian locally symmetric space if for each \( p \in M \) \( \exists \) a normal neighborhood \( N_p \) of \( p \) on which the geodesic symmetry \( s_p \) is an isometry, i.e. \( s_p(g) = g \), where \( g \) is the pseudo-Riemannian structure on \( N_p \subset M \). (\( s_p \) is a local isometry from \( M \) to itself).

2 **Definition of Symmetric Space**

**Definition 7 (Riemannian Globally Symmetric Space)** Let \( M \) be an analytic Riemannian manifold, \( M \) is called Riemannian globally symmetric if each \( p \in M \) is an isolated fixed point of an involutive (its square but not the mapping itself is the identity) isometry \( s_p \) of \( M \). Or equivalently \( \forall p \in M \) there is some \( s_p \in I(M) \) with the properties: \( s_p(p) = p \), \( (ds_p)_p = -I \).

**Example 1: Euclidean Space**

Let \( M = \mathbb{R}^n \) with the Euclidean metric. The geodesic symmetry at any point \( p \in \mathbb{R}^n \) is the point reflection \( s_p(p + v) = p - v \). The isometry group is the Euclidean group \( E(n) \) generated by translations and orthogonal linear maps; the isotropy group of the origin is the orthogonal group \( O(n) \). Note that the symmetries do not generate the full isometry group \( E(n) \) but only a subgroup which is an order-two extension of the translation group.

**Example 2: The Sphere**

Let \( M = S^n \) be the unit sphere with the standard scalar product. The symmetry at any \( x \in S^n \) is the reflection at the line \( \mathbb{R}x \subset \mathbb{R}^{n+1} \), i.e. \( s_x(y) = -y + 2\langle y, x \rangle x \) (the component of \( y \) in \( x \)-direction, \( \langle y, x \rangle x \), is preserved while the orthogonal complement \( y(y, x)x \) changes sign). In this case, the symmetries generate the full isometry group which is the orthogonal group \( O(n+1) \). The isotropy group of the last standard unit vector \( e_{n+1} = (0, ..., 0, 1)^T \) is \( O(n) \subset O(n+1) \).

**Example 3: Compact Lie groups**

Let \( M = G \) be a compact Lie group with biinvariant Riemannian metric, i.e. left and right translations \( L_g, R_g : G \to G \) acts as isometries for any \( g \in G \). Then \( G \) is a symmetric space where the symmetry at the unit element \( e \in G \) is the inversion \( s_e(g) = g^{-1} \). Then \( s_e(e) = e \) and \( ds_e(v) = -v \) for any \( v \in \mathfrak{g} = T_eG \), so the involutive condition is satisfied. We have to check that \( s_e \) is an isometry,
i.e. \((ds_e)_g\) preserves the metric for any \(g \in G\). This is certainly true for \(g = e\), and for arbitrary \(g \in G\) we have the relation \(s_e \circ L_g = R_{g^{-1}} \circ s_e\) which shows \((ds_e)_g \circ (dL_g)_e = (dR_{g^{-1}})_e \circ (ds_e)_e\). Thus \((ds_e)_g\) preserves the metric since so do the other three maps in the above relation.

**Example 4: Projection model of the Grassmannians**

Let \(S = G_k(\mathbb{R}^n)\) be the set of all \(k\)-dimensional linear subspaces of \(\mathbb{R}^n\). The group \(O(n)\) acts transitively on this set. The symmetry \(s_E\) at any \(E \in G_k(\mathbb{R}^n)\) will be the reflection \(s_E\) with fixed space \(E\), i.e. with eigenvalue 1 on \(E\) and \(-1\) on \(E^\perp\).

But what is the manifold structure and the Riemannian metric on \(G_k(\mathbb{R}^n)\)? One way to see this is to embed \(G_k(\mathbb{R}^n)\) into the space \(S(n)\) of symmetric real \(n \times n\) matrices: We assign to each \(k\)-dimensional subspace \(E \in \mathbb{R}^n\) the orthogonal projection matrix \(p_E\) with eigenvalues 1 on \(E\) and 0 on \(E^\perp\). Let \(P(n) = \{p \in S(n)\mid p^2 = p\}\) denote the set of all orthogonal projections. This set has several mutually disconnected subsets, corresponding to the trace of the elements which here is the same as the rank:

\[
P(n)_k = P(n) \subset S(n)_k, \quad S(n)_k = \{x \subset S(n) \mid \text{trace } x = k\}
\]

Now we may identify \(G_k(\mathbb{R}^n)\) with \(P(n)_k \subset S(n)_k\), using the embedding \(E \mapsto p_E\) which is equivariant in the sense \(g p_E g^T = p_{gE}\) for any \(g \in O(n)\). In fact, each \(p_E\) lies in this set, and vice versa, a symmetric matrix \(p\) satisfying \(p^2 = p\) has only eigenvalues 1 and 0 with eigenspaces \(E = \text{im } p\) and \(E^\perp = \ker p\), hence \(p = p_E\), and the trace condition says that \(E\) has dimension \(k\). \(P(n)_k\) is a submanifold of the affine space \(S(n)_k\) since it is the conjugacy class of the matrix

\[
p_0 = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}
\]

i.e. the orbit of \(p_0\) under the action of the group \(O(n)\) on \(S(n)\) by conjugation. The isotropy group of \(p_0\) is \(O(k) \times O(n-k) \subset O(n)\). A complement of \(T_l(O(k) \times O(n-k))\) in \(T_lO(n)\) is the space of matrices of the type

\[
\begin{pmatrix} 0 & -L^T \\ L & 0 \end{pmatrix}
\]

with arbitrary \(L \in R^{(n-k)\times k}\), thus \(P(n)_k = G_k(\mathbb{R}^n)\) has dimension \(k(n-k)\).

Define \(F: S(n) \rightarrow S(n), F(p) = p^2 - p\), then \(P(n)_k = G_k(\mathbb{R}^n) \subset F^{-1}(0)\), the kernel \(\ker dF_p\) is contained in \(T_pG_k(\mathbb{R}^n)\). But the subspace \(\ker dF_p = \{v \in S(n)\mid vp + pv = v\}\) is isomorphic to \(\text{Hom}(E, E^\perp)\) since it contains precisely the symmetric matrices mapping \(E = \text{im } p\) into \(E^\perp\) and vice versa. Thus \(\ker dF_p = T_pG_k(\mathbb{R}^n)\).

Now we equip \(P(n)_k \subset S(n)\) with the metric induced from the trace scalar product \(<x, y> = \text{trace}(x^T y) = \text{trace}(xy)\) on \(S(n)\). The group \(O(n)\) acts isometrically on \(S(n)\) by conjugation and preserves \(P(n)_k\), hence it acts isometrically on \(P(n)_k\). In particular, let \(s_E \in O(n)\) be the reflection at the subspace
$E$ and let $\hat{s}_E$ be the corresponding conjugation, $\hat{s}_E(x) = s_E x s_E$. This is an isometry fixing $p_E$, and since $s_E$ fixes $E$ and reflects $E^\perp$, the conjugation $\hat{s}_E$ maps any $x \in T_{p_E}G_k(\mathbb{R}^n)$ into $-x$ ($x$ is a linear map from $E$ to $E^\perp$ and vice versa). Thus $\hat{s}_E$ is the symmetry at $p_E$.

3 Homogeneous description

For a symmetric space, we have theorem:

**Theorem 1** Let $M$ be a Riemannian symmetric space and $p_0$ any point in $M$. Let $G = I(M)_0$ be the identity component of the isometry group and $K$ be the isotropy group of $G$ at $p_0$. Then $K$ is a compact subgroup of the connected group $G$ and $G/K$ is analytically diffeomorphic to $M$ under the mapping $gK \to g(p_0)$, $g \in G$.

So we can get think of symmetric spaces as the homogeneous space of the isometry group $G$. A natural question to ask is what group $G$ and subgroup $K$ will lead to a symmetric space. To answer this question, we need the definition of Riemannian symmetric pairs.

3.1 From Symmetric space to Symmetric Pairs

**Definition 8** (symmetric pair) Let $G$ be a connected Lie group and $K$ a closed subgroup. The pair $(G, K)$ is called a symmetric pair if there exists an involutive analytic automorphism $\sigma$ of $G$ s.t. $(G_{\sigma})_0 \subset K \subset G_{\sigma}$, where $G_{\sigma}$ is the set of fixed points of $\sigma$ in $G$ and $(G_{\sigma})_0$ is the identity component of $G_{\sigma}$.

If in addition, $\text{Ad}_{G}(K)$ (the adjoint group of $K$ in $G$) is compact, $(G, K)$ is said to be a Riemannian symmetric pair.

We can get symmetric pairs from symmetric spaces:

**Theorem 2** Let $M$ be a symmetric space with a fixed point $p_0$, $G = I(M)_0$ be the identity component of the isometry group and let $K$ be the isotropy group of $G$ at $p_0$. Then the map $G/K \to M$ with $K \mapsto g(p_0)$ is a bijection. The group $G$ has an involutive automorphism $\sigma$ given by $\sigma : G \to G$, $g \mapsto s_{p_0} \circ g \circ s_{p_0}$ with stabilizer $(G_{\sigma})_0 \subset K \subset G_{\sigma}$.

$\text{Ad}(K)$ is compact since $K$ is closed and bounded and $\text{Ad}$ is a homeomorphism. So $(G, K)$ is a Riemannian symmetric pair.

3.2 From Symmetric Pairs to Symmetric Space

In fact symmetric pairs lead to symmetric spaces.

**Theorem 3** Let $M$ be a Riemannian manifold and $I(M)$ the set of all isometries of $M$. Then

(1) The compact open topology of $I(M)$ turns $I(M)$ into a locally compact topological transformation group.
(2) Let \( p \in M \) and let \( \tilde{K} \) denote the subgroup of \( I(M) \) which leaves \( p \) fixed. Then \( \tilde{K} \) is compact.

**Theorem 4** Let \((G, K)\) be a Riemannian symmetric pair. Let \( \pi \) denote the natural mapping of \( G \) onto \( G/K \) and put \( o = \pi(e) \). Let \( \sigma \) be any analytic, involutive automorphism of \( G \) on \( M = G/K \) s.t. \((G_o) \subset K \subset G_o\). Then there is a \( G \)-invariant Riemannian structure \( Q \) on \( M \) that makes \((M, Q)\) a Riemannian globally symmetric space. The geodesic symmetry \( s_o \) satisfies
\[
    s_o \circ \pi = \pi \circ \sigma, \quad \tau(\sigma(g)) = s_o \tau(g) s_o, \quad g \in G
\]
where \( \tau \) is the parallel translation. In particular, \( s_o \) is independent of the choice of \( Q \).

So we get a Riemannian symmetric space \( M = G/K \) from a symmetric pair \((G, K)\).

**Example 4: The Compact Grassmannian**

First consider the Grassmannian of oriented \( k \)-planes in \( \mathbb{R}^{k+l} \), denoted by \( M = \widetilde{G}_k(\mathbb{R}^{k+l}) \). Thus, each element in \( M \) is a \( k \)-dimensional subspace of \( \mathbb{R}^{k+l} \) together with an orientation. We shall assume that we have the orthogonal splitting \( \mathbb{R}^{k+l} = \mathbb{R}^k \oplus \mathbb{R}^l \), where the distinguished element \( p = \mathbb{R}^k \) takes up the first \( k \) coordinates in \( \mathbb{R}^{k+l} \) and is endowed with its natural positive orientation.

Let us first identify \( M \) as a homogeneous space. Observe that \( O(k+1) \) acts on \( \mathbb{R}^{k+l} \). As such, it maps \( k \)-dimensional subspaces to \( k \)-dimensional subspaces, and does something uncertain to the orientations of these subspaces. We therefore get that \( O(k+1) \) acts transitively on \( M \). This is, however, not the isometry group as the matrix \(-I \in SO(k+l)\) acts trivially if \( k \) and \( l \) are even.

The isotropy group consists of those elements that keep \( R^k \) fixed as well as preserving the orientation. Clearly, the correct isotropy group is then: \( SO(k) \times O(l) \subset O(k+1) \).

The tangent space at \( p = \mathbb{R}^k \) is naturally identified with the space of \( k \times l \) matrices \( Mat_{k \times l} \), or equivalently, with \( \mathbb{R}^k \otimes \mathbb{R}^l \). The isotropy action of \( SO(k) \times O(l) \) on \( Mat_{k \times l} \) now acts as follows:
\[
    SO(k) \times O(l) \times Mat_{k \times l} \to Mat_{k \times l}, \quad (A, B, X) \mapsto AXB^{-1} = AXB^T
\]
The representation, when seen as acting on \( \mathbb{R}^k \otimes \mathbb{R}^l \), is denoted by \( SO(k) \otimes O(l) \).

To see that \( M \) is a symmetric space, we have to show that the isotropy group contains the required involution. On the tangent space \( T_p M = Mat_{k \times l} \) it is supposed to act by multiplication by \(-1\). Thus, we have to find \((A, B) \in SO(k) \times O(l)\) such that for all \( X, AXB^T = -X \).

Clearly, we can just set: \( A = I_k \), \( B = -I_l \). Depending on \( k \) and \( l \), other choices are possible, but they will act in the same way.

We have now exhibited \( M \) as a symmetric space, although we didn’t use the isometry group of the space. Instead, we used a finite covering of the isometry group and then had some extra elements that acted trivially.
Remark: If we define X to be the matrix that is 1 in the (1, 1) entry and otherwise zero, then \( AXB^T = A_1(B_1^1)T \), where \( A_1 \) is the first column of \( A \) and \( B_1 \) is the first column of \( B \). Thus, the orbit of \( X \), under the isotropy action, generates a basis for \( \text{Mat}_{k \times l} \) but does not cover all of the space. This is an example of an irreducible action on Euclidean space that is not transitive on the unit sphere.

Note: Example 2(sphere) and 4(Grassmannian) arises as so are called extrinsic symmetric spaces: A submanifold \( S \subset \mathbb{R}^N \) is called extrinsic symmetric if it is preserved by the reflections at all of its normal spaces. More precisely, let \( s_p \) be the isometry of \( \mathbb{R}^N \) fixing \( p \) whose linear part \( ds_p \) acts as identity \( I \) on the normal space \( \nu_p S \) and as \(-I\) on the tangent space \( T_p S \), then \( S \) is extrinsic symmetric if \( s_p(S) = S \) for all \( p \in S \). Extrinsic symmetric spaces are classified.

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Figure 1: some classification

References


