1. Introduction

Let $M$ be a differentiable manifold. We may consider the de Rham cohomology groups of $M$, $H^\ast_{\text{deR}}(M)$ which are built based on the smooth structure on $M$. The content of de Rham’s theorem is that these cohomology groups are isomorphic to the singular cohomology groups of $M$, $H^\ast(M, \mathbb{R})$, and therefore de Rham cohomology is a topological invariant.

There are two common ways of proving de Rham’s theorem. The first one involves using an integration map on differentiable forms to build an isomorphism between the de Rham cohomology groups and singular cohomology groups for each degree $p$. The second one uses Čech cohomology to prove that these groups are isomorphic.

Here we will take the second approach. We will define the basic concepts needed to do an outline of the proof. In particular I want to focus on the main ideas used in the proof and not on the technical details. For reference on the details and topics that I outline here, [1] gives a treatment of sheaf theory and cohomology, [2] follows Cartan and focuses on developing the necessary amount of sheaf theory for the proof of de Rham’s theorem. To see how this tools can be used in the holomorphic setting I suggest [3].

2. Sheaves

In this paper, we use sheaves to keep track of the local information of the manifold $M$. Later, using sheaf cohomology we will use the local information carried by sheaves to understand global properties of $M$.

I will define only the basic notions of sheaves and presheaves which we need for the outline of the proof of de Rham’s theorem. Let $X$ be a topological space. A presheaf $\mathcal{F}$ of abelian groups over $X$ is given by an abelian group $\mathcal{F}(U)$ for each open set $U$ of $X$ together with a restriction morphism

$$\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$$

for each pair of open sets $V \subset U$ which is a morphism of abelian groups. We require that $\mathcal{F}(\emptyset) = 0$. For any three open sets $W \subset V \subset U$ of $X$ we also require compatibility of the restriction morphisms

$$\rho_{UV} = \rho_{WV} \circ \rho_{UV} \to \mathcal{F}(U) \to \mathcal{F}(V).$$

The elements of $\mathcal{F}(U)$ are called the sections of $F$ on $U$. A presheaf of abelian groups is a sheaf if it satisfies the condition that giving a section of $\mathcal{F}$ on $U$ is equivalent to giving a
collection of sections on each open set of the covering, whose restrictions coincide on the intersections.

**Example 1.** If $X$ is the manifold $M$, we may consider the following sheaves on $M$:

- The sheaf $\Omega^k_M$ of smooth $k$-forms that assigns to each open set $U$ of $M$ the set $\Omega^k_M(U)$ of smooth $k$-forms on $U$.
- $\mathbb{R}_M$ the constant sheaf on $M$ that assigns to each open set $U$ the set $\mathbb{R}_M(U)$ of locally constant functions on $U$.

A map of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is a collection, for each open set $U$, of morphisms $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for $\sigma \in \mathcal{F}(U)$ and $V \subset U$, we have $\phi_U(\sigma)|_V = \phi_V(\sigma|_V)$.

We may define the kernel, image, cokernel of a map of sheaves. This allows us to talk about exact sequences of maps of sheaves

**Example 2.** The exterior derivative on $k$-forms on $M$ can be computed locally, this means we get a map of sheaves $d : \Omega^k_M \to \Omega^{k+1}_M$.

Since $d^2 = 0$ we have a complex of sheaves

$$0 \to \mathbb{R}_M \to \Omega^0_M \xrightarrow{d} \Omega^1_M \to \Omega^2_M \to \cdots$$

where the first map is the inclusion of the sheaf of locally constant functions to the sheaf of smooth functions on $M$. Using Poincaré lemma we know that any closed form is locally exact, hence the above complex is in fact exact.

### 3. Čech cohomology

Let $\mathcal{I}$ be a totally ordered set. We define an abstract simplicial complex on $\mathcal{I}$ to be a collection $\Delta$ of finite subsets of $\mathcal{I}$, closed under taking subsets. Each $F \in \Delta$ is called a face of $\Delta$. If $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ is a locally finite open cover of $M$, then we can associate to $\mathcal{U}$ an abstract simplicial complex $\mathcal{N}(\mathcal{U})$, called the nerve of $\mathcal{U}$. The faces of $\mathcal{N}(\mathcal{U})$ are the sets $\{U_{i_1}, \ldots, U_{i_k}\}$ with $U_{i_1} \cap \ldots \cap U_{i_k} \neq \emptyset$.

**Definition 1.** Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of $M$. Then $\mathcal{V}$ is a refinement of $\mathcal{U}$ if every element of $\mathcal{V}$ is contained in an element of $\mathcal{U}$. A choice of inclusions $\tau : \mathcal{V} \to \mathcal{U}$ is called a refining map.

We have the following theorem proved by Čech.

**Theorem 1.** Let $\mathcal{U}$ be a locally finite open cover of $M$. If all intersections of sets in $\mathcal{U}$ (including elements of $\mathcal{U}$) are contractible, then $\mathcal{N}(\mathcal{U})$ is homotopy equivalent to $M$. 
A cover which satisfies the hypothesis of the theorem above is called a Čech cover of $M$. It is a fact that we won’t prove, that all manifolds admit Čech covers.

Now we will define Čech cohomology for a sheaf $F$ on $M$ and an open cover $U$ of $M$, but I won’t emphasize too much on the technical details. The point is that for an open cover $U$ of $M$, we may define Čech $k$-cochains and a boundary operator that puts them together in a complex which we call the Čech complex. Then the Čech cohomology of $F$ with respect to $U$ is the homology of the Čech complex.

**Definition 2.** Let $F$ be a sheaf on $M$, and let $U = \{U_i\}$ be a locally finite open cover of $M$. A Čech $k$-cochain is a function $\alpha$ on the $k$-faces of $N(U)$ such that the value on the face $U_{i_0}, \ldots, U_{i_k}$ lies in $F(U_{i_0} \cap \cdots \cap U_{i_k})$. Thus the group of $k$-cochains is

$$\check{C}^k(U, F) = \bigoplus_{i_0 < \cdots < i_k} F(U_{i_0} \cap \cdots \cap U_{i_k})$$

The coboundary operator $d$ sends $k$-cochains to $(k+1)$-cochains by the formula

$$d\alpha(U_{i_0} \cdots U_{i_k} \cap U_{i_{k+1}}) = \sum_{j=0}^{k+1} (-1)^j \alpha(U_{i_0} \cdots \cap \hat{U}_{i_j} \cap \cdots \cap U_{i_k}),$$

the hat indicates that we omit that set from the intersection.

We observe that the coboundary operator $d$ satisfies $d^2 = 0$ so we get a chain complex of chains which we call the Čech complex of $F$ associated to $U$. Then the Čech cohomology of $F$ with respect to $U$ is defined to be the homology of the Čech complex, we denote it by $\check{H}^*(U, F)$.

Using a refinement map $\tau$ as defined earlier, we map get a morphism of the Čech cohomologies with respect to different covers. If $V$ is a refinement of $U$ and $\tau$ is a refining map, $\tau$ induces a map

$$\tau^* : \check{H}^k(U, F) \to \check{H}^k(V, F)$$

for all $k$. It is a fact that the map above does not depend on the choice of the refining map.

**Definition 3.** Let $F$ be a sheaf on $M$. The $k$-th Čech cohomology group of $F$ is

$$\check{H}^k(M, F) = \lim_{\to \mathcal{U}} \check{H}^k(U, F)$$

where the covers are ordered by refinement.

To end this section we note that with some extra work we can show that $\check{H}(M, \mathbb{R}_M)$ and $H^k_{\text{sing}}(M, \mathbb{R})$ (the singular cohomology) are isomorphic. The main point in the proof of this fact, is to note that singular cochains can be approximated by Čech cochains for a very refined cover $U$ of $M$. Then we obtain a homotopy between the Čech complex for the cover $U$ and the complex of singular cochains which induces a map on cohomology groups. Finally, taking a direct limit shows isomorphism.
4. de Rham’s Theorem

Definition 4. A sheaf $\mathcal{F}$ on $M$ is fine if its sections can be glued by partitions of unity, i.e. for every open set $U \subset M$ and for every locally finite cover $\mathcal{V}$ of $U$, there exist extension maps $\varphi_V^U : \mathcal{F}(V) \to \mathcal{F}(U)$ such that

$$\sum_{V \in \mathcal{V}} \varphi_V^U \circ \rho_V = \text{id}_{\mathcal{F}(U)}$$

We see that for manifolds, we can use partitions of unity to show that the sheaves $\Omega^k_M$ of smooth $k$-forms are fine. Note that the constant sheaf $\mathbb{R}_M$ is not fine.

Definition 5. Let $\mathcal{F}$ be a sheaf on $M$. A fine resolution of $\mathcal{F}$ is an exact sequence of sheaf maps

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$$

where every $\mathcal{F}^k$ is exact.

Example 3. The exact sequence of sheaves induced by the exterior derivative operator is a fine resolution of $\mathbb{R}_M$

$$0 \to \mathbb{R}_M \to \Omega^0_M \xrightarrow{d} \Omega^1_M \to \Omega^2_M \to \cdots$$

The next theorem is one of the key results that allows us to go from local information captured by sheaves to global information given by the cohomology groups.

Theorem 2. Let $\mathcal{F}$ be a sheaf on $M$ and let

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$$

be a fine resolution of $\mathcal{F}$. Suppose that $\mathcal{U}$ is an open cover of $M$ such that the sequence of homomorphisms

$$\mathcal{F}^j(U_{i_0} \cap \cdots \cap U_{i_k}) \to \mathcal{F}^j(U_{i_0} \cap \cdots \cap U_{i_k}) \to \mathcal{F}^{j+1}(U_{i_0} \cap \cdots \cap U_{i_k})$$

is exact for every face $\{U_{i_0}, \ldots, U_{i_k}\}$ of $\mathcal{N}(\mathcal{U})$. Then

$$\hat{H}^k(\mathcal{U}, \mathcal{F}) = \frac{\ker d : \mathcal{F}^k(M) \to \mathcal{F}^{k+1}(M)}{\text{im} d : \mathcal{F}^{k-1}(M) \to \mathcal{F}^k(M)}$$

Corollary 1. Let $\mathcal{F}$ be a sheaf on $M$, and suppose $\mathcal{U}$ is an open cover of $M$ such that $\hat{H}^k(U_{i_0} \cap \cdots \cap U_{i_j}, \mathcal{F}) = 0$ for all finite intersections $U_{i_0} \cap \cdots \cap U_{i_j}$ of elements in $\mathcal{U}$. Then the canonical map

$$\hat{H}^k(\mathcal{U}, \mathcal{F}) \to \hat{H}^k(M, \mathcal{F})$$

is an isomorphism for all $k$.

Corollary 2. Let $\mathcal{F}$ be a sheaf on $M$, and let

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$$

be any fine resolution of $\mathcal{F}$. Then

$$\hat{H}^k(M, \mathcal{F}) \cong \frac{\ker d : \mathcal{F}^k(M) \to \mathcal{F}^{k+1}(M)}{\text{im} d : \mathcal{F}^{k-1}(M) \to \mathcal{F}^k(M)}$$
Theorem 3 (de Rham). Let $M$ be a paracompact $C^\infty$-manifold. Then
\[ \hat{H}^k(M, \mathbb{R}_M) \cong H^k_{dR}(M) \cong H^k_{\text{sing}}(M, \mathbb{R}) \]
for all $k$.

Proof. By definition, the de Rham cohomology is the homology of the resolution of $\mathbb{R}_M$ by sheaves of smooth $k$-forms evaluated on $M$. Using the previous corollary this is the same as $\hat{H}^k(M, \mathbb{R}_M)$. Then we use the (nontrivial) fact that $\hat{H}^k(M, \mathbb{R}_M) = H^k_{\text{sing}}(M, \mathbb{R})$ and we are done. \qed

References