1 Introduction

Manifolds at any regularity level, from continuous to smooth, can be studied. It’s natural to wonder how distinct these concepts are – if there is a fundamental difference between, say, a $C^1$ manifold and a smooth one. A function space topology helps answer this question. With a topology in place, one can also prove various useful approximation results.

In the first section, the weak and strong topologies are introduced using coordinate-based definitions, and some of their basic properties discussed. In the next section, we give some of the major approximation results, e.g. that every $C^r$ map can be approximated by a smooth map and every $C^r$ manifold is $C^r$-diffeomorphic to a smooth manifold. The last section introduces jets to give equivalent definitions of the weak and strong topologies. As an application, this definition is used to prove another approximation result.

This exposition is based on that in Hirsch’s *Differential Topology*, Chapter 2. Munkres is a useful reference for topological background. In the following, $M$ and $N$ are $C^r$ manifolds, with $r \in [0, \infty]$ depending on the context. All manifolds are without boundary.

2 Coordinate-Based Definitions

We begin by defining the weak topology on $C^r(M, N)$.

**Definition 1** Let $f \in C^r(M, N)$. Take a chart $(\phi, U)$ on $M$, a compact set $K \subset U$, a chart $(\psi, V)$ on $N$ such that $f(K) \subset V$ and $\epsilon > 0$. The corresponding weak subbasic neighborhood of $f$ is the set

$$B^r(f; (\phi, U), K, (\psi, V), \epsilon)$$

of functions $g \in C^r(M, N)$ such that $g(K) \subset V$ and

$$\|D^k(\psi f \phi^{-1})(x) - D^k(\psi g \phi^{-1})(x)\| < \epsilon \text{ for all } x \in \phi(K) \text{ and each } k = 0, 1, \ldots, r.$$ 

Here $\| \cdot \|$ is a standard Euclidean norm.

Using these sets as a subbasis, we obtain a topology on $C^r(M, N)$. The resulting topological space is denoted $C^r_{\text{wt}}(M, N)$. This topology has the advantage of being quite easy to define and understand, but there is a possible drawback: A basic neighborhood of $f$ in this topology consists of functions $g$ which are close to $f$ on some compact set. If $M$ is
not compact, this may not be a good measure of how close the functions are. The strong topology defined below remedies this issue.

**Definition 2** Let \( f \in C^r(M, N) \) and take

- \( \Phi = \left\{ \left( \phi_\alpha, U_\alpha \right) \right\}_\alpha \) a family of charts on \( M \) such that \( \{U_\alpha\}_\alpha \) is locally finite,
- \( K = \{K_\alpha\}_\alpha \) a family of compact sets with \( K_\alpha \subset U_\alpha \),
- \( \Psi = \left\{ \left( \psi_\alpha, V_\alpha \right) \right\}_\alpha \) a family of charts on \( N \) such that \( f(K_\alpha) \subset V_\alpha \), and
- \( E = \{\epsilon_\alpha\}_\alpha \) a collection of positive numbers.

The corresponding strong basic neighborhood of \( f \) is the set

\[
B^r(f; \Phi, K, \Psi, E)
\]

of functions \( g \in C^r(M, N) \) such that for every \( \alpha \), we have \( g(K_\alpha) \subset V_\alpha \) and

\[
\left\| D^k(\psi_\alpha f \phi_\alpha^{-1})(x) - D^k(\psi_\alpha g \phi_\alpha^{-1})(x) \right\| < \epsilon_\alpha \text{ for all } x \in \phi(K_\alpha) \text{ and each } k = 0, 1, \ldots, r.
\]

To see that this defines a basis, suppose \( g \in B^r(f; \Phi, K, \Psi, E) \cap B^r(f'; \Phi', K', \Psi', E') \). Then the neighborhood \( B^r(g; \Phi \cup \Phi', K \cup K', \Psi \cup \Psi', E \cap E') \) is contained in the intersection provided the distances in \( E \) are sufficiently small. The set \( C^r(M, N) \) with the strong topology is denoted \( C^r_S(M, N) \).

We can now define the strong and weak topologies on \( C^\infty(M, N) \). The strong topology is the union of all the topologies induced by the inclusions \( C^\infty(M, N) \rightarrow C^r_S(M, N) \) for \( r \in \mathbb{N} \), and similarly the weak topology is the union of the topologies induced by the inclusions \( C^\infty(M, N) \rightarrow C^r_W(M, N) \).

To validate the use of the strong topology, one can show that various important sets, such as submersions, embeddings, proper maps, etc., are open in \( C^r_S(M, N) \) for any \( 1 \leq r \leq \infty \):

**Theorem 3** The following sets are open in \( C^r_S(M, N) \) for any \( r \geq 1 \):

i) \( C^r \) submersions,

ii) \( C^r \) immersions,

iii) \( C^r \) embeddings,

iv) proper \( C^r \) maps,

v) closed \( C^r \) embeddings,

vi) \( C^r \) diffeomorphisms.

**Proof of i).** It is enough to prove this in the case \( r = 1 \) since

\[
\{C^r(M, N) \text{ submersions} \} = \{C^1(M, N) \text{ submersions} \} \cap C^r(M, N)
\]

and the inclusion \( C^1_S(M, N) \rightarrow C^1_S(M, N) \) is continuous.

Let \( f \in C^1(M, N) \) be an immersion. Choose an atlas \( \Phi = \{\phi_\alpha, U_\alpha\}_\alpha \) of \( M \) such that each \( U_\alpha \) has compact closure and \( f(U_\alpha) \subset V_\alpha \) for some chart \( (\psi_\alpha, V_\alpha) \) on \( N \). Let \( \Psi = \{(\psi_\alpha, V_\alpha)\} \),
and let \( K = \{ K_\alpha \}_\alpha \) be a compact cover of \( M \) such that \( K_\alpha \subset U_\alpha \).

The set
\[
A_\alpha = \{ D(\psi_\alpha f \psi_\alpha^{-1})(x) : x \in \phi(K_\alpha) \}
\]
is then compact in the set \( L(\mathbb{R}^m, \mathbb{R}^n) \) of linear maps from \( \mathbb{R}^n \) to \( \mathbb{R}^m \). Each map is surjective and surjectivity is an open condition in \( L(\mathbb{R}^n, \mathbb{R}^m) \), so there is \( \epsilon_\alpha \) such that if \( S \in A_\alpha \), \( T \in L(\mathbb{R}^n, \mathbb{R}^m) \), and \( \| T - S \| < \epsilon_\alpha \), then \( T \) is surjective. Set \( E = \{ \epsilon_\alpha \} \). Then each element in \( B^1(f; \Phi, \Psi, K, E) \) is a submersion. \( \square \)

**Proof of iv)**. This property in fact holds for \( r = 0 \) also, as the proof will show. Let \( f : M \to N \) be a proper map. Take an atlas \( \Phi = \{(\phi_\alpha, U_\alpha)\}_\alpha \) on \( M \) such that the \( U_\alpha \) are locally finite and have compact closure. Let \( K = \{ K_\alpha \}_\alpha \) be a compact cover of \( M \) with \( K_\alpha \subset U_\alpha \). Using the fact that \( f \) is proper, it is possible to ensure, by decomposing the \( U_\alpha \) and \( K_\alpha \) into smaller sets if necessary, that there is an atlas \( \Psi = \{(\psi_\alpha, V_\alpha)\}_\alpha \) of \( N \) such that the \( V_\alpha \) are locally finite and \( f(U_\alpha) \subset V_\alpha \). We claim that \( B^r(f; \Phi, \Psi, K, E) \) consists of proper maps, for arbitrary \( E = \{ \epsilon_\alpha \}_\alpha \). Indeed, if \( g \) is in this neighborhood of \( f \) and \( L \subset N \) is compact, then \( g(K_\alpha) \subset V_\alpha \) for each \( \alpha \) and \( L \) intersects only finitely many \( V_\alpha \). Then \( g^{-1}(L) \) is a closed set which intersects only finitely many \( K_\alpha \), so \( g^{-1}(L) \) is a finite union of compact sets. \( \square \)

The proofs of the remaining statements are omitted. The proof of ii) is parallel to that of i); the proof of iii) is more technical. Property iv) follows since the set of closed embeddings is the set of proper embeddings, while vi) holds by reducing to the case of connected manifolds and noting that diffeomorphisms are exactly the proper embeddings which are submersions.

**3 Approximations**

With a topology on \( C^r(M, N) \) established, we can make precise statements about the relationship between \( C^\infty \) and \( C^r \) manifolds. The first results deal with approximating \( C^r \) functions by higher-regularity maps.

Suppose first that \( M \) and \( N \) are subsets of Euclidean space. In this case, smooth approximations are constructed using the convolution: If \( f \in C^r(M, N) \) and \( K \subset M \) is compact, choose a radius \( \delta \) such that \( B(x, \delta) \subset M \) for each \( x \in K \) and such that \( \| D^k f(y) - D^k f(x) \| \leq \epsilon \) for all \( x, y \in K \) such that \( |x - y| \leq \delta \). Let \( \theta \) be a smooth bump function with \( \text{supp} \ \theta \subset B(0, \delta) \) and \( \int \theta = 1 \). Then the convolution of \( f \) with \( \theta \) is a smooth function which \( C^r \) approximates \( f \) on the neighborhood \( \cup_{x \in K} B(x, \delta) \).

A partition of unity is used to extend this result beyond compact sets.

**Theorem 4** Let \( M \subset \mathbb{R}^m \) and \( N \subset \mathbb{R}^n \) be open. Then \( C^\infty(M, N) \) is dense in \( C^r_S(M, N) \) for any \( r \geq 0 \).
We will choose the $\delta$. Note that for each $k$, not on $\lambda$-neighborhood of the compact set $\text{supp} \alpha$, this is a collection of positive numbers. The elements of $B^r(f; K, E)$ are the functions $g \in C^r(M, N)$ such that
\[
\|D^k g(x) - D^k f(x)\| < \epsilon_k \quad \text{for all } x \in K_\alpha \text{ and each } k = 0, 1, \ldots, r.
\]

Given $K$ and $E$, we will construct a smooth function which lies in $B^r(f; K, E)$. Let $\{\lambda_\alpha\}_\alpha$ be a smooth partition of unity on $M$ such that $K_\alpha \subseteq \text{Int} (\text{supp} \lambda_\alpha)$ and $\text{supp} \lambda_\alpha$ is compact. For any $\delta_\alpha > 0$, there is a smooth function $g_\alpha$ which approximates $f$ on a neighborhood of the compact set $\text{supp} \lambda_\alpha$ satisfying
\[
\|D^k g_\alpha(x) - D^k f(x)\| < \delta_\alpha \quad \text{for all } x \in K_\alpha \subseteq \text{supp} \lambda_\alpha, \ k = 0, 1, \ldots, r.
\]

We will choose the $\delta_\alpha$ later. Define $g : M \to N$ by
\[
g(x) = \sum_\alpha \lambda_\alpha(x) g_\alpha(x).
\]

Note that for each $k$, the derivative $D^k(\lambda_\alpha g_\alpha)$ is a sum of bilinear functionals in the $D^j \lambda_\alpha$ and $D^j g_\alpha$. This functional depends only on $r$ and the dimensions of the Euclidean spaces -- not on $\lambda_\alpha$ or $g_\alpha$. Thus for some $C_k > 0$, we have the estimate
\[
\|D^k \lambda_\alpha(x) g_\alpha(x)\| \leq C_k \max_{0 \leq j \leq k} \|D^j \lambda_\alpha(x)\| \max_{0 \leq j \leq k} \|D^j g_\alpha(x)\|.
\]

Let $C = \max\{C_0, C_1, \ldots, C_r\}$, and let $A_\alpha = \{\beta : K_\alpha \cap K_\beta \neq \emptyset\}$. Local finiteness of $K = \{K_\alpha\}_\alpha$ implies that this a finite set.

Now let $x \in K_\alpha$. We have
\[
\|D^k g(x) - D^k f(x)\| = \left\| \sum_{\beta \in A_\alpha} D^k \left( \lambda_\beta(x) (g_\beta(x) - f(x)) \right) \right\|
\leq C_k \sum_{\beta \in A_\alpha} \max_{0 \leq j \leq k} \|D^j \lambda_\beta(x)\| \cdot \delta_\beta.
\]

We can make this as small as we like, i.e. smaller than $\epsilon_\alpha$, by choosing the $\delta_\alpha$ sufficiently small. This choice ensures $g \in B^r(f; K, E)$ as desired. \qed

As one might expect, exactly the same statement holds for general manifolds $M$ and $N$. The idea of the proof is to take a countable atlas $\{(\phi_n, U_n)\}_n$ of $M$ and apply the Euclidean space result to define an approximating function on each coordinate domain. By repeating this process inductively, one can obtain an approximating function on the entire manifold $M$. There’s a fair bit of work involved in making this precise though, as one must show that the cobbled-together function is sufficiently smooth and remains a good approximation.

From this approximation result for general manifolds and the openness theorem stated in the previous section, its clear that the set of $C^r(M, N)$ diffeomorphisms is dense in the
The set of $C^s$ diffeomorphisms for any $1 \leq r \leq s$. From this it follows that two manifolds are $C^r$ diffeomorphic if and only if they are $C^s$ diffeomorphic. One direction of this statement is obvious, of course, but the other is more surprising.

Furthermore, the following result tells us that if $M$ is a $C^r$ manifold, we can make it into a $C^s$ manifold by eliminating some of the charts in its atlas.

**Theorem 5** Let $M$ be a manifold with a $C^r$ differential structure $β$. Then for any $s \geq r$, there is a $C^s$ differential structure $γ$ on $M$ such that every chart of $γ$ is a chart of $β$.

The proof uses Zorn’s lemma to find a maximal subset of $B \subset M$ on which there is such a $C^s$ structure. One then proves by contradiction that $B$ must in fact be the entire manifold $M$. Using this theorem, we conclude that for $1 \leq r < s \leq \infty$, any $C^r$ manifold is $C^s$ diffeomorphic to a $C^s$ manifold. This indicates that generally it suffices to study $C^\infty$ manifolds – the results for $C^r$ manifolds with $r < \infty$ will be parallel.

## 4 Topologies through Jets

This section introduces an alternative, more abstract, definition of the topologies on $C^r(M, N)$. The construction is based on identifying $C^r(M, N)$ with a set of continuous maps from $M$ into a new manifold $J^r(M, N)$, called the set of $r$-jets from $M$ to $N$. Though constructing the space of jets takes some work, the resulting definition of the topology on $C^r(M, N)$ is elegant and yields some useful properties.

**Definition 6** Let $M$ and $N$ be $C^r$ manifolds with $r < \infty$. The set of $r$-jets from $M$ to $N$ is the set of equivalence classes

$$J^r(M, N) = \{ j_x^r f : x \in M \text{ and } f \in C^r(U, N) \text{ for some open set } U \text{ containing } x \}. $$

The equivalence relation is $j_x^r f = j_{x'}^r f'$ if and only if $x = x'$ and for any charts $(φ, V)$ of $M$ and $(ψ, W)$ of $N$ such that $x \in V$ and $f(V) \subset W$, the derivatives of $ψfφ^{-1}$ and $ψf'φ^{-1}$ at $φ(x)$ agree up to order $r$.

The set of $r$-jets based at a point $x$ is denoted $J^r_x(M, N)$.

It is useful to consider the set of jets between two Euclidean spaces, $J^r(\mathbb{R}^m, \mathbb{R}^n)$. In this case, there is an obvious choice of representative for the jet $j_x^r f$, namely the $r$-th order Taylor approximation of $f$ at $x$. This is a polynomial completely determined by the derivatives of $f$ at $x$, a finite list of numbers. Therefore it makes sense to identify $J^r_x(\mathbb{R}^m, \mathbb{R}^n)$ with a vector space:

$$J^r_x(\mathbb{R}^m, \mathbb{R}^n) = \mathbb{R}^n \times L^1_{\text{sym}}(\mathbb{R}^m, \mathbb{R}^n) \times L^2_{\text{sym}}(\mathbb{R}^m, \mathbb{R}^n) \times \cdots \times L^r_{\text{sym}}(\mathbb{R}^m, \mathbb{R}^n).$$

Here $L^k_{\text{sym}}(\mathbb{R}^m, \mathbb{R}^n)$ is the space of symmetric $k$-linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$, the space containing the $k$–th derivative at $x$. Allowing the basepoint $x$ to vary yields an identification

$$J^r(\mathbb{R}^m, \mathbb{R}^n) = \mathbb{R}^m \times \left(\mathbb{R}^n \times L^1_{\text{sym}}(\mathbb{R}^m, \mathbb{R}^n) \times L^2_{\text{sym}}(\mathbb{R}^m, \mathbb{R}^n) \times \cdots \times L^r_{\text{sym}}(\mathbb{R}^m, \mathbb{R}^n)\right).$$
The same construction works for $J^r(U,V)$ when $U$ and $V$ are open subsets of Euclidean space, and we see that $J^r(U,V)$ is an open subset of a finite-dimensional vector space. This is used to construct a topology on $J^r(M,N)$ for general $C^{r+s}$ manifolds $M$ and $N$ as follows. Let $(\phi,U)$ and $(\psi,V)$ be charts on $M$ and $N$ respectively. There is a bijection

$$
\Theta_{\phi\psi} : J^r(U,V) \rightarrow J^r(\phi(U),\psi(V))$

$$
j^r_{f,x} \mapsto j^r_{\phi(x)}(\psi f \phi^{-1}).$$

Furthermore, note that

$$\Theta_{\phi\psi}^{-1} : j^r_g \mapsto j^r_{\phi^{-1}(\psi^{-1})}(g(\phi^{-1})).$$

The transition functions $\tilde{\psi}^{-1}$ and $\tilde{\phi}^{-1}$ are $C^{r+s}$ diffeomorphisms, so $\Theta_{\phi\psi}^{-1}$ is a $C^r$ map between Euclidean spaces if we identify $J^r(\phi(U),\psi(V))$ and $J^r(\tilde{\phi}(U),\tilde{\psi}(V))$ with Euclidean spaces as described above. Thus the maps $\Theta_{\phi\psi}$ can be taken as charts for $J^r(M,N)$, where $J^r(M,N)$ is endowed with the topology induced by these maps.

We are now prepared to identify $C^r(M,N)$ with a set of continuous maps from $M$ to $J^r(M,N)$ via the following injection:

$$j^r : C^r(M,N) \rightarrow C^0(M,J^r(M,N))$$

$$f \mapsto [x \mapsto j^r_{x}f].$$

To see that this is indeed an injective map, note that $j^r_g = j^r f$ if and only is $j^r_{x}f = j^r_{x}g$ for every $x$ in $M$, which in particular implies that $f(x) = g(x)$ for each $x$.

Through this map, a topology on $C^0(M,J^r(M,N))$ yields a topology on $C^r(M,N)$. Recall the compact-open topology on $C(X,Y)$, which is generated by subbasic sets of the form $\{f \in C(X,Y) : f(K) \subset V\}$, where $K \subset X$ is compact and $V \subset Y$ is open. This topology on $C^0(M,J^r(M,N))$ leads to the weak topology on $C^r(M,N)$ defined previously.

We also have the strong topology on $C(X,Y)$, defined using the graph: If $G_f \subset X \times Y$ is the graph of $f$ and $W \subset X \times Y$ is a neighborhood of $G_f$, then $\{g \in C(X,Y) : G_g \subset W\}$ is a basic neighborhood of $f$. This topology on $C^0(M,J^r(M,N))$ gives the strong topology on $C^0(M,N)$.

We now show that $C^r(M,N)$ is identified with a weakly closed set in $C^0(M,J^r(M,N))$, which has some useful consequences.

**Theorem 7** The image of $j^r$ in $C^0(M,J^r(M,N))$ is a weakly closed set.

**Proof.** The image is weakly closed if it is closed under uniform convergence on compact sets (by the definition of weak convergence when the target space is metric). Any compact set in $M$ can be decomposed into a collection of compact convex coordinate neighborhoods. Thus, in the local representation, it suffices to show that if $U \subset \mathbb{R}^m$ is an open set and
\{f_n\} \subset C^r(U, \mathbb{R}^n) is a sequence such that the derivative sequences \{D^k f_n\}, \ k = 0, 1, 2, \ldots, r converge uniformly to \(g_k : U \to L^k(\mathbb{R}^m, \mathbb{R}^n)\) on \(U\), then the \(g_k\) came from a single function in \(C^r(U, \mathbb{R}^m)\). In particular, we must show that \(g_k = D^k g_0\). For \(k = 1\), write
\[
g_0(x + y) = \lim_{n \to \infty} f_n(x + y) = \lim_{n \to \infty} f_n(x) + \int_0^1 Df_n(x + ty) y \, dt = g_0(x) + \int_0^1 g_1(x + ty) y \, dt \quad \text{by uniform convergence.}
\]
This implies that \(g_1 = Dg_0\) as desired. The cases \(k = 2, \ldots, r\) are parallel. \(\Box\)

Defining the topology on \(C^\infty(M, N)\) using jets is a bit more complicated, since we must first define \(J^\infty(M, N)\). Repeating the above construction won’t work; instead, \(J^\infty(M, N)\) is defined to be the inverse limit (in the category of topological spaces) of the sequence
\[
J^0(M, N) \leftarrow J^1(M, N) \leftarrow J^2(M, N) \leftarrow \cdots.
\]
The maps \(j^r\) yield a map \(j^\infty : C^\infty(M, N) \to C^0(M, J^\infty(M, N))\). We can use this correspondence as above to construct weak and strong topologies on \(C^\infty(M, N)\). Note that the image in \(C^0(M, J^\infty(M, N))\) is still weakly closed.

With this established, the properties of the weak and strong topologies give that \(C^\infty_s(M, N)\) is a Baire space in the strong topology, i.e. countable intersections of open dense sets are dense. This means that the space \(C^\infty_s(M, N)\), though it is not second-countable and has infinitely many components, is well-behaved in some sense. More concretely, one can use the Baire property to prove approximation results like those given in the previous section, as follows.

**Theorem 8** Let \(M\) and \(N\) be \(C^r\) manifolds, \(r \geq 1\). If \(\dim N \geq 2 \dim M + 1\), then embeddings are dense in the set of proper maps \(\{f \in C^r(M, N) : f\ \text{is proper}\}\).

**Proof.** Let \(f \in C^r(M, N)\) be a proper map. Given a neighborhood \(B = B^r(f; \Phi, K, \Psi, E)\), we want find an embedding in \(B\). We may assume, by taking a smaller neighborhood if needed, that \(B\) consists of proper maps (since the set of proper maps is open) and that the collection \(\{K_\alpha\}_\alpha\) covers \(M\). For each \(\alpha\), let
\[
X_\alpha = \{g \in B : g|_{K_\alpha} \text{ is an embedding}\}.
\]
This is open since the set of embedding is open in \(C(K_\alpha, N)\) (this holds for manifolds with boundary such as \(K_\alpha\), and the restriction map \(C^r(M, N) \to C^r(K_\alpha, N)\) is continuous. We claim that \(X_\alpha\) is also dense in \(B\). To check this, take an open set \(W_\alpha\) with compact closure such that \(K_\alpha \subset W_\alpha \subset \overline{W_\alpha} \subset U_\alpha\) and fix \(g \in B\). By the proof of [a version of] the Whitney embedding theorem, we can approximate \(g\) by a \(C^r\) embedding \(h_\alpha\) on \(W_\alpha\). Then let \(\{\lambda, \theta\}\) be a partition of unity subordinate to \(\{M \setminus K_\alpha, W_\alpha\}\). The function \(\lambda g + \theta h_\alpha\) is an element of \(X_\alpha\) which approximates \(g\). Similarly, if \(K_\alpha \cap K_\beta \neq \emptyset\), then the collection
\[
X_{\alpha\beta} = \{g \in B : f|_{K_\alpha \cup K_\beta} \text{ is an embedding}\}
\]
is open and dense in $B$.

Now let $K^1, K^2, \ldots$ be a sequence of locally finite refinements of the cover $K$ such that for any pair of distinct points $x, y \in M$, we have $x \in K^j_\alpha \in K^j$ and $y \in K^j_\beta \in K^j$ for some disjoint $K^j_\alpha, K^j_\beta \in K^j$, i.e., points eventually have disjoint neighborhoods. Since $M$ is second-countable, the $K^j$ can be chosen to be countable.

Let $X^j \subset B$ be the set of functions $g$ such that $f|_{K^j_\alpha \cup K^j_\beta}$ is an embedding for any disjoint $K^j_\alpha, K^j_\beta \in K^j$. Each $X^j$ is a countable intersection of dense open sets in $B$ (using the above argument and the fact that $K^j$ is countable), and hence so is $\bigcap_j X^j$. Since $B$ is Baire, the intersection is an open dense set in $B$. Elements of this intersection are functions which are embeddings when restricted to $K^j_\alpha \cup K^j_\beta$ for any $j, \alpha$, and $\beta$. This means that they are immersions, and the separating property of the refinements $K^j$ implies injectivity. □

References
