Solve the following exercises from the Lecture Notes:

4.5) Show that $T^*M$ has a smooth structure of manifold of dimension $2 \dim M$, for which the projection $\pi : T^*M \to M$ is a smooth map.

4.6) Check that if $M$ and $N$ are smooth manifolds and $\Psi : M \to N$ is a smooth map, then $d\Psi : TM \to TN$ is also smooth.

5.1) Show that $\{(x, |x|) : x \in \mathbb{R}\}$ is not the image of an immersion $\Phi : \mathbb{R} \to \mathbb{R}^2$.

5.2) Show that there exists a diffeomorphism $\Psi : T\mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{R}^3$, which makes the following diagram commutative:

\[
\begin{array}{ccc}
T\mathbb{S}^3 & \xrightarrow{\Psi} & \mathbb{S}^3 \times \mathbb{R}^3 \\
\pi \downarrow & & \downarrow \tau \\
\mathbb{S}^3 & & \mathbb{S}^3
\end{array}
\]

where $\tau : \mathbb{S}^3 \times \mathbb{R}^3 \to \mathbb{S}^3$ is the projection in the first factor and the restriction $\Psi : T_p\mathbb{S}^3 \to \mathbb{R}^3$ is linear for every $p \in \mathbb{S}^3$.

Hint: The 3-sphere is the set of quaternions of norm 1.

5.3) Let $\{y^1, \ldots, y^e\}$ be some set of smooth functions on a manifold $M$. Show that:

(a) If $\{d_p y^1, \ldots, d_p y^e\} \subset T^*_p M$ is a linearly independent set, then the functions $\{y^1, \ldots, y^e\}$ is a part of a coordinate system around $p$.

(b) If $\{d_p y^1, \ldots, d_p y^e\} \subset T^*_p M$ is a generating set, then a subset of $\{y^1, \ldots, y^e\}$ is a coordinate system around $p$.

(c) If $\{d_p y^1, \ldots, d_p y^e\} \subset T^*_p M$ is a basis, then the functions $\{y^1, \ldots, y^e\}$ form a coordinate system around $p$.

5.4) Show that a submersion is an open map. What can you say about an immersion?