1. (25 Points) Prove or disprove the following statement: if $M$ and $N$ are smooth manifolds with boundary then $M \times N$ is also a smooth manifold with boundary.

The statement is false, as shown by the following example:

$M = [0, +\infty]$ - manifolds with boundary

$N = [0, +\infty]$  

$M \times N = [0, +\infty] \times [0, +\infty]$ not a manifold with boundary
2. (25 Points) If $M$ is a manifold and $A \subset O \subset M$ where $A$ is closed set and $O$ is an open set, then there is a smooth function $f : M \rightarrow \mathbb{R}$, with $f(m) = 1$ if $m \in A$ and $f(m) = 0$ if $m \notin O$.

\[ U_1 = O \]
\[ U_2 = M \setminus A \]

Choose an open cover \( \{ U_1, U_2 \} \) subordinate to the cover \( \{ O_1, O_2 \} \). We claim that we can take \( f = \varphi_1 \):

- \( \text{supp } \varphi_1 \subset U_1 = O \) \( \implies \) \( f(m) = \varphi_1(m) = 0 \) if \( m \notin O \)

- \( \text{supp } \varphi_2 \subset U_2 = M \setminus A \) \( \implies \) \( \varphi_2(m) = 0 \) if \( m \in A \)

- \( \varphi_1(m) + \varphi_2(m) = 1 \), \( \forall m \implies f(m) = \varphi_1(m) = 1 - \varphi_2(m) = 1 \)

if \( m \in A \)
3. (25 Points) Let \( \Phi : \mathbb{P}^2 \to \mathbb{P}^2 \) be the smooth map

\[
\Phi([x : y : z]) = \left[ \frac{2x}{x^2 + y^2 + z^2} : \frac{y}{x^2 + y^2 + z^2} : \frac{2z}{x^2 + y^2 + z^2} \right]
\]

Write a basis for \( T_{[0:1:0]}\mathbb{P}^2 \) and determine what \( d_{[0:1:0]}\Phi \) does to each basis element.

Choose the chart \((U, \varphi)\) for \( \mathbb{P}^2 \) centered at \([0:1:0]:\)

\[
U = \{ [x:y:z] \in \mathbb{P}^2 \mid y \neq 0 \}
\]

\[
\varphi : U \to \mathbb{R}^2, \quad [x:y:z] \mapsto \left( \frac{x}{y}, \frac{z}{y} \right)
\]

Set \( \alpha^1 = \frac{x}{y}, \quad \alpha^2 = \frac{z}{y} \). Then

\[
\begin{align*}
\left\{ e_1 = \frac{\partial}{\partial \alpha^1} \bigg|_{[0:1:0]} \right. & , e_2 = \frac{\partial}{\partial \alpha^2} \bigg|_{[0:1:0]} \right. \\
& \text{basis for } T_{[0:1:0]} \mathbb{P}^2
\end{align*}
\]

Since \( \overline{\Phi}([0:1:0]) = [0:1:0] \), we have that \( d_{[0:1:0]} \overline{\Phi} : T_{[0:1:0]} \mathbb{P}^2 \to T_{[0:1:0]} \mathbb{P}^2 \) and we compute:

\[
\begin{align*}
d_{[0:1:0]} \overline{\Phi} \cdot e_1 (\alpha^i) &= e_1 (\alpha^i \circ \overline{\Phi}) = \left\{ \begin{array}{l} e_1 (2x^1) = 2 \\
\end{array} \right. \Rightarrow d_{[0:1:0]} \overline{\Phi} \cdot e_4 = 2 e_4 \\

d_{[0:1:0]} \overline{\Phi} \cdot e_2 (\alpha^i) &= e_2 (\alpha^i \circ \overline{\Phi}) = \left\{ \begin{array}{l} e_2 (2x^1) = 0 \\
\end{array} \right. \Rightarrow d_{[0:1:0]} \overline{\Phi} \cdot e_2 = 2 e_2
\end{align*}
\]
4. (25 Points) Let \( \Phi_1 : M_1 \to N \) and \( \Phi_2 : M_2 \to N \) be smooth maps, and assume that at least one of them is a submersion. Show that the fiber product:

\[
M_1 \times_N M_2 := \{(m_1, m_2) \in M_1 \times M_2 : \Phi_1(m_1) = \Phi_2(m_2)\}
\]

is a smooth manifold and compute its dimension.

(HINT: Show first that for any smooth manifold \( N \) the diagonal

\[
\Delta = \{(x, x) : x \in N\} \subset N \times N
\]

is an embedded submanifold.)

\[
\Delta = \operatorname{graph}(id : N \to N) \implies \Delta \text{ is an embedded submanifold with } \dim \Delta = \dim N
\]

\[
\Phi : M_1 \times M_2 \to N \times N, (m_1, m_2) \mapsto (\Phi_1(m_1), \Phi_2(m_2))
\]

Hence, if \( \Phi \downarrow \Delta \) we have that \( \Phi^{-1}(\Delta) \) is an embedded submanifold of \( \Pi_1 \times M_2 \) and

\[
\operatorname{codim}(\Pi_1 \times M_2) = \operatorname{codim} \Delta \implies \dim(N \times N) = \dim \Pi_1 + \dim M_2 - \dim \Delta
\]

In fact, we claim that:

\[
\operatorname{Im} \operatorname{d}_{\Phi^{-1}(\mu_1, \mu_2)} \Phi + \operatorname{T}_{\Phi^{-1}(\mu_1, \mu_2)} \Delta = \operatorname{T}_{\Phi^{-1}(\mu_1, \mu_2)} (N \times N), \forall (\mu_1, \mu_2) \in \Phi^{-1}(\Delta)
\]

\[
\therefore \left\{ (d_{\mu_1} \Phi, u_1, d_{\mu_2} \Phi, u_2) : u_1 \in T_{\mu_1} M_1, u_2 \in T_{\mu_2} M_2 \right\} + \left\{ (v, w) : v \in T_{\mu_1} N \right\} = \left\{ (w_1, w_2) : w_1 \in T_{\mu_1} N, w_2 \in T_{\mu_2} N \right\}
\]

where \( \mu = \Phi_1(\mu_1) = \Phi_2(\mu_2) \).
\( \Leftarrow \) For any \( w_1, w_2 \in T_m N \), there exists \( v \in T_m N \), \( u_1 \in T_{w_1} H_1 \), \( u_2 \in T_{w_2} H_2 \) such that:

\[
\begin{align*}
    w_1 &= v + d_{w_1} \Phi_1 \cdot u_1 \\
    w_2 &= v + d_{w_2} \Phi_2 \cdot u_2
\end{align*}
\]

(\#)

Assume, e.g., that \( \Phi_1 \) is a submersion. Given \( w_1, w_2 \in T_m N \), choose any \( u_2 \in T_{w_2} M_2 \) and define \( v \in T_m N \) by:

\[
v := w_2 - d_{w_2} \Phi_2 \cdot u_2
\]

Since \( d_{w_1} \Phi_1 \) is surjective, there exists some \( u_1 \in T_{w_1} H_1 \) such that:

\[
    w_1 - v = d_{w_1} \Phi_1 \cdot u_1
\]

This shows that (\#) has a solution, as claimed.

\[
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\]