1. (40 Points) On $\mathbb{R}^4$ with coordinates $(x, y, z, w)$ consider the following 3 vector fields:

$$X_1 = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \quad X_2 = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \quad X_3 = x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x}.$$ 

Show that they span a 2-dimensional integrable distribution on $M = \mathbb{R}^4 - \{(0,0,0,w) : w \in \mathbb{R}\}$. What are the integral leaves?
2. (40 Points) Let $G \subset \text{GL}(2)$ be the 2-dimensional Lie group:

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x > 0, y \in \mathbb{R} \right\}$$

Show that the vector fields $\frac{x}{\partial x}$ and $\frac{y}{\partial y}$ form a basis for the Lie algebra $\mathfrak{g}$ of $G$. 

3. (40 Points) Let $M$ be an orientable manifold and let $\Phi : M \to \mathbb{R}$ be a smooth map. Show that if $0$ is a regular value of $\Phi$ then $\Phi^{-1}(0) \subset M$ is also an orientable manifold.

Hint: Observe that $N = \Phi^{-1}([0, +\infty))$ is a manifold with boundary $\Phi^{-1}(0) = \partial N$. 
4. (40 Points) In $\mathbb{R}^4$ with coordinates $(x, y, z, w)$, let $X = \cos w \frac{\partial}{\partial x} + \sin y \frac{\partial}{\partial z}$ and $\omega = x dx \wedge dy + z dz \wedge dw$. Compute the following:

(a) $i_X \omega$.

(b) $d\omega$.

(c) $\mathcal{L}_X \omega$.

(d) $\Phi^* \omega$, where $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ is the map $\Phi(t, u) = (t \cos t, u, t \sin t, u)$. 
5. (40 Points) Let $M$ be a smooth manifold and fix an orientation for $S^1$. Given a smooth map $\gamma : S^1 \to M$ and a differential 1-form $\alpha \in \Omega^1(M)$ define:

$$\int_{\gamma} \alpha := \int_{S^1} \gamma^* \alpha.$$ 

(a) Show that if $\alpha = df$, then for any $\gamma : S^1 \to M$:

$$\int_{\gamma} \alpha = 0.$$ 

(b) Show that if $d\alpha = 0$, and $H : [0, 1] \times S^1 \to M$ is a smooth map, then:

$$\int_{\gamma_0} \alpha = \int_{\gamma_1} \alpha,$$

where $\gamma_0(\theta) = H(0, \theta)$ and $\gamma_1(\theta) = H(1, \theta)$. 
