Let \( f: \mathbb{R}^n \to \mathbb{R}^p \) be of class \( C^1 \) on a neighborhood of \( a \in \mathbb{R}^n \) and assume that:

\[ f(a) = 0 \quad \& \quad \text{RANK} \left( \text{D}f(a) \right) = p \]

Then there is an open set \( A \subset \mathbb{R}^n \) and \( h: A \to \mathbb{R}^m \) of class \( C^1 \) with inverse of class \( C^1 \), such that \( h(a) = 0 \) and

\[ f \circ h^{-1}(x^1, \ldots, x^n) = (x^{n+1}, \ldots, x^p) \]

**Note:** A function \( h: \mathbb{R}^n \to \mathbb{R}^n \) of class \( C^1 \), with inverse of class \( C^1 \) is called a diffeomorphism. The theorem says that \( f \) is locally, up to a diffeomorphism, projection in the last \( p \) components.
Proof: Consider the jacobian matrix of $f : \mathbb{R}^n \to \mathbb{R}^p$ at $a$:

$$ f'(a) = \begin{bmatrix} \star & \ldots & \star \end{bmatrix} \text{ p rows } \text{ m columns} $$

Since this matrix has rank $p$, there are $p$ columns which are linearly independent, corresponding to variables $x_i, \ldots, x_p$. Let $\pi : \mathbb{R}^m \to \mathbb{R}^n$ be a permutation of the variables sending these variables to the last $p$ entries:

$$ \pi(x_1, \ldots, x_m) = (\ldots, x_i, \ldots, x_p) $$

Note that $\pi$ is of class $C^1$ with inverse of class $C^1$ (it is a linear map).

If we consider the map $f \circ \pi^{-1} : \mathbb{R}^{m+p} \times \mathbb{R}^p \to \mathbb{R}^p$, at the point $\pi^*(a) = (b, c)$, we have by the chain rule:

$$ (f \circ \pi^{-1})'(b, c) = \begin{bmatrix} \star & L \end{bmatrix} \text{ p rows } \text{ (m-p)columns } \text{ p-columns} $$

where $L$ is invertible. Hence, the map $F : \mathbb{R}^{m+p} \times \mathbb{R}^p \to \mathbb{R}^{m+p} \times \mathbb{R}^p$:

$$ F(x, b) = (x \circ b, f \circ \pi^{-1}(x, b)) , \quad F(b, c) = (0, f(a) = 0 $$

has jacobian matrix:

$$ F'(b, c) = \begin{bmatrix} a & 0 \\ \star & L \end{bmatrix} \Rightarrow \text{det}(F'(b, c)) \neq 0 $$
By the inverse function theorem, we find $F : \mathbb{R}^n \to \mathbb{R}^m$
of class $C'$ defined in some open set containing $0 \in \mathbb{R}^n$.

Note that:

$$F \circ F^{-1}(x,y) = (x,y) \iff F \circ F^{-1}(0,0) = y$$

Therefore, we may take $h = F \circ \pi : \mathbb{R}^m \to \mathbb{R}^m$, so

that $h(u) = F(b,c) = 0$ & $F \circ h(x,y) = y$. \[\square\]