Department of Mathematics
University of Illinois at Urbana-Champaign

MATH 417 – SPRING 2017 – SECTION B1
MIDTERM 3

APRIL 21, 2017

SOLUTIONS

1. (25 points) Determine all the ideals of $\mathbb{Z}$ that contain the ideal $I = \langle 24, 36, 60\rangle$.

SOLUTION: Since we have the prime factorizations:

$$24 = 2^3 \cdot 3, \quad 36 = 2^2 \cdot 3^2, \quad 60 = 2^2 \cdot 3 \cdot 5,$$

it follows that $\gcd(24, 36, 60) = 2^2 \cdot 3 = 12$. Hence, we have:

$$I = \langle 24, 36, 60\rangle = \langle 12\rangle.$$

Now, every ideal in $\mathbb{Z}$ is principal, and we have $\langle a\rangle \subset \langle b\rangle$ if and only if $b|a$. Hence, the ideals that contain I are the principal ideals $\langle b\rangle$ with $b|12$, namely:

$$\langle 1\rangle, \quad \langle 2\rangle, \quad \langle 3\rangle, \quad \langle 4\rangle, \quad \langle 6\rangle, \quad \langle 12\rangle.$$

2. (25 points) Determine integers x and y such that

$$\gcd(135, 1987) = x135 + y1987.$$

SOLUTION:

We use Euclides Algorithm. First, using the division algorithm, we obtain:

$$\begin{aligned} 1987 &= 14 \times 135 + 97 \\ 135 &= 1 \times 97 + 38 \\ 97 &= 2 \times 38 + 21 \\ 38 &= 1 \times 21 + 17 \\ 21 &= 1 \times 17 + 4 \\ 17 &= 4 \times 4 + 1 \\ 4 &= 4 \times 1 + 0 \end{aligned}$$

so we have:

$$\begin{aligned} \gcd(1987, 135) &= \gcd(135, 97) = \gcd(97, 38) = \gcd(38, 21) = \\ &= \gcd(21, 17) = \gcd(17, 4) = \gcd(4, 1) = 1. \end{aligned}$$

By solving back the equations, we find:

$$\begin{aligned} 1 &= 17 - 4 \times 4 \\ &= 17 - 4 \times (21 - 1 \times 17) = 5 \times 17 - 4 \times 21 \\ &= 5 \times (38 - 1 \times 21) - 4 \times 21 = 5 \times 38 - 9 \times 21 \\ &= 5 \times 38 - 9 \times (97 - 2 \times 38) = -9 \times 97 + 23 \times 38 \\ &= -9 \times 97 + 23 \times (135 - 1 \times 97) = 23 \times 135 - 32 \times 97 \\ &= 23 \times 135 - 32 \times (1987 - 14 \times 135) = 471 \times 135 - 32 \times 1987 \end{aligned}$$

So the answer is $x = 471$ and $y = -32$.

3. (25 points) Does the following equation have any solutions? If so, what are they?

$$x^3 + 1 \equiv 0 \pmod 5.$$

SOLUTION:

Observe that:

x	$x^3 + 1$
0	1
1	2
3	28
4	65

Since any x is congruent $\pmod 5$ to one, and only one number, 0, 1, 2, 3 or 4, we conclude that the solutions of the equation are $x \equiv 4 \pmod 5$, i.e., $x = 4 + 5k$ with $k \in \mathbb{Z}$.

4. (25 points) Let A be a commutative unitary ring of characteristic n. Show that the ring $M_k(A)$ consisting of $k \times k$-matrices with entries in A also has characteristic n.

Solution:

If 1 denotes the identity of the ring A, then the identity $\mathbf{1}$ of the ring $M_k(A)$ is the $k \times k$ diagonal matrix with 1 in each entry:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & \cdots & 1 \end{pmatrix}$$

Obviously, if we add n-times the matrix $\mathbf{1}$ we obtain the diagonal matrix:

$$n\mathbf{1} = \begin{pmatrix} n1 & 0 & \cdots & 0 \\ 0 & n1 & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & \cdots & n1 \end{pmatrix}$$

It follows that:

$$n\mathbf{1} = 0 \quad \iff \quad n1 = 0.$$

Recalling that the characteristic of a unitary ring is $n > 0$ if n is the smallest integer such that $n1 = 0$, and is $n = 0$ if no multiple of the identity is zero, we conclude that A and $M_k(A)$ have the same characteristic.

5. (Extra Credit: 10 points) Find all solutions of the system of equations:

$$\begin{cases} 3x \equiv 2 \pmod 5 \\ 2x \equiv 3 \pmod 7 \end{cases}$$

Solution:

Since 5 is prime, the number 3 is invertible $\pmod 5$. Its inverse is 2, since $2 \times 3 \equiv 1 \pmod 5$. Hence, the first equation has solution:

$$\begin{aligned} 3x \equiv 2 \pmod 5 \quad &\iff \quad 2 \times 3x \equiv 2 \times 2 \pmod 5 \\ &\iff \quad x \equiv 4 \pmod 5 \\ &\iff \quad x = 4 + 5y \quad (y \in \mathbb{Z}) \end{aligned}$$

Replacing in the second equation, we obtain:

$$\begin{aligned} 2x \equiv 3 \pmod 7 \quad &\iff \quad 2 \times (4 + 5y) \equiv 3 \pmod 7 \\ &\iff \quad 10y \equiv -5 \pmod 7 \\ &\iff \quad 3y \equiv 2 \pmod 7 \end{aligned}$$

Since 7 is prime, the number 3 is invertible $\pmod 7$. Its inverse is 5, since $5 \times 3 \equiv 1 \pmod 7$. Therefore:

$$\begin{aligned} 3y \equiv 2 \pmod 7 \quad &\iff \quad 5 \times 3y \equiv 5 \times 2 \pmod 7 \\ &\iff \quad y \equiv 10 \pmod 7 \\ &\iff \quad y = 10 + 7k \quad (k \in \mathbb{Z}) \end{aligned}$$

Finally, replacing back in the expression for x, we conclude that the solutions of the system are:

$$x = 4 + 5y = 4 + 5(10 + 7k) = 54 + 35k, \quad (k \in \mathbb{Z}).$$